Even the Minkowski space is holed

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Abstract

To cure the lack of predictive power of general relativity, Geroch proposed to complete the theory with an additional postulate that only "hole-free" spacetimes are permitted. This postulate (or, rather, that obtained from it by some disambiguating) seems physically well-founded and at the same time appropriately restrictive. I show, however, that it is too strong — it prohibits even the Minkowski space.

1 Introduction

Consider a space obtained by removing from the Minkowski plane the angle

$$|t/x| > 2,$$

[see Fig. 1(a)], and by gluing then the rays $t/x = \pm 2$, $t > 0$ together (the points are identified if they have the same t-coordinate). Evidently, M, though being a legitimate spacetime[1], is singular and it is the singularities of this type that are our subject (a simple example of a non-flat spacetime M with a similar singularity is this: from an arbitrary n-dimensional space-time remove an $(n-2)$-dimensional sphere and let M be the double covering of the resulting space, see Fig. 1b). Their peculiarity is that they, in fact, deprive general relativity of its predictive power. Indeed, in contrast to the "usual", curvature singularities, these "topological" ones are absolutely "sudden": *nothing* would tell an observer approaching such a singularity that his world line will terminate in a moment. In the presence of such singularities, everything (the geometry of the universe, its topology, causal structure, etc.) may change whimsically and (apparently) causelessly. For example, the spacetime depicted in Fig. 1(a) is just the Minkowski space up to some

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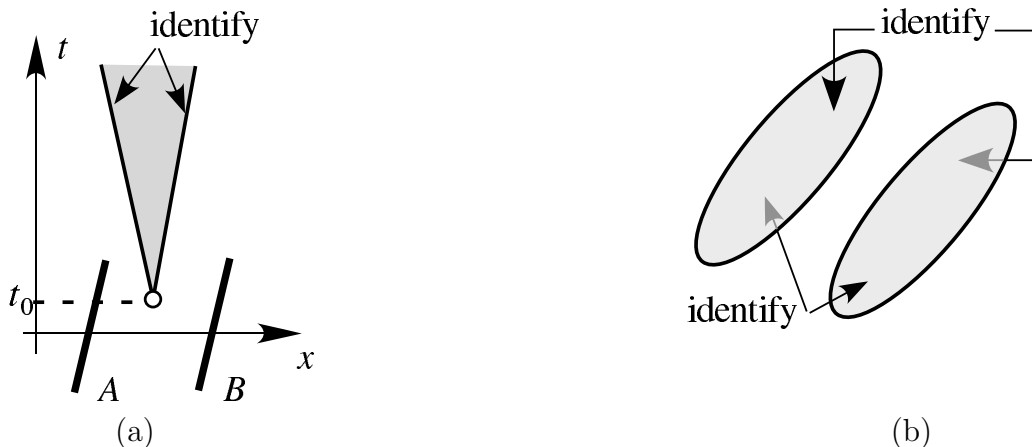[1]It is a smooth connected pseudo-Riemannian manifold.

Figure 1: (a) The spacetime is obtained by cutting out the shadowed sector from the Minkowski space and gluing together the rays bounding the sector (the rays do not comprise the vertex of the angle). (b) Remove a disc D from a domain N, take a copy of this incised spacetime, and identify the upper bank of either cut with the lower bank of the other. If N and $\tilde{N}$ are different spacetimes, the result is a double covering of $N - S$, where $S = \mathrm{Bd}\, D$ is a sphere of co-dimension 2. If they are just different regions of the same spacetime, then, depending on the position of the disks, the resulting spacetime is either a time machine [1] or a wormhole [2].

moment t_0. But after that moment some observers (A and B in the figure) will discover that without experiencing any acceleration they started to move towards each other. Figure 1(b) shows how in the otherwise Minkowskian space a time machine or a wormhole may appear with no visible cause (more of bizarre examples can be found in [3]). "Thus general relativity, which seemed at first as though it would admit a natural and powerful statement at prediction, apparently does not" [4].

The real problem with the just described singularities is that none of them have ever been observed (see [5], though). So, it seems that we are overlooking some fundamental law of nature. Arguably this law may be formulated as an additional (non-local, of course) requirement on the structure of spacetime which would explicitly prohibit some singularities. In looking for such a requirement it is desirable, first, to comprehend what *exactly* is to be prohibited. In particular, each of the singular spacetimes mentioned

2

above can be viewed as a quasiregular singularity (i. e., a singularity with bounded curvature, see [6] for a rigorous definition), or as an "absolutely mild singularity" [3] (it has a finite covering by open sets, each of which can be extended to a singularity-free spacetime). It is also a "locally extendible" spacetime (i. e., it contains an open set $U \subset M$ isometric to a subset U of some other spacetime M such that the closure of U in M is compact while the closure of U in M is not [7]). And, finally, it is a "hole" in the following sense (up to two subtle points, the definition below is that of [4]; the difference is brought about by disambiguation of the latter, see the Remark below).

Definition. Denote by $D^+(S)$ the collection of all points p of M such that every future-directed timelike curve in M, having future endpoint p and no past endpoint, meets $S \subset M$. A space-time (M, g) is called *hole-free* if it has the following property: given any achronal hypersurface[2] S in M and any metric preserving embedding π of an open neighbourhood U of $D^+(S)$ into some other spacetime (M, g), then $\pi(D^+(S)) = D^+(\pi(S))$.

So, which of these — quite different — classes of spacetimes should be excluded?

Two first possibilities seem too *ad hoc*. The third one once seemed more promising, but Beem and Ehrlich demonstrated [8] that even the Minkowski space is locally extendible — which, in my belief, rules out this variant: a postulate is definitely *too* strong if it excludes the Minkowski space. So, we are left[3] with Geroch's proposal to "modify general relativity as follows: the new theory is to be general relativity, but with the additional condition that only hole-free spacetimes are permitted". The proposal seems to be physically well motivated, the idea behind it being to remedy the following "defect" of general relativity: "...although S determines what happens in $D^+(S)$, what this $D^+(S)$ will be, and in particular how "large" it is, requires knowledge not only of S, but also of the space-time M, g in which S is embedded" [4].

Recently, Manchak has constructed a spacetime which is inextendible and globally hyperbolic but fails to be hole-free [9]. This spacetime possesses a

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[2]In Geroch's paper, in contrast to this one, the four-dimensional case is considered, but the generalization of what follows to four dimensions is trivial.

[3]As long, that is, as we are restricted to the classes listed above. There are also holes* [9], Clarke holes [10], and more.

3

nasty singularity, so one might wonder if its exclusion is that great loss for the theory. Still, some suspicion appeared that the requirement for a physically reasonable spacetime to be hole-free might be too strong. In the present paper I show that this is the case.

Remarks on terminology. 1) In giving the definition to $D^+(S)$, Geroch [4] refers to [7], where that object is defined for *arbitrary* S. On the other hand, *explicitly* the definition in [4] is formulated only for sets S which are three-dimensional achronal surfaces and this is in accord with [11], to which a reference is given, too. This discrepancy is immaterial within the range of problems studied in [11][4], but becomes important in our case: in defining hole-freeness one could — without contradicting [4], formally at least — restrict oneself to S and π such that $\pi(S)$ is achronal too. One would then arrive at a *different* property — let us call it *hole-freeness* — and, correspondingly, at the postulate that only hole-free spacetimes are permitted[5]. This postulate is less restrictive and fails to fulfill its function: the existence of a point is now determined not only by S, but also by the entire $I^-(S)$, so by requiring that spacetimes would be hole-free one does not eliminate the above-mentioned "defect". Which suggests that hole-freeness in Geroch's proposal must be understood in the sense of the above-formulated definition. 2) We have chosen the domain of π to be an open neighbourhood of $D^+(S)$ [not just $D^+(S)$, as in [4]] to avoid some ambiguity arising when the term "embedding" is applied to a set which is not a manifold. This must not, however, lead to any weakening of the result, because a metric preserving embedding of U is at the same time a metric preserving embedding of $D^+(S)$ whichever way the latter is defined.

Proposition. The Minkowski plane is not hole-free.

2 Proof

Let S be the hyperbolae $t = -\sqrt{x^2 + 1}$. Then obviously $D^+(S)$ is the closed set bounded by this hyperbolae from below and by the angle $t = -|x|$ from

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[4]"[. . .] the definition of the domain of dependence could be applied equally well when S is not achronal, but [. . .] the consequence is to introduce additional complications without adding anything really new" [11].

[5]It is probably this approach that eventually leads to the concept of hole-freeness* [9].
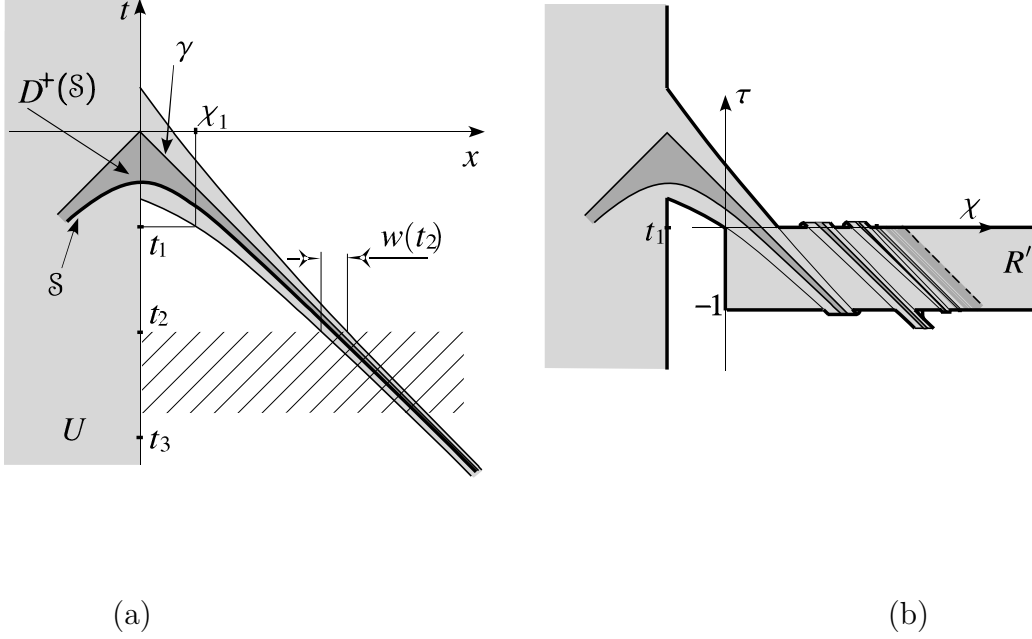
(a)

(b)

Figure 2: (a) The light-gray area U is a neighbourhood of the dark-gray area $D^+(\mathsf{S})$. The intersection of U with the hatched strip is O_2. (b) The light-gray area is M — an extension of U alternative to the Minkowski plane.

above, see figure 2a. The angle consists of two rays and the right one (i. e., the null geodesic $x = -t$ from the origin) we shall denote by γ. The neighbourhood U of $D^+(\mathsf{S})$ is defined to be the union of the left half-plane and the strip bounded by the graphs of the functions

$$x(t) = \sqrt{t^2 - 1} - e^t \quad \text{and} \quad x(t) = e^t - t$$

The mentioned strip at, say, $t < -2$ is characterized by width w:

$$w(t) \equiv -\sqrt{t^2 - 1} - t + 2e^t,$$

which monotonically decreases with $|t|$.

Now pick a sequence of negative numbers such that $t_1 < -2$ and

$$t_{k+1} < t_k - 1, \quad w(t_k) < 2^{-k} \qquad k = 1, 2, \ldots$$

and define the following portions of the strip:

$$O_k \equiv \{p \in U: \quad x(p) > 0, \quad t_k - 1 < t(p) < t_k\}, \qquad O \equiv \bigcup_k O_k.$$

5

To build the spacetime M, consider a flat strip

$$R: \qquad \mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\chi^2,$$
$$\tau \in (-1, 0), \quad \chi > 0$$

and the isometry Ψ which sends, for every k, each point $p \in O_k$ to the point $q \in R$ according to the rule[6]

$$\tau(q) = t(p) - t_k, \qquad \chi(q) = x(p) - \chi_k$$

$$\chi_1 \equiv \sqrt{t_1^2 - 1} - e^{t_1}, \qquad \chi_k \equiv \chi_1 - t_k + t_1 - \sum_{i=1}^{k-1} w_i \quad \text{at } k > 1.$$

Now M is defined as the quotient of $U \cup R$ over Ψ [i. e., as the result of gluing together U and R by O, see Fig. 2(b)]:

$$M \equiv U \cup_\Psi R.$$

The natural projection $\pi \colon U \to M$ is an isometric embedding, so it only remains to prove that $\pi(D^+(\mathsf{S})) \neq D^+(\pi(\mathsf{S}))$.

To this end consider the null geodesic $\pi(\gamma)$ and denote by L the part of it that lies in $R \equiv \pi(R)$. L is the set of the geodesic segments γ^k which, in the convenient coordinates

$$u(q) \equiv \chi(p) + \tau(p), \quad v(q) \equiv \chi(p) - \tau(p), \qquad p \in R, \quad q = R,$$

are the segments $\gamma^k \colon \quad u = u_k$, where

$$u_k \equiv -t_k - \chi_k = |t_1| - \chi_1 + \sum_{i=1}^{k-1} w_i, \qquad k = 1, 2 \ldots$$

(at $k = 1$ the last term is understood to be zero). Now note that u_k grow with k and converge to some u. Let us check that the existence of the geodesic $\gamma \colon \quad u = u$ proves our claim (in fact, this is obvious from Fig. 2(b), where γ is shown by the dashed line).

Indeed, any $p \in R \cap \pi(D^+(\mathsf{S}))$ belongs to some O_k. So, $u(p)$ must be less than u_k and hence strictly less than u. Thus $p \notin \gamma$. On the other hand,

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[6]The recurrent form of the last formula may be more transparent: $\chi_k = \chi_{k-1} + (t_{k-1} - t_k) - w_{k-1}$

any past directed timelike curve α through a point $a \in \gamma$ must have points with the u-coordinate less than u, hence α meets a γ_k in some point s. If $s \notin D^+(\pi(\mathsf{S}))$ (which is, in principle, imaginable, though not, perhaps, in our case), then the proof is completed, because s is a point of $\pi(D^+(\mathsf{S}))$. And if $s \in D^+(\pi(\mathsf{S}))$, then α being extended sufficiently far to the past must meet $\pi(\mathsf{S})$, which implies that a (and thus the entire γ, too) are in $\pi(D^+(\mathsf{S}))$.

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