

Book review

The Haskell Road to Logic, Maths and Programming, Kees Doets and Jan van Eijck, Texts in Computing Series, King's College Publications, London, 2004. Price: £14.00 pounds, \$ 25.00, 444 pages, Paperback, ISBN: 0-9543006-9-6.

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1 Preamble

In university curricula, the subjects of programming and maths and logic tend to be separated. For instance, in the typical computer-science curriculum, mathematics and logic are taught according to the (dry) mathematical tradition, perhaps with modern concessions to help the less abstract minds. One sort of concession is to illustrate the implementability of selected mathematical concepts in some programming language, be it C, Fortran, or others. Another sort of concession is to hint at the usefulness of systems like Maple or Mathematica — systems that are probably more appreciated by mathematicians than applied, programming-minded computer scientists. By contrast, programming courses are almost certainly focused on a specific programming paradigm and on a specific language. Logic and maths may show up in such courses as a sort of “application domain”, while competing with many other domains. For instance, a typical course on the programming language Haskell is likely to emphasize Haskell’s strengths in the areas of abstract-data type specification and combinator libraries for domain-specific languages.

The authors of the book at hand have succeeded in amalgamating the themes of maths, logic and Haskell programming in a mutually beneficial manner. The book really stands out in so far that it *leverages* Haskell in diving deep into maths and logic (as deep as covering power series or co-inductive proofs). The use of Haskell makes the mathematical concepts quite digestible; Haskell lends itself so well as a strongly typed and executable modeling language. The authors’ style is quite relaxing. The mathematical concepts are explained in a way that carefully helps with an intuitive understanding.

The “Haskell road to Logic, Maths and Programming” by Kees Doets and Jan van Eijck does *not* (mean to) comprise a detailed and comprehensive introduction to functional programming or the programming language Haskell. There exist such introductions to Haskell; cf. the excellent textbooks by Thompson and Hudak [14, 6], and these introductions should still be useful for readers who want to deeply appreciate Haskell as a general-purpose programming language and Haskell’s strengths other than maths and logic. The book at hand does not assume prior knowledge of Haskell, though; all Haskell concepts are introduced incrementally. The combination of texts on maths/logic with computer programming is not unprecedented. For instance, Barwise and Etchemendy’s “Language, Proof, and Logic” [2] accompanies a text on logic with several computer programs. Even more to the point, Hall and O’Donnell’s “Discrete math-

ematics using a computer” [5] covers maths and logic in similar ways than the book at hand, and it also facilitates Haskell.

2 Chapter by chapter

Preface

This is an outstanding book, but the preface could be improved. In particular, the preface seems to be somewhat ambiguous about the purpose and the subject of the book. Citation: “The purpose of the book is to teach logic and mathematical reasoning in practice and to connect logical reasoning with computer programming.” The mentioning of “practice” may count as confusing. Elsewhere in the preface: “The subject of the book is the use of logic in practice, more in particular the use of logic in reasoning about programming tasks.” It is not immediately obvious to the reader what sorts of “programming tasks”, what sorts of “reasoning” are meant here. (For instance, one may think that program validation and verification is included, but this turns out not to be the case. Likewise, one may expect reasoning about complexity.) It would help if the following questions were answered more clearly: What are the *mathematical fields* covered by the book? What *sorts of reasoning* about either mathematical notions or programs or both are carried out?

According to the preface, Haskell is a member of the LISP family with the lambda calculus as foundation. This characterization does not hint at two of Haskell’s strengths: its advanced type system and its lazy semantics. (LISP, in particular, does not share these assets.) Both strengths are exploited in the book, and this is worth emphasizing. The choice of Haskell is less arbitrary than it may seem from the preface.

The outline of the chapters could be more helpful. Regarding chapters 1–7, there is basically one sentence. The explanations for the remaining chapters may overwhelm newcomers with technical terms (cf. Chapter 9: “closed forms for polynomial sequences”). It would be wonderful to read a preface that provides an accessible overview of the chapters, including a story how these subjects fit together, making brief links to the three components of the book’s title: logic, maths, programming.

Chapter 1: Getting started

The text starts with some basic explanations of literate programming, lambda calculus and typed functional programming. Then, the text picks up the problem of prime-number testing for a first illustration of the “Haskell Road to Maths”. The text is pleasant to read, and it introduces a good deal of mathematical conventions and Haskell idioms including primitive types, products and lists. The text also states clearly an important goal of the book early on: to communicate skills for the formulation of properties about programs and their proofs. The textbook style of the book also settles immediately: a very good

mix of definitions, explanations, examples and exercises.

Eventually, the chapter describes prime factorization and prime-number generation. Surprisingly, the authoritative definition of the former problem is given in an imperative language notation with assignments! The Haskell implementation of the latter puts the higher-order combinators `map` and `filter` to work. This is an indication that the first chapter has come to real speed. Some bits of the Haskell exposition in this chapter lack context in the sense that idioms and examples do not contribute to the running theme of algorithms around prime numbers.

The chapter pleasantly closes by emphasizing a major strength of Haskell: this pure, declarative, functional language admits equational reasoning, which is out of reach for imperative languages like Java due to destructive assignments and other side effects in these languages. Like most chapters in the book, Chapter 1 carries a short tail “Further Reading”.

Chapter 2: Talking about Mathematical Objects

The chapter describes the basics of logic, in particular the usual connectives (not, if, and, or, iff, for all, there exists) complete with Haskell encodings of connectives, formulas, and validity checks for formulas. The text is very pleasant to read; the alternation between mathematical notation and Haskell encoding really makes sense. The text carefully explains the many aspects of quantification: free variables, open and closed formulas, domains for quantification. The text makes a laudable effort to help the beginning logician. For instance, the text identifies “bad habits” regarding the ambiguous use of quantifiers in textual formulations of mathematical statements.

The text makes clear that Platonism is adopted in this book: every mathematical statement is either true or false. Other forms of logic, such as intuitionistic logic or fuzzy logic are not covered.

A side note on Haskell’s type classes

Throughout the book, there are a few occasions where Haskell’s type classes could show off nicely. For instance, when checks for validity or logical equivalence are provided (cf. Chapter 2), they are coded separately for *several* arities, as indicated by the following examples:

```
valid1 :: (Bool -> Bool) -> Bool
valid1 bf = (bf True)
           && (bf False)
```

```

valid2 :: (Bool -> Bool -> Bool) -> Bool
valid2 bf = (bf True  True)
            && (bf True  False)
            && (bf False True)
            && (bf False False)

valid3 :: (Bool -> Bool -> Bool -> Bool) -> Bool
-- ... and so on ...

```

A single, generic `valid` function would suffice for all arities, as defined below. The same holds for logical equivalence. (It is left as an exercise for the reader to eliminate the minor violation of strict Haskell 98 rules.)

```

class Valid f
  where
    valid :: f -> Bool

instance Valid Bool
  where
    valid = id

instance Valid f => Valid (Bool -> f)
  where
    valid f = valid (f True)
              && valid (f False)

```

It is understandable that the text does not want to overwhelm the reader in the early chapters. However, the power of Haskell’s type classes could have been discussed in a later chapter. Fortunately, the reader will still hear about this important part of Haskell because a few chapters take advantage of some type classes from the Haskell library.

Chapter 3: The Use of Logic: Proof

The chapter communicates style recommendations and recipes for simple proofs. This is done without much reference to Haskell or Haskell examples. The idea seems to be that the chapter provides the foundation for carrying out mathematical proofs and proofs about program properties in all subsequent chapters.

A typical style recommendation is “Make sure the reader knows exactly what you are up to.” The various recipes basically cover all logical connectives — their introduction and elimination. And then again, even the use of the recipes is complemented by further recommendations such as not to consider *proof by contradiction* all too easily.

The authors deserve extra credit for their clear and detailed enumeration and rationalization of the proof recipes, complete with careful coverage of style issues. This textbook may really enable readers to acquire proving skills. Some of

the examples may be potentially too demanding for the beginning mathematician. For instance, the first example for $\exists$ -elimination refers to transcendent reals and their definition as not being roots of a polynomial with integer coefficients. (Polynomials and number theory are only covered much later in the book.)

The text readily admits that programming languages like Haskell are of little help with proving general properties about programs. One may use the computer though to refute single cases. Perhaps, this is too much to ask for, but two topics could have been touched upon in the context of this chapter:

- The Curry-Howard isomorphism, i.e., the correspondence between types and logical formulas, as much as expressions and logical proofs; see [13] for a functional-programming-minded discussion of this correspondence.
- The use of theorem provers to check or even to find proofs.

Chapter 4: Sets, Types and Lists

The chapter, according to the authors, presents ample opportunity to exercise (perhaps better: acquire) the reader's skills in writing implementations for set operations and for proving things about sets. This is perhaps a bit misleading. The text does certainly not discuss different (efficient) implementations of sets or other data types. In reality, the chapter focuses on the axiomatization of sets as well as pairs, products and lists. The text also explains Russell's paradox and the provisions by the Haskell type system that help avoiding such problems in actual Haskell programs. (The text also touches upon the relation between the Halting problem and Russell's paradox.) Eventually, several list- and set-processing primitives are encoded. These implementations facilitate the type classes `Eq` ("equality") and `Ord` ("ordering") from the Haskell prelude. Afterwards, the chapter spends some time on Haskell's special idioms for list comprehensions, which are clearly related to the mathematical notion of set comprehensions. List comprehensions are illustrated with a detailed example in the realm of relational database queries.

Regarding "Further Reading", some more references spring to mind. The reader could appreciate pointers to existing functional-programming-minded or Haskell-specific work on the efficient implementation of data types [11, 12]. The chapter's major example for list comprehensions, a database query scenario, is weakly typed: the type `String` is used as the universal data type for all cells in the database instance and all column names. Haskell can do much better using more sophisticated type-system features. Admittedly, substantially more strongly typed approaches require non-Haskell 98 features [8, 7]. It is an asset of the book at hand that it stays with the Haskell 98 standard.

Chapter 5: Relations

The chapter describes the theory of relations ("sets of ordered pairs"); it defines the common properties of relations and other related terms (reflexivity,

transitivity, equivalence relations, quotients, characteristic functions, etc.). The properties are also implemented as checks that can be effectively computed for relations that are given as finite sets of pairs. The chapter focuses on elementary material about relations. N-ary relations and the corresponding relational algebra are not covered. The chapter offers a huge amount of simple but insightful exercises with which the reader can develop skills in mathematical proofs. These exercises are particularly manageable since they assume mostly basic properties of sets and the chapter’s simple definitions for relations.

Chapter 6: Functions

The chapter is about functions as *mathematical objects*, not to be confused with Haskell functions. There is clearly a plethora of important mathematical notions around functions: function definition by case discrimination, the relational view on functions, surjections, injections, bijections, inverse, partiality, congruences, and so on.

As in previous chapters, the text captures properties of mathematical objects as Haskell expressions that perform “checks”. That is, a certain property of a function, such as injectivity, is checked under the assumption that the domain of the function is given as a (finite) list:

```
injective :: Eq b => (a -> b) -> [a] -> Bool
injective f [] = True
injective f (x:xs) =
    notElem (f x) (image f xs) && injective f xs
```

One could take this idea a tiny step further, by writing down QuickCheck [4] properties in Haskell. The QuickCheck tool would automatically check all these properties. For instance, rather than stating the mere expectation that `fromEnum` should be the left-inverse of `toEnum` (cf. page 222), one could instead formulate QuickCheck properties for different instances of `Enum` (i.e., for different enumeration types). Here is a sample property for the `Char` type:

```
prop_EnumChar = forAll charInts $ \x ->
    fromEnum ((toEnum x) :: Char) == x
charInts :: Gen Int
-- restrict Int generator to valid character codes
```

Chapter 7: Induction and Recursion

The chapter starts with the basic proof method for mathematical induction. The method is explained in all detail and with great care. In particular, the text clarifies that induction proofs are particularly straightforward whenever the function “under study” is defined by (primitive) recursion on the natural numbers. Induction and recursion is then also demonstrated for trees and lists. The text clearly hints at the full generality of the concepts: recursion schemes

can be advised for a large class of data types, and corresponding induction proof schemes are thereby enabled.

The “Further Reading” section would be more complete with some references to seminal work on recursion schemes, e.g., [9]. There is also a wonderful tutorial by Augusteijn, which applies morphisms (such as primitive recursion) to the rigorous design of sorting algorithms and their encoding in Haskell [1].

Chapter 8: Working with Numbers

The chapter presents the different number systems that exist: natural numbers, integers, rational numbers, irrational numbers, and complex numbers. By developing these number systems for Haskell, the text also provides deeper insight into Haskell’s built-in and library support for numbers.

As usual, natural numbers are shown to suffice as basis for integers and rational numbers. In fact, different representations for these two derived number systems are discussed and implemented in Haskell. Irrational numbers are defined informally (as infinite, non-periodic decimal expansions). Implementation-wise, the Haskell primitive type for reals is used. Key notions for reasoning about reals are defined: continuous functions, limits, Cauchy sequences.

Throughout the chapter, it is shown how the different number systems participate in standard type classes for equality, ordering and arithmetic operations. Standard algebraic properties of the operations on the numbers (associativity etc.) are routinely proved. For several of the number representations, geometric interpretations are visualized. These illustrations should be of great help for the reader.

Chapter 9: Polynomials

The chapter discusses polynomials with the initial goal to automate the search for polynomials as closed forms for sequences of integers. To this end, Babbage’s classic difference method is explained and implemented in Haskell. (Recall that the method works as follows: suppose we are given a “long enough” sequence of integers $a_0, \dots, a_n$, for which we presume that it may be computed from a polynomial $f(x)$ such that $a_0 = f(0), \dots, a_n = f(n)$, then we may attempt to find the closed form $f(x)$ in two steps. (i) Determine the degree of $f(x)$ by difference analysis on the given sequence of integers. (ii) Perform Gaussian elimination to determine the coefficients of $f(x)$.)

The authors succeed in capturing the beauty of this method, making clear that no magic is involved. Again, the combination of mathematical notation and Haskell encoding helps enormously to get all the fine details across. This chapter is a master piece: a concise but still comprehensive and insightful explanation of the method. All of Babbage, complete with the Haskell encoding on just 13 pages!

The chapter also discusses the link between polynomials and combinatorics. In this context, Newton’s binomial theorem is derived and put to work. These

parts of the chapter are slightly less clear in stating *up-front* what they aim to achieve.

We recall the simple version of Newton’s binomial theorem:

$$(z + 1)^n = \sum_{k=0}^n \binom{n}{k} z^k$$

The left-hand side is not in the form of a polynomial (it uses a binomial — a sum of two terms), whereas the right-hand side is in the form of a polynomial. Hence, one way to read Newton’s theorem is that it enables binomial expansions such that pure polynomials are derived.

It is straightforward to prove Newton’s theorem (by induction). The text however conveys the beauty of Newton’s findings by deriving the theorem through operations on polynomials: substitution and calculus. Again, the text communicates deep insight combined with a Haskell model.

The final part of the chapter discusses the representation of polynomials as lists of their coefficients. Polynomial arithmetics and calculus is then derived on top of this representation. The main goal is to reason about combinatorial problems. In particular, polynomials (as lists of coefficients) are derived as solutions to combinatorial problems. For instance, the polynomial expansion of $(z + 1)^{10}$ solves the problem of picking k elements from a set of 10.

Chapter 10: Corecursion

The chapter discusses infinite data structures (mostly streams). Haskell laziness makes this language clearly suited to study such data structures even programmatically. One important goal of the chapter is to describe proof methods for corecursive functionality. The text also makes excursions to the following two related subjects. Non-deterministic processes are modeled as functions from random integer streams (for coding “decisions”) to streams (for coding the actions performed). The earlier discussion of the link between polynomials and combinatorics is generalized to power series (so to say polynomials with infinite series of coefficients). Power series naturally show up once polynomials are divided; the results of division may have an infinite degree.

The authors make the point (in the introduction) that they deliver the first, general textbook treatment of recursion. This already indicates the challenge of this section. Some readers may expect from the chapter to become well-versed in coinduction proofs. Unfortunately, the text is scarce in this respect. There are essentially just 5 pages for the recipe of coinduction proofs. There is essentially just one example of a coinduction proof for streams, and one may argue whether or not the proof is detailed enough. The “Further reading” section should perhaps refer to more scenarios for coinductive reasoning, and more sample proofs. In particular, Barwise and Moss’ book on “Vicious circles” [3] springs to mind.

Chapter 11: Finite and Infinite Sets

The chapter can be seen as a more advanced continuation of Chapter 4: Sets, Types and Lists. The text begins with a deeper analysis of mathematical induction. Many additional insights are provided, when compared to the intuitive use of induction proofs in earlier chapters. For instance, the notion of strong induction is presented, which does not require a base step and uses a stronger induction hypothesis, thereby enabling proofs of more involved problems. Another subject of the chapter is equipollence, preparing ultimately for the discussion of cardinalities of sets including infinite cardinalities such as $\aleph_0$ (i.e., the cardinality of the natural numbers) and $2^{\aleph_0}$ (i.e., the cardinality of the powerset of the natural numbers). A few ideas are nicely illustrated through Haskell code, e.g., the diagonalization for the rational numbers, which shows that this set of numbers is countably infinite.

3 Non-Haskell roads to logic and maths

When reading this book, it is a useful exercise to question every now and then whether or not the Haskell road is a convenient one, what the issues would be when mainstream programming languages were used at times.

It seems to be undisputed that Haskell allows for a more or less immediate transcription of mathematical notation to program code. This is made possible by Haskell’s declarativeness, purity and conciseness. Haskell’s laziness further helps with encoding mathematical problems that involve infinite data structures. Haskell’s higher-orderness is essential to provide mathematical concepts as library functionality. (For instance, think of the higher-order function that turns a sequence of a polynomial’s coefficients in the actual function for the polynomial.) Finally, Haskell’s type system serves an important documentary purpose; machine-checked type information and executability are in fact the two major ways in which Haskell notation *complements* mathematical notation.

Without any claim of completeness, we would like to illustrate some of the issues that arise when encoding maths in mainstream languages such as C# or Java. For the code examples that follow, we will use C# (2.0), but we could have used Java (1.5) just as well, without changing the tenor of the observations.

Consider the following Haskell function, `difs`, for computing a difference sequence. This operation is at the heart of Babbage’s difference engine. (Recall: one uses the function `difs` to determine the degree of a potential polynomial $f(x)$ that should be derived as the closed form from a “long enough” sequence of integers $a_0, \dots, a_n$, for which we assume that it corresponds to $f(0), \dots, f(n)$. The degree of $f(x)$ is the number of iterations of applying `difs` until the resulting sequence is a constant sequence, if this happens at all.)

```
difs :: [Integer] -> [Integer]
difs []    = []
difs [_]   = []
difs (m:n:ks) = n-m : difs (n:ks)
```

The following C# encoding may serve as an efficient transcription.

```
static Decimal[] difs(Decimal[] given)
{
    Decimal[] result = new Decimal[
        given.Length == 0 ? 0 : given.Length-1];
    for (int i=1; i<given.Length; i++)
        result[i-1] = given[i]-given[i-1];
    return result;
}
```

The C# function operates on arrays of values of type `Decimal` (which is C#'s counterpart to Haskell's arbitrary precision integers). The array for the result is explicitly allocated, leveraging the observation that the `difs` function returns a sequence that is one element shorter than the (non-empty) input sequence. All sense of equational reasoning is gone. The allocation code requires an extra proof in order to establish its correctness. There is actually no straightforward way to retain the recursive formulation of the problem. (For the record, with C++ pointer arithmetics, we were able to process the input array recursively. However, the recursive synthesis of the result would not work so easily.)

Hence, the mainstream road to maths is bumpy because of the encoding efforts; it will be rather painful once we wanted to reason about these programs. We attempt some variations in the sequel. The following C# variation uses the generic collection type `List` in place of arrays:

```
static List<Decimal> difs(List<Decimal> given)
{
    List<Decimal> result = new List<Decimal>();
    if (given.Count > 1)
    {
        result.Add(given[1]-given[0]);
        result.AddRange(difs(given.GetRange(1,given.Count-1)));
    }
    return result;
}
```

The earlier array location reduces to a simple object construction for the initially empty list. (Hence, we eliminated the need for the extra insight regarding the length of the result.) The `List` type is rich enough to re-establish the recursive coding style. The exercised style is rather inefficient. For instance, the `GetRange` method involves cloning of the “tail”. We also note that we still fail to provide any sort of equational style; so equational reasoning remains out of reach. Furthermore, we fail to obtain the conciseness of pattern matching.

Yet another deficiency is worth mentioning: The previous C# encodings are all eager. This is perhaps acceptable for the use of `difs` in the context of Babbage's method, but eventually all sorts of similar list processing functionality needs to be lazy. (Think of moving from polynomials to power series.) Here is

a more clever C# solution, which engages in less cloning, and which is lazy at the same time:

```
static IEnumerable<Decimal> difs(IEnumerable<Decimal> given)
{
    Decimal? i = null;
    foreach (Decimal j in given)
    {
        if (i.HasValue) yield return j-i.Value;
        i = j;
    }
}
```

That is, we use the `IEnumerable` interface and the `yield return` idiom to compute the resulting list lazily. (We also use C#'s nullable types; cf. “?”.) It is not an accident that this solution uses an imperative loop again. That is, C#'s streams (or lazy lists) are idiomatically restricted. Re-establishing recursion is a non-trivial undertaking. Here is a (prohibitively inefficient) attempt:

```
static IEnumerable<Decimal> difs(IEnumerable<Decimal> given)
{
    if (empty(given)) yield break;
    if (empty(tail(given))) yield break;
    Decimal m = head(given);
    Decimal n = head(tail(given));
    yield return n-m;
    foreach (Decimal i in difs(tail(given)))
        yield return i;
}
```

One major problem is that the streams returned from the recursive calls needs to be pulled into the local result stream explicitly through the (expensive) `foreach` loop. Also, the code is baroque anyhow, even though we anticipated the possibility of helpers to take apart lists; cf. `empty`, `head`, `tail` (definitions omitted).

Bottom line This detour has illustrated that relatively simple pieces of mathematical definitions can be difficult to encode in mainstream programming languages. So anyone in search of a road to logic and maths, is well advised to favor the Haskell road, not to get lost in the programming bits of such an endeavor. It is a unique capability of Haskell that mathematical notation (in the form of equations, sums, products, compound expressions, etc.) is type-able and executable in this programming language more or less as is. The book under review leverages this capability very appropriately.

4 Concluding remarks

Doets and van Eijck’s “The Haskell Road to Logic, Maths and Programming” is an astonishingly extensive and accessible textbook on logic, maths, and Haskell. The book adopts a systematic but relaxed mathematical style (definition, example, exercise, ...); the text is very pleasant to read due to a small amount of anecdotal information, and due to the fact that definitions are fluently integrated in the running text. The overall selection of mathematical subjects is non-surprising (which is good), except that some advanced topics are included such as corecursion and Cantor’s infinities. One would perhaps expect to see a more substantial treatment of calculus and some coverage of statistics.

Regarding the targeted audience, the text seems to be valuable and advisable for undergraduates with mathematics and logic as minor subjects. However, even mathematically ambitious readers should find this book interesting, be it then more for the Haskell part of it. The book was indeed developed through undergraduate courses at the University of Amsterdam.

The Haskell samples in the book appear to be highly idiomatic, and readers can certainly expect to learn Haskell programming, so to say as a side effect. Not surprisingly, the book does not cover all of a typical functional programming text. For instance, type classes are used only in a minimalistic way. Also, monads do not show up anywhere in the book. While this is normally “unimaginable” for a Haskell introduction, it makes total sense for this specific book. The book’s write-up and program development has been based on literate programming. (However, the sources that are obtainable from the authors’ website do not currently comprise the text of the chapters; presumably for copyright reasons.)

Regarding the availability of the book: the “Texts in Computing Series” of *King’s College Publications* are published on a “Print on Demand” basis. This has as the benefit that the price is much lower than with regular publishers — at the time of writing: 25\$ vs. 14£. Some may argue that it also has the minor drawback that the book is not for sale in local bookstore, but has to be ordered through Amazon instead.

I take the liberty to close this review with an admittedly personal comment. I would have wanted this very book during my computer science studies. The Haskell way of modeling mathematical concepts in a declarative, executable and strongly typed manner is very supportive in learning. The small but telling distance between mathematical notation and Haskell notation makes it also more rewarding to actually engage in reasoning about mathematical problems and programs. I still remember wasting time on engaging in Pascal encodings for some of my maths homework. Should I end up giving lectures on maths and logic, I will love to use this book in such a course.

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