

Strict detector-efficiency bounds for n -site Clauser-Horne inequalities

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An analysis of detector-efficiency in many-site Clauser-Horne inequalities is presented, for the case of perfect visibility. It is shown that there is a violation of the presented n -site Clauser-Horne inequalities if and only if the efficiency is greater than $\frac{n}{2n-1}$. Thus, for a two-site two-setting experiment there are no quantum-mechanical predictions that violate local realism unless the efficiency is greater than $\frac{2}{3}$. Secondly, there are n -site experiments for which the quantum-mechanical predictions violate local realism whenever the efficiency exceeds $\frac{1}{2}$.

The Bell, Clauser-Horne-Shimony-Holt (CHSH), and Clauser-Horne (CH) [1] inequalities state what correlations or probabilities are to be expected from a local realistic model. The experimental system used to test these inequalities (or rather, their prerequisites) is shown schematically in figure 1. Tests made to date are subject to different loopholes such as reduced efficiency, reduced visibility, and certain problems of obtaining strict locality, making the application of the above inequalities depend on extra assumptions. The search for experiments where less assumptions are needed have reached quite far, but one loophole remains: detector inefficiency, which is usually dealt with by making the no-enhancement assumption. The present paper is an analysis without use of this assumption of efficiency bounds for the CH inequality and some n -site generalizations of it.

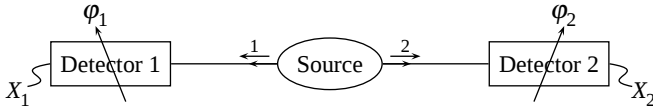


FIG. 1. A schematic picture of the physical system used in the two-site case. The source emits entangled pairs of particles, one at each detector site. The detectors have a local (angular) setting $\varphi_i \in \{a, b\}$.

The formalism of probability theory will be used below, where the event space Λ is a space of points λ corresponding to a certain value of the “hidden variable”. The measurement results are described by random variables which take their values in the value space V (here $\{0, 1\}$), and on the space there is a probability measure P , which gives the probabilities of events, i.e., subsets of Λ .

The Bell and CHSH inequalities are relations on correlations, and contain various restrictions on the measurement results. For example, in the case of the Bell inequality, perfect correlation at equal detector settings and the labeling ± 1 of the results, which makes it unsuitable for use in experiments where noise and inefficiency

occurs. The CHSH inequality may be used in noisy situations, but still cannot directly accommodate inefficiencies of the measurement setup (this can be remedied; see [2]). The CH inequality, on the other hand, is a relation on probabilities and does not need such restrictions on the results [3] but only the prerequisites of Realism and Locality.

Theorem 1: (Clauser-Horne) If we have

- (i) *Realism.* Measurement results can be described by probability theory, using (two families of) random variables.

$$\begin{aligned} X_1(\varphi_1, \varphi_2) : \Lambda &\rightarrow V \\ \lambda &\mapsto X_1(\varphi_1, \varphi_2, \lambda) \\ X_2(\varphi_1, \varphi_2) : \Lambda &\rightarrow V \\ \lambda &\mapsto X_2(\varphi_1, \varphi_2, \lambda) \end{aligned} \quad \forall \varphi_1, \varphi_2.$$

- (ii) *Locality.* A measurement result at one site should be independent of the detector orientation at the other site,

$$\begin{aligned} X_1(\varphi_1, \lambda) &\stackrel{\text{def}}{=} X_1(\varphi_1, \varphi_2, \lambda), \text{ independently of } \varphi_2 \\ X_2(\varphi_2, \lambda) &\stackrel{\text{def}}{=} X_2(\varphi_1, \varphi_2, \lambda), \text{ independently of } \varphi_1 \end{aligned}$$

except at a null set, then, with $A_i = X_i(a, \lambda)$ and $B_i = X_i(b, \lambda)$,

$$\begin{aligned} P(A_1 = B_2 = 1) + P(B_1 = A_2 = 1) - P(B_1 = B_2 = 1) \\ \leq P(A_1 = 1) + P(A_2 = 1) - P(A_1 = A_2 = 1) \end{aligned} \quad (1)$$

Proof: Given (i), the following inequality is obviously true:

$$P(A_1 = B_2 = 1 \cup B_1 = A_2 = 1) \leq P(A_1 = 1 \cup A_2 = 1). \quad (2)$$

On the right-hand side we have

$$\begin{aligned} P(A_1 = 1 \cup A_2 = 1) \\ = P(A_1 = 1) + P(A_2 = 1) - P(A_1 = A_2 = 1), \end{aligned} \quad (3)$$

and on the left-hand side, by use of (ii),

$$\begin{aligned}
P(A_1=B_2=1 \cup B_1=A_2=1) \\
&= P(A_1=B_2=1) + P(B_1=A_2=1) \\
&\quad - P(A_1=B_1=A_2=B_2=1) \\
&\geq P(A_1=B_2=1) + P(B_1=A_2=1) \\
&\quad - P(B_1=B_2=1). \quad \blacksquare
\end{aligned} \tag{4}$$

This inequality is well suited for direct inclusion of noise and inefficiency, as can be seen in [4]. There, the Clauser-Horne inequality given in a slightly different form is used to investigate if it is violated by quantum mechanics at lower efficiencies than the 82.83% bound obtained from the CHSH inequality [2], and at which level of background noise this is possible. The calculation in [4] uses the quantum-mechanical expressions for the (ideal) probabilities, e.g.

$$P_{|\psi\rangle}(A_1=A_2=1) = \langle\psi|1_{A_1}1_{A_2}|\psi\rangle, \tag{5a}$$

$$P_{|\psi\rangle}(A_1=1) = \langle\psi|(|1_{A_1}\rangle\langle 1_{A_1}| \otimes \mathbb{I}_2)|\psi\rangle, \tag{5b}$$

to derive a quantum measurement operator $\mathcal{B}$ corresponding to the probabilities in the CH inequality including detector inefficiency:

$$\begin{aligned}
\langle\psi|\mathcal{B}|\psi\rangle &= \eta^2 \left(P_{|\psi\rangle}(A_1=A_2=1) + P_{|\psi\rangle}(A_1=B_2=1) \right. \\
&\quad \left. + P_{|\psi\rangle}(B_1=A_2=1) - P_{|\psi\rangle}(B_1=B_2=1) \right) \\
&\quad - \eta \left(P_{|\psi\rangle}(A_1=1) + P_{|\psi\rangle}(A_2=1) \right). \tag{6}
\end{aligned}$$

$$\eta \leq \frac{P_{|\psi\rangle}(A_1=1) + P_{|\psi\rangle}(A_2=1)}{P_{|\psi\rangle}(A_1=A_2=1) + P_{|\psi\rangle}(A_1=B_2=1) + P_{|\psi\rangle}(B_1=A_2=1) - P_{|\psi\rangle}(B_1=B_2=1)}. \tag{7}$$

Clearly,

$$P_{|\psi\rangle}(A_1=A_2=1) \leq \min_{i=1,2} P_{|\psi\rangle}(A_i=1) \tag{8a}$$

$$P_{|\psi\rangle}(A_1=B_2=1) \leq P_{|\psi\rangle}(A_1=1) \tag{8b}$$

$$P_{|\psi\rangle}(B_1=A_2=1) \leq P_{|\psi\rangle}(A_2=1) \tag{8c}$$

$$P_{|\psi\rangle}(B_1=B_2=1) \geq 0. \tag{8d}$$

The lowest possible bound in (7) would be obtained when we have equality in (8a–d), and when $P_{|\psi\rangle}(A_1=1) = P_{|\psi\rangle}(A_2=1)$ (which gives the best possible value in (8a) [5]). We then would have

$$\eta \leq \frac{2P_{|\psi\rangle}(A_1=1)}{3P_{|\psi\rangle}(A_1=1)} = \frac{2}{3}. \tag{9}$$

A lower bound cannot be obtained; we have proved the *only if* part.

It is not possible to devise a quantum state giving equality in (8a–d) that violates the CH inequality [6], but there are states that come arbitrarily close, and such

Note that the assumption of independent errors at a constant rate is used in equation (6), and this assumption will be used from here on in this paper. In [4], the parameters of $\mathcal{B}$ (efficiency η and detector settings φ_i) and $|\psi\rangle$ are varied randomly to show that there is a violation at no background if $\eta > \frac{2}{3}$.

To obtain this bound analytically, the eigenvalues of $\mathcal{B}$ would be needed, and the calculation is manageable. Unfortunately, this involves solving a fourth-degree polynomial equation, and since the degree of the polynomial will increase rapidly with n , a method more suitable for the purpose of this paper will be presented. The two-site bound $\frac{2}{3}$ will be obtained analytically without too much calculation, and this will be generalized below.

Theorem 2: (The lowest possible efficiency bound from the CH inequality) In the case of independent errors at a constant rate, there is a violation of the CH inequality if and only if

$$\eta > \frac{2}{3}.$$

Proof: An important observation is that the CH inequality with the quantum probabilities from (6) inserted is equivalent to

states will be used in the proof of the *if* part. Given $\epsilon > 0$ and using $\theta = 2 \arctan(\epsilon)$, the quantum state (see [7]),

$$\begin{aligned}
|\delta\rangle &= C \left((1 - 2 \cos(\theta)) |0_{B_1}0_{B_2}\rangle \right. \\
&\quad \left. + \sin(\theta) (|1_{B_1}0_{B_2}\rangle + |0_{B_1}1_{B_2}\rangle) \right) \tag{10}
\end{aligned}$$

and the rotation

$$\begin{pmatrix} |0_{A_i}\rangle \\ |1_{A_i}\rangle \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} |0_{B_i}\rangle \\ |1_{B_i}\rangle \end{pmatrix} \tag{11}$$

yields

$$P_{|\delta\rangle}(A_1=A_2=1) = K \neq 0 \tag{12a}$$

$$P_{|\delta\rangle}(A_1=B_2=1) = K \tag{12b}$$

$$P_{|\delta\rangle}(B_1=A_2=1) = K \tag{12c}$$

$$P_{|\delta\rangle}(B_1=B_2=1) = 0 \tag{12d}$$

$$P_{|\delta\rangle}(A_1=1) = P_{|\delta\rangle}(A_2=1) = K(1 + \epsilon^2), \tag{12e}$$

that is, (7) reduces to

$$\eta \leq \frac{2K(1+\epsilon^2)}{3K} = \frac{2}{3}(1+\epsilon^2). \quad (13)$$

So when $\eta > \frac{2}{3}$, there exists a quantum state that will give a violation of the CH inequality, which proves the *if* part. ■

A note here is that the $|\delta\rangle$ used above is *not* an eigenvector to $\mathcal{B}$ as used in [4]. A comparison shows that using the best possible $|\delta\rangle$ at a given η gives a violation about 80% of the violation from the eigenvector [8].

In the many-site case, a number of extensions of the CH inequality are possible. In the two-site case above, the inequality uses one- and two-detection probabilities. To generalize, a choice is made yielding a simple expression that contains only $(n-1)$ - and n -detection probabilities. The sums used below denote a summation over all possible combinations.

Theorem 3: (An n -site Clauser-Horne inequality) If we have Theorem 1 (i)–(ii) except at a null set, then

$$\begin{aligned} & \sum P(\text{One } B \text{ and all other } A\text{'s}=1) \\ & - \sum P(\text{Even number of } B\text{'s and all other } A\text{'s}=1) \\ & \leq \sum P(\text{All } A\text{'s}=1 \text{ except one}) - (n-1)P(\text{All } A\text{'s}=1) \end{aligned} \quad (14)$$

Proof: The following inequality is obviously true:

$$\begin{aligned} & P(\cup \{\text{One } B \text{ and all other } A\text{'s}=1\}) \\ & \leq P(\cup \{\text{All } A\text{'s}=1 \text{ except one}\}) \end{aligned} \quad (15)$$

On the right-hand side we have

$$\begin{aligned} & P(\cup \{\text{All } A\text{'s}=1 \text{ except one}\}) \\ & = \sum P(\text{All } A\text{'s}=1 \text{ except one}) - (n-1)P(\text{All } A\text{'s}=1), \end{aligned} \quad (16)$$

and on the left-hand side,

$$\eta \leq \frac{\sum P_{|\psi\rangle}(\text{All } A\text{'s}=1 \text{ except one})}{\sum P_{|\psi\rangle}(\text{One } B \text{ and all other } A\text{'s}=1) + (n-1)P_{|\psi\rangle}(\text{All } A\text{'s}=1) - \sum P_{|\psi\rangle}(\text{Even number of } B\text{'s and all other } A\text{'s}=1)}. \quad (20)$$

Clearly,

$$P_{|\psi\rangle}(\text{All } A\text{'s}=1) \leq \min P_{|\psi\rangle}(\text{All } A\text{'s but one}=1) \quad (21a)$$

$$\begin{aligned} & P_{|\psi\rangle}(\text{One } B \text{ and all other } A\text{'s}=1) \\ & \leq P_{|\psi\rangle}(\text{The same } A\text{'s}=1) \end{aligned} \quad (21b)$$

$$P_{|\psi\rangle}(\text{Two or more } B\text{'s}=1) \geq 0, \quad (21c)$$

The lowest possible bound in (20) would be obtained

$$\begin{aligned} & P(\cup \{\text{One } B \text{ and all other } A\text{'s}=1\}) \\ & = \sum P(\text{One } B \text{ and all other } A\text{'s}=1) \\ & \quad - \sum P(\text{Two } B\text{'s and all } A\text{'s}=1) \\ & \quad + \sum P(\text{Three } B\text{'s and all } A\text{'s}=1) \\ & \quad - \dots \\ & \quad + (-1)^n P(\text{All } B\text{'s and all } A\text{'s}=1). \end{aligned} \quad (17)$$

Now

$$\begin{aligned} 0 & \leq P(\text{Some set of } B\text{'s and all } A\text{'s}=1) \\ & \leq P(\text{The same } B\text{'s and all other } A\text{'s}=1), \end{aligned} \quad (18)$$

and therefore, taking the signs into account

$$\begin{aligned} & P(\cup \{\text{One } B \text{ and all other } A\text{'s}=1\}) \\ & \geq \sum P(\text{One } B \text{ and all other } A\text{'s}=1) \\ & \quad - \sum P(\text{Two } B\text{'s and all other } A\text{'s}=1) \\ & \quad - \sum P(\text{Four } B\text{'s and all other } A\text{'s}=1) \\ & \quad - \dots \\ & \quad - \begin{cases} P(\text{All } B\text{'s}=1), & n \text{ even,} \\ \sum P(\text{One } A \text{ and all other } B\text{'s}=1), & n \text{ odd.} \end{cases} \end{aligned} \quad (19)$$

which concludes the proof. ■

In the same spirit as in the two-site case (Theorem 2), we have

Theorem 4: (The lowest possible efficiency bound from the above n -site CH inequality) In the case of independent errors at a constant rate, there is a violation of the above n -site CH inequality if and only if

$$\eta > \frac{n}{2n-1}.$$

Proof: Again, the n -site CH inequality above with quantum probabilities inserted is equivalent to

when we have equality in (21a–c), and when all $P_{|\psi\rangle}(\text{All } A\text{'s but one}=1)$ are equal (which gives the best possible value in (21a)). We then would have

$$\eta \leq \frac{nP_{|\psi\rangle}(\text{All } A\text{'s}=1 \text{ but } A_n)}{(2n-1)P_{|\psi\rangle}(\text{All } A\text{'s}=1 \text{ but } A_n)} = \frac{n}{2n-1}. \quad (22)$$

A lower bound cannot be obtained; we have proved the *only if* part.

Again, states arbitrarily close to giving equality will be used in the proof of the *if* part. Given $\epsilon > 0$, and using $\theta = 2 \arctan(\epsilon)$, the quantum state (see [7]),

$$|\delta\rangle = C \left((1 - n \cos(\theta)) |0_{B_1} \cdots 0_{B_n}\rangle + \sin(\theta) (|1_{B_1} 0_{B_2} \cdots 0_{B_n}\rangle + \cdots + |0_{B_1} \cdots 0_{B_{n-1}} 1_{B_n}\rangle) \right) \quad (23)$$

(where C is a normalization constant), and the rotation

$$\begin{pmatrix} |0_{A_i}\rangle \\ |1_{A_i}\rangle \end{pmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} |0_{B_i}\rangle \\ |1_{B_i}\rangle \end{pmatrix} \quad (24)$$

yields

$$P_{|\psi\rangle}(\text{All } A\text{'s}=1) = K \neq 0 \quad (25a)$$

$$P_{|\psi\rangle}(\text{One } B \text{ and all other } A\text{'s}=1) = K \quad (25b)$$

$$P_{|\psi\rangle}(\text{Two or more } B\text{'s}=1) = 0 \quad (25c)$$

$$P_{|\psi\rangle}(\text{All } A\text{'s but one}=1) = K(1 + \epsilon^2), \quad (25d)$$

that is, (20) reduces to

$$\eta \leq \frac{nK(1 + \epsilon^2)}{(2n-1)K} = \frac{n}{2n-1}(1 + \epsilon^2). \quad (26)$$

So when $\eta > \frac{n}{2n-1}$, there exists a quantum state that will give a violation of the CH inequality. ■

This $|\delta\rangle$ is in a similar fashion as in Theorem 2 not an eigenstate, and a check in the three-site case shows that the violation from the best $|\delta\rangle$ is just below 70% of that obtained from the eigenstate. Unfortunately, the computational effort required for these comparisons increases rapidly with the number of sites.

To conclude, we have seen that in the two-site CH inequality, quantum mechanics yields a violation only if $\eta > \frac{2}{3}$, and thus, by the result in [9], the lowest bound we can obtain in *any* Bell inequality for two sites and two detector settings is $\frac{2}{3}$. For n -site situations the picture is not so clear, because of the lack of similar results.

It should be said that although the current paper yields low bounds on efficiency, these bounds are only proven here for 100% visibility. While the inequalities are valid irrespective of visibility, extra background events will reduce, and then prevent, a violation of the inequality. The usable visibility depends on the absolute size of the violation of the inequalities (i.e., K). In [4] it is clear that lowering the bounds on η puts higher demand on the visibility. In general, the same behavior is to be expected in an n -particle setup.

Nevertheless, the best bound possible to obtain from the n -particle CH inequalities presented here is that when

$$\eta > \frac{n}{2n-1}, \quad (27)$$

there is a quantum state yielding a violation in the corresponding n -particle experiment. This enables us to state

Theorem 5: (A bound on η) Whenever

$$\eta > \frac{1}{2}, \quad (28)$$

there exists an n and an ϵ , so that the n -particle quantum state $|\delta\rangle$ given above violates the corresponding n -particle CH inequality.

Or in other words, there are experiments for which the quantum-mechanical predictions violate local realism whenever the efficiency exceeds $\frac{1}{2}$.

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- [2] A. Garg and N. D. Mermin, *Phys. Rev. D* **35**, 3831 (1987), J.-Å. Larsson, *Phys. Rev. A* **57**, 3304 (1998).
- [3] The theorems really talk about probabilities of certain *events*. It is possible to use them in situations where the actual measurement results are e.g. real-valued, such as position and momentum variables. The event $A_1 = 1$ might then correspond to a certain range of values of the position measurement.
- [4] P. H. Eberhard, *Phys. Rev. A* **47**, R747 (1993).
- [5] If $P_{|\psi\rangle}(A_1 = 1) \neq P_{|\psi\rangle}(A_2 = 1)$, equality in (8a–d) implies one of the terms in the numerator is less than the other, while two of the terms in the denominator are less than the third, yielding a higher bound.
- [6] Surprisingly, there *is* a state giving the probabilities in eqns. (8a–d): to obtain equality in (8a–c) we have to use $A_i = B_i$ (no rotation), and then $P(B_1 = B_2 = 1) = P(A_1 = 1)$, which has to be 0 to fulfill (8d). This yields a *product state* obeying (8a–d), $|0_{A_1} 0_{A_2}\rangle$ for which there is no violation (e.g., (7) is on the form $\frac{0}{0}$). Indeed, it was noted in [4] that the state which gave the highest violation approaches this product state as η goes to $\frac{2}{3}$, and so does the $|\delta\rangle$ state used here when ϵ decreases.
- [7] We want a symmetrical state $|\delta\rangle$ such that

$$\langle 1_{B_i} 1_{B_j} | \delta \rangle = 0, \quad i \neq j \quad (29a)$$

$$\langle 1_{A_1} \cdots 1_{A_{n-1}} 1_{B_n} | \delta \rangle = \langle 1_{A_1} \cdots 1_{A_n} | \delta \rangle, \quad (29b)$$

$$\langle 1_{A_1} \cdots 1_{A_{n-1}} 0_{A_n} | \delta \rangle = \epsilon \langle 1_{A_1} \cdots 1_{A_n} | \delta \rangle. \quad (29c)$$

Given (29a) and symmetry, we have

$$|\delta\rangle = a |0_{B_1} \cdots 0_{B_n}\rangle + b (|1_{B_1} 0_{B_2} \cdots 0_{B_n}\rangle + \cdots + |0_{B_1} \cdots 0_{B_{n-1}} 1_{B_n}\rangle). \quad (30)$$

Inserting this in (29b), using the general rotation in equation (24) and solving for a , we obtain

$$a = b \frac{1 - n \cos(\theta)}{\sin(\theta)}. \quad (31)$$

By use of (29c), we may solve for ϵ :

$$\epsilon = \frac{\cos(\theta) - 1}{\sin(\theta)} = \frac{2 \sin^2(\frac{\theta}{2})}{2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})} = \tan(\frac{\theta}{2}). \quad (32)$$

We then have the probabilities we want from $|\delta\rangle$ and our rotation.

- [8] Another choice is made in W. Y. Hwang, I. G. Koh, and Y. D. Han, Phys. Lett. A **212**, 309 (1996), which uses the quantum state described in L. Hardy, Phys. Rev. Lett. **71**,

1665 (1993), where (in the notation used here) $A_1 = 1$ implies $B_2 = 1$ and vice versa, so that the probabilities are (compare with (12a–e))

$$P_{|\delta\rangle}(A_1 = A_2 = 1) = K \neq 0 \quad (33a)$$

$$P_{|\delta\rangle}(A_1 = B_2 = 1) = K(1 + \epsilon^2) \quad (33b)$$

$$P_{|\delta\rangle}(B_1 = A_2 = 1) = K(1 + \epsilon^2) \quad (33c)$$

$$P_{|\delta\rangle}(B_1 = B_2 = 1) = 0 \quad (33d)$$

$$P_{|\delta\rangle}(A_1 = 1) = P_{|\delta\rangle}(A_2 = 1) = K(1 + \epsilon^2). \quad (33e)$$

This choice is also usable, but yields a lower absolute violation in the CH inequality than the choice in the proofs above.

- [9] I. Pitowsky, *Quantum probability – quantum logic, Lecture notes in Physics* (Springer, Berlin, 1989).