Thesis for the degree of Master of Science in Engineering Physics

Algebraic Structures in M-theory

Ling Bao

Department of Theoretical Physics
Chalmers University of Technology and Göteborg University
Göteborg, Sweden 2004

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Ling Bao
Department of Theoretical Physics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden

Abstract

In this thesis we reformulate the bosonic sector of eleven dimensional supergravity as a simultaneous nonlinear realisation based on the conformal group and an enlarged affine group called G_{11}. The vielbein and the gauge fields of the theory appear as space-time dependent parameters of the coset representatives. Inside the corresponding algebra g_{11} we find the Borel subalgebra of e_7, whereas performing the same procedure for the Borel subalgebra of e_8 we have to add some extra generators. The presence of these new generators lead to a new formulation of gravity, which includes both a vielbein and its dual. We make the conjecture that the final symmetry of M-theory should be described by a coset space, with the global and the local symmetry given by the Lorentzian Kac-Moody algebra e_{11} and its subalgebra invariant under the Cartan involution, respectively. The pure gravity itself in D dimensions is argued to have a coset symmetry based on the very extended algebra A_{D-3}^{+++}. The tensor generators of a very extended algebra are divided into representations of its maximal finite dimensional subalgebra. The space-time translations are thought to be introduced as weights in a basic representation of the Kac-Moody algebra. Some features of the root system of a general Lorentzian Kac-Moody algebra are discussed, in particular those related to even self-dual lattices.

Acknowledgments

In the ancient Crete, Ariadne's clew guided Theseus through Minotaur's maze and helped him escape the dreadful death.

I would like to express my deepest gratitudes to my supervisor and source of inspiration throughout this thesis, professor Bengt E.W. Nilsson, a fantastic world has been opened before my eyes. Many thanks to all the members of the Elementary Particle Theory and Mathematical Physics group for surviving my endless questions. Special thanks to my roommate Ulrik Svensson for those invaluable discussions. Finally I would like to thank my dear family, you have always given me the courage through the sunshine and the rain.

Contents

1

Introduction

A central theme in fundamental physics during the twentieth century was the search for ways to bring together gravity and quantum mechanics, that is, to find a gravitational theory at the Planck length. Until 1995 the best candidates for the *theory of everything* were the ten-dimensional superstrings. Although the superstrings succeeded to shine new light on the issue of quantum gravity, one of the major problems was that the number of different self-consistent superstring theories was not one, but five. Also the required ten-dimensional space-time is somewhat startling, since supersymmetry permits a supersymmetric field theory in eleven dimensions. The discoveries of the supermembrane and the superfivebrane then made it clear that the superstring theories cannot be the whole story. In the last few years just before the new millennium, Edward Witten opened a new window revealing a mysterious theory called M-theory. It is a non-perturbative theory in eleven dimensions, where the various superstrings correspond to different extremal regions of the coupling constants arising after compactification and related by dualities to one another. At the low energy limit, M-theory is described by eleven dimensional supergravity.

This thesis work deals with the possible symmetries of M-theory, having eleven dimensional supergravity as the starting point. The symmetry groups appear manifestly by using a technique called nonlinear realisation. Typically, one has a Lie group as the global symmetry group, while the local symmetry is its maximal compact subgroup. Certain features of eleven dimensional supergravity and its dimensionally reduced theories seem to point towards a rank eleven Lorentzian Kac-Moody algebra named e_{11}.

The outline of this thesis is the following: **Chapter 2** deals with nonlinear realisations. First the general theory is reviewed, detailed calculations are then carried out for the scalar fields of type IIB supergravity and the bosonic part of eleven dimensional supergravity. The correct field equations of the latter are obtained by simultaneous nonlinear realisation based on a group called G_{11} and the conformal group, both having the Lorentz group as the local subgroup.

In **Chapter 3** some basic properties of Lorentzian Kac-Moody algebras are presented, with the focus being mainly on the root system. Conditions are found for a Lorentzian Kac-Moody algebra to include a real principal so(2,1) subalgebra. Also, a section is devoted to the so called very extension of a finite dimensional Lie algebra g, which is obtained from g by adding three more simple roots in a specific way. The very extended Lie algebras related to self-dual lattices are of special interest, e.g., e_{11}, m_{19} and k_{27}.

The conjecture about e_{11} being a symmetry of M-theory is made in **Chapter 4**. It is shown that the eleven dimensional supergravity contains the finite dimensional Lie groups of exceptional type as symmetries. The concept of level is introduced, and the root space of e_{11} is divided into tensor representations of its subalgebra A_{10}. The translation generators are found to be included in a basic representation of e_{11}. An alternative formulation of gravity, involving a field dual to the vielbein, is also presented in this chapter.

Conclusions are made in **Chapter 5**, together with some brief words about relevant issues not covered in this thesis.

There are some loosely attached subjects included as appendices, providing short introductions to differential forms, the conformal group, general coordinate transformations, real principal subalgebras and the Lorentzian self-dual lattices $\Pi_{D-1;1}$. Last but not least, there is a summary about field transformations under the conformal group.

2

Nonlinear realisations

When the eleven dimensional supergravity is dimensionally reduced on a n-torus to (11-n) dimensions, for $n = 1, \ldots, 8$, the scalars in the resulting theory can be associated with the coset space formed by a non-compact symmetry group E_n divided by its maximal compact subgroup F_n, see Table 2.1. The local subgroup F_n can be used to choose the coset repres-

Dimension	E_n	F_n
11	1	1
10, IIB	SL(2)	SO(2)
10, IIA	SO(1,1)/Z_2	1
9	GL(2)	SO(2)
8	$\mathrm{E}_3 \sim \mathrm{SL}(3)\times\mathrm{SL}(2)$	U(2)
7	$\mathrm{E}_4 \sim \mathrm{SL}(5)$	USp(4)
6	$\mathrm{E}_5 \sim \mathrm{SO}(5,5)$	USp(4)$\times$USp(4)
5	E_6	USp(8)
4	E_7	SU(8)
3	E_8	SO(16)

Table 2.1: *Coset spaces of the supergravities.*

entatives of $\mathrm{E}_n/\mathrm{F}_n$ to belong to the Borel subgroup of E_n, i.e., exponentials of the Cartan and the positive root generators in E_n. The number of scalar fields is then just the number of generators in the Borel subalgebra. Dimensional reduction of eleven dimensional supergravity to two dimensions and one dimension is thought to be theories invariant under the affine extension of E_8, denoted E_9, and the hyperbolic group E_{10}, respectively. It has been proposed that the eleven dimensional supergravity itself has some kind of exceptional geometry. The local subgroups F_n for $n = 1, \ldots, 8$ are the subgroups invariant under the Cartan involution, and for the cases of $n = 9, 10$ one needs to introduce an involution in order to define possible local subgroups. The gauge fields and the scalars transform non-trivially under the E_n group, whereas gravity is inert under this global internal symmetry group. Our goal is to treat the gauge fields and gravity on the same footing as nonlinear realisations.

We will in this chapter first look at the general theory of nonlinear realisations with a coset space symmetry. More detailed calculations are then carried out for the scalar part of type IIB supergravity and the bosonic sector of eleven dimensional supergravity. In the latter case we will have to make simultaneous realisation of the conformal group and the G_{11} group, which is the affine group IGL(11) extended with some extra generators.

The technique of nonlinear realisations can also be applied to the bosonic sector of IIA and IIB supergravities, the closed bosonic string as well as the branes in M-theory, by using different coset symmetries simultaneously realised with the conformal group. To include supersymmetry in the eleven dimensional supergravity, we have to consider simultaneous realisation of the groups IGL(11|32) and OSp(1|64), where OSp(1|64) is replacing the conformal group [1].

2.1 General theory

There is a complete duality between sets of fields, which transform linearly while being subject to certain nonlinear constraints, and equivalent sets of unconstrained fields which transform according to nonlinear realisation. The bridge between these two sets of fields are some massless Goldstone bosons called preferred fields, which generally enter nonlinearly. If nonlinear realisations are introduced by considering linear realisations of the preferred fields together with a constraint, the constraint implies that the vacuum state in the theory must be non-invariant, and the symmetry is spontaneously broken. The non-vanishing of some vacuum expectation values indicates the vacuum asymmetry. If some long-range vector fields are present and the currents of the spontaneously broken symmetries are coupled to a gauge field of the Yang-Mills type, then the symmetry breaking manifests itself not through the appearance of Goldstone bosons, but in the generation of mass for some of the components of the gauge field.

Consider a general Lie group G with some specified subgroup H. Suppose that the linear irreducible representations of G

$$\Psi \to \Psi' = D(g_0)\Psi, \quad g_0 \in G \tag{2.1}$$

and their decomposition into linear irreducible representations of H are known. $D(*)$ denote the representations of the group, to which the field Ψ belongs, and g_0 is a group element of G. We want to find covariant objects undergoing spontaneous symmetry breaking from G to H. A representative g of the coset G/H is called a reducing matrix, and will obey the following requirements

i. g belongs also to the group G, thus for any finite-dimensional representation $g_0 \to D(g_0)$ the functional $D(g)$ shall be well-defined. The number of independent fields ϕ_a needed to parametrise g is equal to or less than the dimensionality of G. ϕ_a are called the preferred fields.

ii. Under the action of the group G, the reducing matrix transforms as
$$g \to g' = g_0 g h^{-1}(\phi, g_0); \quad g_0 \in G \ , \ h(\phi, g_0) \in H. \tag{2.2}$$

iii. Under the operation of the subgroup H, the reducing matrix transforms as
$$g \to hgh^{-1}. \tag{2.3}$$

For any finite-dimensional representation, the transformation (2.2) will become $D(g) \to D(g_0)D(g)D(h^{-1})$. We can project the nonlinear realisations out of the linear ones by defining

$$\psi \equiv D(g^{-1})\Psi, \tag{2.4}$$

which transforms under G as

$$\psi \to \psi' = D(g'^{-1})\Psi' = D(h)\psi. \tag{2.5}$$

We define the reducing matrix as

$$g = e^{\phi_a K^a}, \tag{2.6}$$

where K^a are the set of infinitesimal generators of G that are not contained in the algebra of H. The matrix representation of g contains at least one column χ transforming linearly

$$D(g)\chi \to D(g')\chi = D(g_0)D(g)\chi, \tag{2.7}$$

i.e., χ is a singlet under H and $D(h^{-1})\chi = \chi$. Expressing the components of χ in terms of the preferred fields, we can calculate transformations of the preferred fields under Eq. (2.2) and the local elements h by $h(\phi, g_0) = g'^{-1} g_0 g$.

The covariant derivatives are found using the outline given in [2]. The ordinary derivative transforms as

$$\partial_\mu \psi \to D(h)\partial_\mu \psi + \partial_\mu D(h)\psi, \tag{2.8}$$

using which we define a new operator

$$\Delta_\mu \psi \equiv D(g^{-1})\partial_\mu \Psi = \partial_\mu \psi + (D(g^{-1})\partial_\mu D(g))\psi \tag{2.9}$$

with the transformation rule

$$\Delta_\mu \psi \to D(h)\Delta_\mu \psi. \tag{2.10}$$

Note that Ψ transforms only globally according to Eq. (2.1). The operator $g^{-1}\partial_\mu g$ belongs to the infinitesimal algebra of G, since

$$g^{-1}(x)g(x+\delta x) = 1 + \delta x^\mu (g^{-1}\partial_\mu g) + \ldots \tag{2.11}$$

for any infinitesimal transformation of G. We can then write $g^{-1}\partial_\mu g = (g^{-1}\partial_\mu g)_i s^i$, where s^i is a chosen basis for the algebra and $(g^{-1}\partial_\mu g)_i$ are the coefficients. In a specific representation of the field Ψ, the relation is $D(g^{-1})\partial_\mu D(g) = (g^{-1}\partial_\mu g)_i S^i$ with S^i being the representations of the basis. Using Eq. (2.2) we find

$$g^{-1}\partial_\mu g \to h(g^{-1}\partial_\mu g)h^{-1} + h\partial_\mu h^{-1}, \tag{2.12}$$

where the inhomogeneous term belongs to the algebra of the subgroup H. We can thus divide $(g^{-1}\partial_\mu g)_i$ into a set transforming covariantly and another set containing the inhomogeneity

$$g^{-1}\partial_\mu g = \Gamma_{\mu\alpha} m^\alpha + \lambda D_\mu \phi_a n^a. \tag{2.13}$$

The preferred fields used to parametrise the reducing matrix are denoted ϕ_a, m^α is a basis for the subalgebra corresponding to H and n^a are the rest of the basis for the total algebra of the group G. It should be noted that m^α and n^a are transforming independently. Eq. (2.13) defines the field quantities $\Gamma_{\mu\alpha}$ and $D_\mu \phi_a$, with the latter to be interpreted as the covariant derivatives of the preferred fields. By doing so, the inhomogeneous term in the transformation will only affect $\Gamma_{\mu\alpha}$. Putting Eq. (2.13) into the definition of $\Delta_\mu \psi$ yields

$$\Delta_\mu \psi = \partial_\mu \psi + \Gamma_{\mu\alpha} M^\alpha \psi + \lambda D_\mu \phi_a N^a \psi, \tag{2.14}$$

where M^α and N^a are the representations of the generators m^α and n^a, respectively. Since the left hand side and the last term on the right hand side of the above equation are covariant, we find the covariant derivative of an arbitrary field under the coset

$$D_\mu \psi = \partial_\mu \psi + \Gamma_{\mu\alpha} M^\alpha \psi. \tag{2.15}$$

If the group G is space-time dependent

$$\Psi(x) \to D(g_0)\Psi(x), \quad g_0 = g_0(x) \in G, \tag{2.16}$$

the transformation of the ordinary derivative on a linear field will be

$$\partial_\mu \Psi \to D(g_0)\partial_\mu \Psi(x) + \partial_\mu D(g_0)\Psi(x), \tag{2.17}$$

where the second term vanishes for a space-time independent group. Using a basis S^i for the generators, the transformation law becomes

$$\partial_\mu \Psi \to D(g_0)[\partial_\mu + (g_0^{-1}\partial_\mu g_0)_i S^i]\Psi(x). \tag{2.18}$$

Introduce now the gauge fields $A_\mu = A_{\mu i} s^i$, which transform as

$$A_\mu \to g_0 A_\mu g_0^{-1} + \frac{1}{f} g_0 \partial_\mu g_0^{-1}, \tag{2.19}$$

where f is a constant. The covariant derivative for the linear field is then

$$\mathfrak{D}_\mu \Psi = (\partial_\mu + f A_{\mu i} S^i)\Psi \tag{2.20}$$

transforming as $\mathfrak{D}_\mu \Psi \to D(g_0)\mathfrak{D}_\mu \Psi$. We can define the antisymmetric field strength associated with the gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f[A_\mu, A_\nu], \tag{2.21}$$

which transforms as $F_{\mu\nu} \to g_0 F_{\mu\nu} g_0^{-1}$.

Introduce the nonlinear fields $\psi = D(g^{-1})\Psi$. As for the space-time independent case, the covariant derivatives should be contained in the quantity

$$\Delta_\mu \psi = D(g^{-1})(\partial_\mu + f A_{\mu i} S^i)\Psi = (\partial_\mu + f B_{\mu i} S^i)\psi. \tag{2.22}$$

The gauge field B_μ is defined as

$$B_\mu \equiv g^{-1} A_\mu g + \frac{1}{f} g^{-1} \partial_\mu g, \tag{2.23}$$

and fulfills

$$B_\mu \to hB_\mu h^{-1} + \frac{1}{f} h\partial_\mu h^{-1}. \tag{2.24}$$

The corresponding field strength is

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + f[B_\mu, B_\nu] = g^{-1}F_{\mu\nu}g \tag{2.25}$$

satisfying $B_{\mu\nu} \to hB_{\mu\nu}h^{-1}$. It is now easy to verify that $\Delta_\mu\psi$ satisfies Eq. (2.10), and we write $\Delta_\mu\psi = D_\mu\psi + \lambda(D_\mu\phi_a)N^a\psi$. In conclusion, the covariant derivatives of an arbitrary nonlinear field and a preferred field are

$$\begin{aligned} D_\mu\psi &= (\partial_\mu + fB_{\mu\alpha}M^\alpha)\psi \\ \lambda D_\mu\phi_a &= fB_{\mu a}, \end{aligned} \tag{2.26}$$

respectively.

Using the covariant objects (ψ, $D_\mu\psi$, $D_\mu\phi_a$ and $B_{\mu\nu}$) we can construct general Lagrangians. Since the preferred fields themselves are not transforming covariantly, we cannot have any mass term ($\phi_a^*\phi^a$) for these, i.e., they are massless bosons. On the other hand, if the group G is space-time dependent, we can have a mass term for the gauge fields B_μ and the preferred fields ϕ_a will be absorbed by the gauge fields. $B_{\mu\alpha}$ is then still massless, while $B_{\mu a}$ has a well-defined mass f/λ.

2.2 Type IIB supergravity

The kinetic terms of the scalar fields in the type IIB supergravity are given in Einstein frame [3] by

$$\mathcal{L}_{IIB} = -\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi\right), \tag{2.27}$$

where ϕ and χ are real scalar fields.

The scalar part of the total Lagrangian possesses a SU(1,1)/U(1) coset space symmetry, with SU(1,1) being the non-compact global symmetry and U(1) its maximal compact subgroup. This invariance can be shown by performing three consecutive coordinate transformations and thus writing the Lagrangian in a manifestly invariant form. The first coordinate transformation is to combine the real scalar fields into a complex field

$$\lambda = \chi + ie^{-\phi} \quad , \quad \bar\lambda = \chi - ie^{-\phi}. \tag{2.28}$$

The Lagrangian becomes

$$\mathcal{L}_{IIB} = \frac{2}{(\lambda-\bar\lambda)^2}\partial_\mu\lambda\partial^\mu\bar\lambda. \tag{2.29}$$

A Möbius transformation is then performed

$$z = \frac{i - \lambda}{i + \lambda} \qquad , \qquad \bar{z} = \frac{-i - \bar{\lambda}}{-i + \bar{\lambda}} \tag{2.30}$$

yielding

$$\mathcal{L}_{IIB} = -2 \frac{\partial_\mu z \partial^\mu \bar{z}}{(1 - z\bar{z})^2}. \tag{2.31}$$

Introducing two complex scalars, ϕ_0 and ϕ_1, satisfying

$$z = \frac{\phi_1}{\phi_0} \qquad \text{and} \qquad |\phi_0|^2 - |\phi_1|^2 = 1 \tag{2.32}$$

brings the Lagrangian to the form

$$\mathcal{L}_{IIB} = -2|\phi_0 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_0|^2. \tag{2.33}$$

We can now define a gauge field A_μ in terms of the complex scalars ϕ_0 and ϕ_1 by

$$A_\mu \equiv \frac{i}{2} \left[(\phi_0 \partial_\mu \bar{\phi}_0 - \phi_1 \partial_\mu \bar{\phi}_1) - (\bar{\phi}_0 \partial_\mu \phi_0 - \bar{\phi}_1 \partial_\mu \phi_1) \right], \tag{2.34}$$

and the Lagrangian finally becomes

$$\mathcal{L}_{IIB} = -2(|\partial_\mu \phi_1 - iA_\mu \phi_1|^2 - |\partial_\mu \phi_0 - iA_\mu \phi_0|^2). \tag{2.35}$$

From the construction of the gauge field, we conclude that A_μ will not give rise to some new independent degrees of freedom.

To see the symmetry of the Lagrangian, we write the fields ϕ_0 and ϕ_1 as components of a group element g of SU(1,1)

$$g = \begin{pmatrix} \phi_0 & \bar{\phi}_1 \\ \phi_1 & \bar{\phi}_0 \end{pmatrix}. \tag{2.36}$$

This is an element of the group SU(1,1) since the complex scalar fields satisfy the condition $|\phi_0|^2 - |\phi_1|^2 = 1$. For the Lagrangian to be invariant under the coset group SU(1,1)/U(1), i.e., to incorporate the local U(1) gauge invariance, we have to define a covariant derivative

$$D_\mu g = \partial_\mu g - ig \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_\mu, \tag{2.37}$$

It is now straightforward to show that the Lagrangian can be written as

$$\mathcal{L}_{IIB} = -\mathrm{Tr}(g^{-1} D_\mu g)(g^{-1} D^\mu g). \tag{2.38}$$

If one uses the usual partial derivatives instead of the covariant derivatives in Eq. (2.38), one will only get a Lagrangian with manifestly global SU(1,1) invariance.

The group element g transforms under SU(1,1)/U(1) as

$$g \to g' = g_0 g h^{-1}, \tag{2.39}$$

where g_0 is a global SU(1,1) transformation whereas h is a local U(1) transformation. The field A_μ will transform as

$$A_\mu \to A'_\mu = h A_\mu h^{-1} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h \partial_\mu h^{-1}, \tag{2.40}$$

the local U(1) group element h are represented here by (2×2) matrices. The resulting Lagrangian is

$$\begin{aligned} \mathcal{L}_{IIB} \to \mathcal{L}'_{IIB} &= -\mathrm{Tr}(g'^{-1}(D_\mu g)')(g'^{-1}(D^\mu g)') \\ &= -\mathrm{Tr}\left([h(g^{-1}D_\mu g)h^{-1}][h(g^{-1}D^\mu g)h^{-1}]\right) \\ &= \mathcal{L}_{IIB}, \end{aligned} \tag{2.41}$$

i.e., the Lagrangian is invariant.

2.3 Eleven dimensional supergravity

It is known since a long time that gravity can be formulated as a simultaneous nonlinear realisation [4]. This is because the group of general coordinate transformations is the closure of the conformal group and the the group of affine transformations [5, 6]. A summary of the conformal generators and its closure with the affine group can be found in Appendix B and C. We will in this section show explicitly how the bosonic sector of eleven dimensional supergravity can be expressed as a nonlinear realisation. But first we will review shortly the field equations of the bosonic sector of eleven dimensional supergravity.

2.3.1 The bosonic gauge field equations

The bosonic sector of the Lagrangian of eleven dimensional supergravity is given by [7]

$$\mathcal{L}_{11} = \frac{e}{4\kappa^2} R(\Omega(e)) - \frac{e}{48} F_{\mu_1\dots\mu_4} F^{\mu_1\dots\mu_4} + \frac{2\kappa}{12^4} \varepsilon^{\mu_1\dots\mu_{11}} F_{\mu_1\dots\mu_4} F_{\mu_5\dots\mu_8} A_{\mu_9\mu_{10}\mu_{11}}, \tag{2.42}$$

where $e = \sqrt{-\det(g_{\mu\nu})}$, R is the curvature scalar, $A_{\mu_1\mu_2\mu_3}$ is the rank three gauge field and $F_{\mu_1\dots\mu_4}$ the corresponding field strength. We use the space-time signature $\eta_{ab} = \text{diag}(-1, 1, 1, \ldots, 1)$. Note that by defining $A'_{\mu\nu\rho} = \kappa A_{\mu\nu\rho}$, the constant κ appears only as an overall factor κ^{-2} in front of the whole action. The field equations can be obtained by varying the action with respect to the metric $g_{\mu\nu}$ and the gauge field $A_{\mu_1\mu_2\mu_3}$, respectively. The calculations are greatly simplified if we write the action in terms of differential forms. (See Appendix A for an introduction to the differential forms.) We define the following p-forms

$$\begin{aligned} d &= dx^\mu \partial_\mu \\ A_3 &= \tfrac{1}{3!} dx^\mu \wedge dx^\nu \wedge dx^\rho \, A_{\mu\nu\rho} \\ F_4 &= \tfrac{1}{4!} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \, F_{\mu\nu\rho\sigma} \\ *F_4 &= \tfrac{e^{-1}}{4!7!} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_7} \varepsilon_{\mu_1 \ldots \mu_{11}} \, F^{\mu_8 \ldots \mu_{11}}. \end{aligned} \tag{2.43}$$

The three and the four forms are related as

$$F_4 = dA_3 \qquad \Rightarrow \qquad F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]}, \tag{2.44}$$

which is invariant under the gauge transformation

$$A_3 \rightarrow A'_3 = A_3 + d\Lambda_2 \qquad \Rightarrow \qquad F_4 \rightarrow F'_4 = dA_3 + d^2\Lambda_2 = F_4. \tag{2.45}$$

More generally the field strength is invariant under a transformation $A'_3 = A_3 + \delta A_3$, such that $d(\delta A_3) = 0$.

The Bianchi identity for the four form F_4 is given by

$$dF_4 = d^2 A_3 = 0 \qquad \Rightarrow \qquad \partial_{[\tau} F_{\mu\nu\rho\sigma]} = 0. \tag{2.46}$$

Using the differential forms defined in Eq. (2.43) we can write the action of the eleven dimensional supergravity as

$$S = \frac{1}{4\kappa^2} \int d^{11}x \sqrt{-g} R(\Omega(e)) + \frac{1}{2} \int F_4 \wedge *F_4 + \frac{\kappa}{3} \int F_4 \wedge F_4 \wedge A_3. \tag{2.47}$$

By varying the action with respect to A_3 we find the equation of motion for the gauge field to be

$$\begin{aligned} d * F_4 + \kappa F_4 \wedge F_4 &= 0 \qquad \text{or} \\ \partial_{\mu_1} F^{\mu_1 \ldots \mu_4} + \tfrac{\kappa e^{-1}}{4 \cdot 12^2} \varepsilon^{\mu_2 \ldots \mu_4 \nu_1 \ldots \nu_8} F_{\nu_1 \ldots \nu_4} F_{\nu_5 \ldots \nu_8} &= 0. \end{aligned} \tag{2.48}$$

Since only the field strength is present in the field equation, the equation of motion is invariant under the gauge transformation in Eq. (2.45).

We can also introduce a dual gauge field A_6 and its corresponding field strength F_7 obeying

$$\begin{aligned} F_7 &= dA_6 + A_3 \wedge F_4 \qquad \text{or} \\ F_{\mu_1 \ldots \mu_7} &= 7\partial_{[\mu_1} A_{\mu_2 \ldots \mu_7]} + 35 A_{[\mu_1 \ldots \mu_3} F_{\mu_4 \ldots \mu_7]}. \end{aligned} \tag{2.49}$$

The Bianchi identity for the seven form F_7 will now automatically yield Eq. (2.48) provided that

$$\begin{array}{rcll} *F_4 & = & -\kappa F_7 & \text{or} \\ F_{\mu_1\ldots\mu_4} & = & \frac{\kappa e^{-1}}{7!}\varepsilon_{\mu_1\ldots\mu_{11}}F^{\mu_5\ldots\mu_{11}}, & \end{array} \tag{2.50}$$

which plays the role of the equation of motion in this dual formulation of supergravity. Note that unlike Eq. (2.48), Eq. (2.50) is a linear equation.

2.3.2 The G_{11} realisation

Consider the Lie algebra with the non-vanishing commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \quad ; \quad [K^a{}_b, P_c] = -\delta^a_c P_b \tag{2.51}$$

together with

$$\begin{array}{rcl} [K^a{}_b, R^{c_1c_2c_3}] & = & 3\delta_b^{[c_1}R^{|a|c_2c_3]} = \delta_b^{c_1}R^{ac_2c_3} + \delta_b^{c_2}R^{ac_3c_1} + \delta_b^{c_3}R^{ac_1c_2} \\ [K^a{}_b, R^{c_1\ldots c_6}] & = & 6\delta_b^{[c_1}R^{|a|c_2\ldots c_6]} \\ [R^{c_1c_2c_3}, R^{c_4c_5c_6}] & = & 2R^{c_1\ldots c_6} \\ [R^{c_1\ldots c_6}, R^{d_1\ldots d_6}] & = & 0 \\ [R^{c_1c_2c_3}, R^{d_1\ldots d_6}] & = & 0. \end{array} \tag{2.52}$$

The generators $K^a{}_b$ and P_a are precisely those of the affine group IGL(11) (see Appendix C), whereas $R^{c_1c_2c_3}$ and $R^{c_1\ldots c_6}$ can be identified as part of the gl(32) automorphism algebra of the eleven dimensional supersymmetry algebra [1]. We use G_{11} to denote the group defined by Eq. (2.51) and Eq. (2.52), while g_{11} is used when referring to the corresponding algebra. Note that g_{11} is not a finite dimensional semisimple Lie algebra, later in Chapter 4 we will identify it as part of the Borel subalgebra of the very extended Kac-Moody algebra e_{11}.

Take the group G_{11} as the global symmetry group and its Lorentz subgroup SO(10,1) as the local symmetry group, i.e., we investigate a theory with G_{11}/SO(10,1) space-time symmetry. The generators of the Lorentz group are the antisymmetric part of the generators $K^a{}_b$ of the conformal group

$$J^a{}_b = K^a{}_b - K_b{}^a. \tag{2.53}$$

Following the general procedure, we take a representative g of the coset space as

$$g = e^{x^f P_f} e^{m_a{}^b K^a{}_b} \exp\left(\frac{1}{3!}A_{c_1c_2c_3}R^{c_1c_2c_3} + \frac{1}{6!}A_{c_1\ldots c_6}R^{c_1\ldots c_6}\right), \tag{2.54}$$

with $m_a{}^b$, $A_{c_1c_2c_3}$ and $A_{c_1\ldots c_6}$ being the coordinate dependent Goldstone fields. Since the antisymmetric part of $K^a{}_b$ belongs to the local symmetry algebra, only the symmetric part of $K^a{}_b$ is needed to write a simplest representative for the coset. However, we will not assume that $m_a{}^b$ is symmetric; a choice that will turn out to make the identification with general relativity easier. The translations are linearly realised, and thus we do not associate any preferred fields with them.

We require the theory to be invariant under

$$g \to g' = g_0 g h^{-1}, \tag{2.55}$$

where g_0 is a constant element of G_{11} and h is a local element of SO(10,1). The manifestly coset invariant Lagrangian takes then the form $\mathcal{L} = \mathrm{Tr}(\mathcal{V}_\mu \mathcal{V}^\mu)$, with the Cartan form $\mathcal{V}$ defined as [1]

$$\mathcal{V} \equiv g^{-1}dg - \omega = dx^\mu\, \mathcal{V}_\mu. \tag{2.56}$$

The Lorentz connection ω is a one-form

$$\omega \equiv \frac{1}{2} dx^\mu\, \omega_{\mu a}{}^b J^a{}_b, \tag{2.57}$$

where $J^a{}_b$ are the Lorentz generators. Note that the Lorentz connection is a quadratic matrix, when writing out the matrix indices explicitly (2.57) becomes $\omega^a{}_b \equiv \frac{1}{2} dx^\mu\, \omega_{\mu c}{}^d (J^c{}_d)^a{}_b$.

To find out how (2.56) transforms under (2.55), we will first investigate the transformation law of the Lorentz connection (2.57). We write the vielbein as a one-form $\mathsf{e}^a = dx^\mu\, \mathsf{e}_\mu{}^a$, and it transforms only under local Lorentz transformations according to

$$\mathsf{e}^a \to \mathsf{e}'^a = h^a{}_b \mathsf{e}^b. \tag{2.58}$$

We can then define the torsion, which is nothing but the covariant derivative of the vielbein

$$T^a \equiv D\mathsf{e}^a = d\mathsf{e}^a + \omega^a{}_b \mathsf{e}^b, \tag{2.59}$$

and express the Lorentz connection in terms of the vielbein by letting the torsion vanish

$$\begin{aligned} \partial_{[\mu}\mathsf{e}_{\nu]}{}^a &+ \omega_{[\mu}{}^a{}_{|b|}\mathsf{e}_{\nu]}{}^b = 0 \qquad \Rightarrow \\ \omega_{\mu bc}(\mathsf{e}) &= \tfrac{1}{2}(\mathsf{e}^\rho{}_b \partial_\mu \mathsf{e}_{\rho c} - \mathsf{e}^\rho{}_c \partial_\mu \mathsf{e}_{\rho b}) - \tfrac{1}{2}(\mathsf{e}^\rho{}_b \partial_\rho \mathsf{e}_{\mu c} - \mathsf{e}^\rho{}_c \partial_\rho \mathsf{e}_{\mu b}) \\ &\quad -\tfrac{1}{2}(\mathsf{e}^\lambda{}_b \mathsf{e}^\rho{}_c \partial_\lambda \mathsf{e}_{\rho a} - \mathsf{e}^\lambda{}_c \mathsf{e}^\rho{}_b \partial_\lambda \mathsf{e}_{\rho a})\mathsf{e}_\mu{}^a. \end{aligned} \tag{2.60}$$

The Lorentz connection is antisymmetric in its last two indices, and using this fact together with Eqs. (2.53) and (2.57) we get

$$\omega = dx^\mu\, \omega_{\mu a}{}^b K^a{}_b. \tag{2.61}$$

By definition the covariant derivative of e^a (i.e., the torsion) has to transform as the vielbein itself

$$T^a \to T'^a = h^a{}_b T^b. \tag{2.62}$$

Using Eqs. (2.58), (2.59) and (2.62) we find the transformation of the Lorentz connection to be

$$\begin{array}{rcl} \omega & \to & \omega' = h\omega h^{-1} + h dh^{-1} \qquad\qquad \text{or} \\ \omega^a{}_b & \to & \omega'^a_b = h^a{}_c \omega^c{}_d (h^{-1})^d{}_b + h^a{}_c d(h^{-1})^c{}_b. \end{array} \tag{2.63}$$

Having found the transformation of the Lorentz connection, we are ready to look at the transformation of the Cartan form $\mathcal{V}$

$$\begin{array}{rcl} \mathcal{V} \to \mathcal{V}' & = & (g')^{-1} dg' - \omega' \\ & = & (g_0 g h^{-1})^{-1} d(g_0 g h^{-1}) - (h\omega h^{-1} + h dh^{-1}) \\ & = & h(g^{-1}dg - \omega)h^{-1} = h\mathcal{V}h^{-1}. \end{array} \tag{2.64}$$

By now it should clear that the Lorentz connection is included in the Cartan form to make it covariant and hence the Lagrangian invariant under (2.55)

$$\mathcal{L} \to \mathcal{L}' = \mathrm{Tr}(h\mathcal{V}_\mu h^{-1} h \mathcal{V}^\mu h^{-1}) = \mathcal{L}, \tag{2.65}$$

where we have omitted all the matrix indices.

Motivated by the knowledge that the Cartan form in Eq. (2.56) will lead to an invariant Lagrangian, we can write it out using the coset representative given in (2.54). The constituents of the Cartan form will then be identified with the covariant objects that appeared in the equations of motion.

$$\begin{array}{rcl} \mathcal{V} & = & g^{-1}dg - \omega \\ & = & dx^\mu\, g^{-1} \partial_\mu e^{x^f P_f} e^{m_a{}^b K^a{}_b} \exp\left(\frac{A_{c_1c_2c_3} R^{c_1c_2c_3}}{3!} + \frac{A_{c_1\ldots c_6} R^{c_1\ldots c_6}}{6!}\right) - \omega \\ & = & dx^\mu\, g^{-1} e^{x^f P_f} [\partial_\mu + (\partial_\mu x^c) P_c] e^{m_a{}^b K^a{}_b} \\ & & \times \exp\left(\frac{A_{c_1c_2c_3} R^{c_1c_2c_3}}{3!} + \frac{A_{c_1\ldots c_6} R^{c_1\ldots c_6}}{6!}\right) - \omega \\ & = & dx^\mu\, g^{-1} e^{x^f P_f} e^{m_a{}^b K^a{}_b} [\partial_\mu + \partial_\mu m_c{}^d K^c{}_d + (e^m)_\mu{}^c P_c] \\ & & \times \exp\left(\frac{1}{3!} A_{c_1c_2c_3} R^{c_1c_2c_3} + \frac{1}{6!} A_{c_1\ldots c_6} R^{c_1\ldots c_6}\right) - \omega \\ & = & dx^\mu\, g^{-1} g\{(e^m)_\mu{}^a P_a + (\partial_\mu m_a{}^b - \omega_{\mu a}{}^b) K^a{}_b \\ & & + \frac{1}{3!}[\partial_\mu A_{c_1c_2c_3} + 3\partial_\mu m_{[c_1}{}^b A_{|b|c_2c_3]}] R^{c_1c_2c_3} \\ & & + \frac{1}{6!}[\partial_\mu A_{c_1\ldots c_6} + 6\partial_\mu m_{[c_1}{}^b A_{|b|c_2\ldots c_6]} \\ & & + 20(\partial_\mu A_{c_1c_2c_3} + 3\partial_\mu m_{[c_1}{}^b A_{|b|c_2c_3]}) A_{c_4c_5c_6}] R^{c_1\ldots c_6}\} \end{array} \tag{2.66}$$

In calculating (2.66) we have used that all the generators commute with the coordinates x^μ. We have also frequently used the Baker-Hausdorff lemma [8]

$$\begin{aligned} Xe^Y &= e^Y \left\{ X + [X,Y] + \tfrac{1}{2!}[[X,Y],Y] + \ldots \right. \\ &\left. + \tfrac{1}{n!}[\ldots[[X,Y],Y],\ldots,Y] + \ldots \right\} . \end{aligned} \tag{2.67}$$

We will now make the following definitions

$$(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_a{}^b = \partial_\mu m_a{}^b \tag{2.68}$$

$$\mathsf{e}_\mu{}^a \equiv (e^m)_\mu{}^a \tag{2.69}$$

$$\Omega_{\mu a}{}^b \equiv \partial_\mu m_a{}^b - \omega_{\mu a}{}^b = (\mathsf{e}^{-1}\partial_\mu \mathsf{e})_a{}^b - \omega_{\mu a}{}^b \tag{2.70}$$

$$\tilde{D}_\mu A_{c_1c_2c_3} \equiv \partial_\mu A_{c_1c_2c_3} + 3(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_{[c_1}{}^b A_{|b|c_2c_3]} \tag{2.71}$$

$$\begin{aligned} \tilde{D}_\mu A_{c_1\ldots c_6} &\equiv \partial_\mu A_{c_1\ldots c_6} + 6(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_{[c_1}{}^b A_{|b|c_2\ldots c_6]} \\ &\quad -20 A_{[c_1c_2c_3}\tilde{D}_{|\mu|}A_{c_4c_5c_6]}, \end{aligned} \tag{2.72}$$

where we identify $\mathsf{e}_\mu{}^a$ with the vielbein. Putting these definitions into Eq. (2.66) we finally get

$$\mathcal{V}_\mu = \mathsf{e}_\mu{}^a P_a + \Omega_{\mu a}{}^b K^a{}_b + \frac{1}{3!}\tilde{D}_\mu A_{c_1c_2c_3} R^{c_1c_2c_3} + \frac{1}{6!}\tilde{D}_\mu A_{c_1\ldots c_6} R^{c_1\ldots c_6}. \tag{2.73}$$

From Eq. (2.73) we can read off the covariant derivatives of the Goldstone fields $m_a{}^b$, $A_{c_1c_2c_3}$ and $A_{c_1\ldots c_6}$

$$\Omega_{ab}{}^c = \mathsf{e}^\mu{}_a \Omega_{\mu b}{}^c = \mathsf{e}^\mu{}_a (\mathsf{e}^{-1}\partial_\mu \mathsf{e})_b{}^c - \omega_{ab}{}^c \tag{2.74}$$

$$\tilde{D}_a A_{c_1c_2c_3} = \mathsf{e}^\mu{}_a \tilde{D}_\mu A_{c_1c_2c_3} \tag{2.75}$$

$$\tilde{D}_a A_{c_1\ldots c_6} = \mathsf{e}^\mu{}_a \tilde{D}_\mu A_{c_1\ldots c_6}, \tag{2.76}$$

where $\mathsf{e}_\mu{}^a \mathsf{e}^\nu{}_a = \delta_\mu^\nu$. These covariant derivatives are not field strengths, since they are not antisymmetrised. Only after a simultaneous nonlinear realisation with the conformal group will the field strengths be recovered. Note that we have defined $\omega_{ab}{}^c \equiv \mathsf{e}^\mu{}_a \omega_{\mu b}{}^c$.

Performing a local Lorentz transformation

$$g \to hgh^{-1} \qquad \text{with} \qquad h^a{}_b = \left(e^{\frac{1}{2}\varsigma_c{}^d(x) J^c{}_d}\right)^a{}_b \tag{2.77}$$

we find that $\mathsf{e}_\mu{}^a$ really transforms in agreement with Eq. (2.58), and also the interpretation of all the contracted indices in $\mathcal{V}_\mu$ as tangent space indices is justified. A matter field B will transform under (2.55) according to $B \to B' = D(h)B$, where D is the representation of the Lorentz group depending on the specific type of the matter field B. The covariant derivative of the matter field is then given by

$$\tilde{D}_a B \equiv \mathsf{e}^\mu{}_a \partial_\mu B + \frac{1}{2}\omega_{ab}{}^c S^b{}_c B, \tag{2.78}$$

where $S^a{}_b$ are the representations of the generators of the Lorentz group associated with B.

2.3.3 The conformal realisation

Let us now turn our attention to the conformal group. The conformal transformations in space-time include translations, Lorentz transformations, dilations and special conformal transformations. The generators and their commutation relations can be found in Appendix B. Here we will first use a slightly different approach to obtain the generators, and then perform the nonlinear realisation based on these. We assume the dimension to be $(10,1)$, but the results can be directly generalised to arbitrary $(d-1,1)$ dimensions.

As mentioned in Appendix B, the conformal group in $(10,1)$ dimensions is equal to the Lorentz group in $(11,2)$ dimensions. A linear representation of O(11,2) acts on a field as

$$\Psi(\zeta) \to \Psi'(\zeta) = D(\Lambda)\Psi(\Lambda^{-1}\zeta), \tag{2.79}$$

where ζ_A denotes the 13-dimensional coordinate vector and Λ is a pseudo-orthogonal transformation on these coordinates

$$\zeta_A \to \zeta'_A = \Lambda_A{}^B \zeta_B. \tag{2.80}$$

We will in this subsection use the signature $\eta_{AB} = \mathrm{diag}(1,-1,\ldots,-1,1)$. The notation $D(*)$ is used to denote the matrix representation of a group element or a generator associated with the field $\Psi(\zeta)$.

The generators of the Lorentz group are as usual defined through the representations of infinitesimal transformations $\Lambda = 1+\epsilon$ as

$$D(\Lambda) = 1 + \epsilon^{AB} J_{AB}. \tag{2.81}$$

The generators J_{AB} are given by

$$(J_{AB})_{CD} = \eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC} \tag{2.82}$$

in the 13-dimensional self-representation and obey the commutation relations

$$[J_{AB}, J_{CD}] = (\eta_{BC}J_{AD} - \eta_{BD}J_{AC} + \eta_{AD}J_{BC} - \eta_{AC}J_{BD}). \tag{2.83}$$

These generators can also be realised using the coordinates in 13 dimensions

$$J_{AB} \equiv \partial_{AB} = \zeta_A\partial_B - \zeta_B\partial_A. \tag{2.84}$$

We can now identify the generators of the conformal group in (10,1) dimensions with different parts of J_{AB} [9]

$$\begin{array}{llcl}
\text{Lorentz generators:} & J_{\mu\nu} & & \\
\text{Translation generators:} & P_\mu & = & J_{13\mu} + J_{12\mu} \\
\text{Special conformal generators:} & K_\mu & = & J_{13\mu} - J_{12\mu} \\
\text{Dilation generator:} & D & = & J_{13,12},
\end{array} \tag{2.85}$$

where the index μ runs from 0 to 10. Introduce a new set of independent coordinate variables

$$x_\mu = \frac{\zeta_\mu}{\zeta_{12} + \zeta_{13}}, \qquad \kappa = \zeta_{12} + \zeta_{13}, \qquad \lambda = \zeta_{12} - \zeta_{13}. \tag{2.86}$$

Calculating the transformations of ζ_A under $e^{\epsilon X}$, with X being the generators in (2.85), verifies that x_μ transforms indeed as a coordinate vector in 11 dimensions. Putting Eq. (2.86) into Eq. (2.84) we find the generators of the conformal group in terms of x_μ, κ and λ. It is possible to eliminate the variable λ by requiring $\zeta^2 = 0$, since this condition is invariant under 13-dimensional Lorentz transformations. This will allow us to set $\lambda = \kappa x^2$, and we finally arrive at

$$\begin{array}{rcl} \partial_{\mu\nu} & = & x_\mu \partial_\nu - x_\nu \partial_\mu \\ \partial_{13\mu} + \partial_{12\mu} & = & \partial_\mu \\ \partial_{13\mu} - \partial_{12\mu} & = & 2x_\mu \left(x^\nu \partial_\nu - \kappa \frac{\partial}{\partial\kappa} \right) - x^2 \partial_\mu \\ \partial_{13,12} & = & x^\nu \partial_\nu - \kappa \frac{\partial}{\partial\kappa}, \end{array} \tag{2.87}$$

which are precisely the generators of the conformal group given in (B.20). We define the degree of homogeneity as $l \equiv \kappa\partial/\partial\kappa$ with the coordinates x^μ having degree zero. For a homogeneous field $\Phi(\zeta)$ of degree l, the dependence of κ will factor out as κ^l on the hypercone $\zeta^2 = 0$, see reference [9].

Let us consider the reducing matrix[1]

$$g = e^{-x^a P_a} e^{\phi^b K_b} e^{\sigma D}, \tag{2.88}$$

having the conformal group as the global and the Lorentz group as the local symmetry, respectively. The preferred fields σ and ϕ^a are coordinate dependent, and we do not associate any field with the translation generators. As will become obvious later, the factor $e^{-x^a P_a}$ is included to give a simpler form for the covariant derivatives. Note also that we have chosen a coset representative, which does not contain any contribution from the local Lorentz subgroup $(J^a{}_b)$. The reducing matrix transforms under the coset according to

$$g(\zeta) \to g'(\zeta') = \Lambda g(\Lambda^{-1}\zeta) h^{-1}(\Lambda^{-1}\zeta, \Lambda), \tag{2.89}$$

where Λ corresponds to a global conformal transformation in (10,1) dimensions, h is a local Lorentz transformation and $\zeta' = \Lambda\zeta$. The nonlinear field ψ is then defined by

$$\psi(\zeta) \equiv D(g^{-1})\Psi(\zeta), \tag{2.90}$$

[1] For the conformal algebra, there is no difference between the algebra index a and the space-time index μ, hence in this subsection both a and μ are raised and lowered using the metric given in (2.113).

where $\Psi(\zeta)$ is the linear field defined in Eq. (2.79). The field $\psi(\zeta)$ transforms irreducibly under the Lorentz group as

$$\psi(\zeta) \to \psi'(\zeta') = D(h)\psi(\Lambda^{-1}\zeta). \tag{2.91}$$

The transformations of the preferred fields ϕ_μ and σ can be found if we require that $(g)_A{}^{12}$ and $(g)_A{}^{13}$ transform as true 13-dimensional vectors, i.e., $h(\zeta,\Lambda)$ acts only in the subspace $\{0,\ldots,10\}$. Using these columns we can define two independent linear fields

$$\begin{aligned}\Phi_A &\equiv \tfrac{1}{2}[g_A{}^{12} - g_A{}^{13}] \\ &= \begin{pmatrix} (\phi_\mu + x_\mu\phi^2)e^{-\sigma} \\ \frac{1}{2}(1 + 2x\cdot\phi + x^2\phi^2 + \phi^2)e^{-\sigma} \\ -\frac{1}{2}(1 + 2x\cdot\phi + x^2\phi^2 - \phi^2)e^{-\sigma} \end{pmatrix}\end{aligned} \tag{2.92}$$

and

$$\Psi_A \equiv \tfrac{1}{2}[g_A{}^{12} + g_A{}^{13}] = \begin{pmatrix} x_\mu e^\sigma \\ \frac{1}{2}(1+x^2)e^\sigma \\ \frac{1}{2}(1-x^2)e^\sigma \end{pmatrix}. \tag{2.93}$$

We are now ready to find the covariant derivatives. It should be stressed that the notation $h_A{}^B$ means the 13-dimensional matrix representation, whereas the notation $D(h)$ means the representation associated with the field $\Psi(\zeta)$. Let us first define the operator $\Delta_{AB}\psi$ as

$$\Delta_{AB}\psi \equiv D(g^{-1})\partial_{AB}\Psi, \tag{2.94}$$

where Ψ is an arbitrary linear field and ∂_{AB} is defined in (2.84). Under the coset, this operator transforms as

$$\Delta_{AB}\psi \to D(g'^{-1})\partial_{AB}\Psi' = \Lambda_A{}^{A'}\Lambda_B{}^{B'}D(h)\Delta_{A'B'}\psi, \tag{2.95}$$

where we have used Eqs. (2.79) and (2.89). To avoid $\Lambda_A{}^B$ in the transformation laws we define a new operator

$$\Delta_{(AB)}\psi \equiv (g^{-1})_A{}^{A'}(g^{-1})_B{}^{B'}\Delta_{A'B'}\psi. \tag{2.96}$$

Using the transformation laws (2.89) and (2.95) we find

$$\Delta_{(AB)}\psi \to h_A{}^{A'}h_B{}^{B'}D(h)\Delta_{(A'B')}\psi. \tag{2.97}$$

To simplify the calculations one observes that

$$\partial_{AB} = \frac{1}{2}(e^{-x\cdot P})_A{}^{A'}(e^{-x\cdot P})_B{}^{B'}(K^\mu\partial_\mu + 2lD)_{A'B'}, \tag{2.98}$$

putting which into Eqs. (2.94) and (2.96) yields

$$\begin{aligned}\Delta_{(AB)}\psi &= (g^{-1})_A{}^{C_1}(g^{-1})_B{}^{C_2}D(g^{-1}) \\ &\quad\times\left[\tfrac{1}{2}(e^{-x\cdot P})_{C_1}{}^{C_3}(e^{-x\cdot P})_{C_2}{}^{C_4}(K^\mu\partial_\mu+2lD)_{C_3C_4}\right]\Psi \\ &= \tfrac{1}{2}D(g^{-1})\{e^{-\sigma}(\partial_\mu+2l\phi_\mu)(K^\mu)_{AB}+2lD_{AB}\}\Psi.\end{aligned} \tag{2.99}$$

It is now clear that the expression (2.99) gets much simplified by putting a factor $e^{-x\cdot P}$ in the definition of g. We separate (2.99) into irreducible parts according to

$$\begin{aligned}\Delta_{(\mu\nu)}\psi &= 0 \\ (\Delta_{(13\mu)}+\Delta_{(12\mu)})\psi &= D(g^{-1})e^{-\sigma}(\partial_\mu+2l\phi_\mu)\Psi \\ (\Delta_{(13\mu)}-\Delta_{(12\mu)})\psi &= 0 \\ \Delta_{(13,12)}\psi &= -l\cdot D(g^{-1})\Psi=-l\psi.\end{aligned} \tag{2.100}$$

It is then natural to find the covariant derivatives inside the term

$$\Delta_\mu\psi\equiv D(g^{-1})e^{-\sigma}(\partial_\mu+2l\phi_\mu)D(g)\psi, \tag{2.101}$$

which transforms as

$$\Delta_\mu\psi\rightarrow\left[h_{13}{}^Ah_\mu{}^B+h_{12}{}^Ah_\mu{}^B\right]D(h)\Delta_{(AB)}\psi=h_\mu{}^\nu D(h)\Delta_\nu\psi(\Lambda^{-1}\zeta). \tag{2.102}$$

To calculate this transformation we have used the fact that $h_A{}^B$ only acts on the $(\mu\nu)$ subspace and takes the form of a 11-dimensional Lorentz transformation.

To obtain the covariant derivatives in 11 dimensions we write (2.101) as

$$\begin{aligned}\Delta_\mu\psi &= e^{-\sigma}(\partial_\mu+2l\phi_\mu)\psi+D(g^{-1})e^{-\sigma}\partial_\mu D(g)\psi \\ &= e^{-\sigma}(\partial_\mu+2l\phi_\mu)\psi+\tfrac{1}{2}(g^{-1}e^{-\sigma}\partial_\mu g)^A{}_BS_A{}^B\psi,\end{aligned} \tag{2.103}$$

where $S_A{}^B$ generates infinitesimal SO(11,2) transformations on $\Psi(\zeta)$. The Cartan form is given by

$$\begin{aligned}\mathcal{V} &= g^{-1}dg \\ &= dx^\mu\, g^{-1}\partial_\mu e^{-x^aP_a}e^{\phi^bK_b}e^{\sigma D} \\ &= dx^\mu\, g^{-1}e^{-x^aP_a}[\partial_\mu-(\partial_\mu x^c)P_c]e^{\phi^bK_b}e^{\sigma D} \\ &= dx^\mu\, g^{-1}e^{-x^aP_a}e^{\phi^bK_b}[\partial_\mu-(\partial_\mu x^c)P_c+\partial_\mu\phi^cK_c-2\phi_\mu D \\ &\quad+2\delta^c_\mu\phi^dJ_{cd}-2\phi_\mu\phi^cK_c+\phi^c\phi_c\delta^d_\mu K_d]e^{\sigma D} \\ &= dx^\mu\{-e^\sigma\delta^a_\mu P_a+e^{-\sigma}(\partial_\mu\phi^a+\phi^b\phi_b\delta^a_\mu-2\phi_\mu\phi^a)K_a \\ &\quad+(\partial_\mu\sigma-2\phi_\mu)D+(\delta^a_\mu\phi^b-\delta^b_\mu\phi^a)J_{ab}\},\end{aligned} \tag{2.104}$$

where the identity element $g^{-1}g$ has been omitted. Observe that $\mathcal{V}_\mu\mathcal{V}^\mu$, with the Cartan form defined without the spin connection, is not invariant under the coset transformations (2.89). The covariant parts of $g^{-1}e^{-\sigma}\partial_\mu g$ are

$$\begin{aligned}(g^{-1}e^{-\sigma}\partial_\mu\Phi)_A &= \tfrac{1}{2}[(g^{-1}e^{-\sigma}\partial_\mu g)_A{}^{12} - (g^{-1}e^{-\sigma}\partial_\mu g)_A{}^{13}] \\ &= \begin{pmatrix} e^{-2\sigma}(\partial_\mu\phi_\nu + \phi^\tau\phi_\tau\eta_{\mu\nu} - 2\phi_\mu\phi_\nu) \\ -e^{-\sigma}(\partial_\mu\sigma - 2\phi_\mu) \\ e^{-\sigma}(\partial_\mu\sigma - 2\phi_\mu)\end{pmatrix}\end{aligned} \tag{2.105}$$

and

$$\begin{aligned}(g^{-1}e^{-\sigma}\partial_\mu\Psi)_A &= \tfrac{1}{2}[(g^{-1}e^{-\sigma}\partial_\mu g)_A{}^{12} + (g^{-1}e^{-\sigma}\partial_\mu g)_A{}^{13}] \\ &= \begin{pmatrix} g_{\mu\nu} \\ e^{-\sigma}(\partial_\mu\sigma - 2\phi_\mu) \\ e^{-\sigma}(\partial_\mu\sigma - 2\phi_\mu)\end{pmatrix}.\end{aligned} \tag{2.106}$$

We use these to define the covariant derivatives of the fields σ and ϕ_μ. The remaining part

$$\begin{aligned}(g^{-1}e^{-\sigma}\partial_\mu g)_{\lambda\rho} &= e^{-\sigma}(\delta^\nu_\mu\phi^\tau - \delta^\tau_\mu\phi^\nu)(J_{\nu\tau})_{\lambda\rho} \\ &= e^{-\sigma}(\delta^\nu_\mu\phi^\tau - \delta^\tau_\mu\phi^\nu)(\eta_{\nu\lambda}\eta_{\tau\rho} - \eta_{\nu\rho}\eta_{\tau\lambda}) \\ &= 2e^{-\sigma}(\eta_{\mu\lambda}\phi_\rho - \eta_{\mu\rho}\phi_\lambda),\end{aligned} \tag{2.107}$$

becomes covariant together with the other terms in Eq. (2.103)

$$\begin{aligned}\mathfrak{D}_\mu\psi &\equiv e^{-\sigma}(\partial_\mu + 2l\phi_\mu)\psi + \tfrac{1}{2}\cdot 2e^{-\sigma}(\delta^\lambda_\mu\phi_\rho - \eta_{\mu\rho}\phi^\lambda)S_\lambda{}^\rho\psi \\ &= e^{-\sigma}(\partial_\mu + 2l\phi_\mu\psi + 2S_\mu{}^\rho\phi_\rho)\psi.\end{aligned} \tag{2.108}$$

We have used that the generators $S_{\mu\nu}$ obey $S_{\mu\nu} = -S_{\nu\mu}$.

In summary, the covariant derivatives of the various fields in 11 dimensions are

$$\mathfrak{D}_\mu\sigma = e^{-\sigma}(\partial_\mu\sigma - 2\phi_\mu) \tag{2.109}$$

$$\mathfrak{D}_\mu\phi_\nu = e^{-2\sigma}(\partial_\mu\phi_\nu + \phi^\tau\phi_\tau\eta_{\mu\nu} - 2\phi_\mu\phi_\nu) \tag{2.110}$$

$$\mathfrak{D}_\mu\psi = e^{-\sigma}(\partial_\mu + 2l\phi_\mu + 2S_\mu{}^\rho\phi_\rho)\psi, \tag{2.111}$$

where $\psi(\zeta)$ is homogeneous of degree l, whereas the fields σ and ϕ_μ are both of degree zero.

We can now calculate the transformations of the preferred fields, or more generally how an arbitrary field transforms under the coset, with the conformal group as the global symmetry and the Lorentz group as the local symmetry. The detailed calculations can be found in Appendix D.

The transformations of the preferred fields with special conformal transformations as the global transformations are

$$\begin{array}{rcl}\sigma'(x') & = & \sigma(x) + \ln|\det\left(\frac{\partial x'}{\partial x}\right)|^{-1/4} \\ \phi'_\mu(x') & = & \frac{\partial x^\nu}{\partial x'^\mu}\left\{\phi_\nu(x) + \frac{1}{2}\frac{\partial}{\partial x^\nu}\left[\ln|\det\left(\frac{\partial x'}{\partial x}\right)|^{-1/4}\right]\right\},\end{array} \tag{2.112}$$

where $x'^\mu = \frac{x^\mu + b^\mu x^2}{1+2b\cdot x + b^2x^2}$ and $\left(\ln|\det\left(\frac{\partial x'}{\partial x}\right)|^{-1/4}\right) = 1 + 2b\cdot x + b^2x^2$. Thus we see that the preferred fields transform with great similarity to the gauge fields, and it is motivated to try assign geometric meanings to these fields.

We would like to create a metric tensor and a connection using only σ and ϕ_μ, i.e., without introducing any new dynamical variable. The easiest choice of a metric tensor is

$$g_{\mu\nu} = \eta_{\mu\nu}e^{2\sigma(x)} \quad \Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu}e^{-2\sigma(x)}, \tag{2.113}$$

using the transformation laws for σ given in Appendix D we can verify that $g_{\mu\nu}$ indeed transforms as a rank two tensor. Having the metric, the Christoffel symbol will be

$$\begin{array}{rcl}\left\{\begin{matrix}\tau \\ \mu\ \nu\end{matrix}\right\} & = & \frac{1}{2}g^{\tau\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \\ & = & \delta^\tau_\mu\partial_\nu\sigma + \delta^\tau_\nu\partial_\mu\sigma - \eta_{\mu\nu}\eta^{\tau\rho}\partial_\rho\sigma.\end{array} \tag{2.114}$$

We can also define an alternative connection using the preferred fields ϕ_μ

$$\Gamma^\tau_{\mu\nu} = C(\delta^\tau_\mu\phi_\nu + \delta^\tau_\nu\phi_\mu \ - g_{\mu\nu}\phi^\tau). \tag{2.115}$$

The constant C can be determined by requiring the connection to transform as

$$\Gamma^\tau_{\mu\nu} \to \Gamma'^\tau_{\mu\nu} = \frac{\partial x'^\tau}{\partial x^{\rho_1}}\frac{\partial x^{\rho_2}}{\partial x'^\mu}\frac{\partial x^{\rho_3}}{\partial x'^\nu}\Gamma^{\rho_1}_{\rho_2\rho_3} - \frac{\partial^2 x'^\tau}{\partial x^{\rho_1}\partial x^{\rho_2}}\frac{\partial x^{\rho_1}}{\partial x'^\mu}\frac{\partial x^{\rho_2}}{\partial x'^\nu}, \tag{2.116}$$

which sets $C = 2$ to yield

$$\Gamma^\tau_{\mu\nu} = 2(\delta^\tau_\mu\phi_\nu + \delta^\tau_\nu\phi_\mu \ - g_{\mu\nu}\phi^\tau). \tag{2.117}$$

Comparing the expression for the Christoffel symbol with $\Gamma^\tau_{\mu\nu}$ leads to

$$\left\{\begin{matrix}\tau \\ \mu\ \nu\end{matrix}\right\} = \Gamma^\tau_{\mu\nu} \quad \Rightarrow \quad \partial_\mu\sigma = 2\phi_\mu. \tag{2.118}$$

If the connection is not equal to the Christoffel symbol, then the geometry is non-Riemannian. This means that the covariant derivative of the metric tensor with respect to this connection will not vanish. Whenever there exist two connections, we can form a general connection

$$
\begin{aligned}
&\left\{ {\tau \atop \mu\ \nu} \right\} + f\left(\left\{ {\tau \atop \mu\ \nu} \right\} - \Gamma^\tau_{\mu\nu} \right) \\
= \; & \left\{ {\tau \atop \mu\ \nu} \right\} + f\left[\delta^\tau_\mu(\partial_\nu\sigma - 2\phi_\nu) + \delta^\tau_\nu(\partial_\mu\sigma - 2\phi_\mu) - g_{\mu\nu}g^{\tau\rho}(\partial_\rho\sigma - 2\phi_\rho) \right],
\end{aligned}
\tag{2.119}
$$

where f is an arbitrary scalar function. From the nonlinear realisation we know that the second part of this general connection transforms as a tensor, since $\mathfrak{D}_\mu\sigma = \partial_\mu\sigma - 2\phi_\mu$ is the covariant derivative of σ. We can thus really put $\partial_\mu\sigma = 2\phi_\mu$ and use the Christoffel symbol as the connection, this choice is the same as to let the torsion vanish. However, we will continue the calculations with a general $\Gamma^\tau_{\mu\nu}$, i.e., assume ϕ_μ to be unrelated to σ.

The covariant derivative of an arbitrary tensor under general coordinate transformations is [4]

$$\mathfrak{D}_\mu T = \partial_\mu T - F^\nu{}_\tau \Gamma^\tau_{\mu\nu} T, \tag{2.120}$$

where $F^\nu{}_\tau$ defines the representation of GL(11,$\mathbb{R}$) to which T belongs, and satisfy the commutation relations

$$[F^\mu{}_\nu, F^\tau{}_\rho] = \delta^\tau_\nu F^\mu{}_\rho - \delta^\mu_\rho F^\tau{}_\nu. \tag{2.121}$$

We see that these generators are precisely $K^a{}_b$ introduced in (2.51). To get the covariant derivatives under conformal transformations, the parts in $F^\mu{}_\nu$ corresponding to Lorentz transformations and dilations are projected out by our specific choice of the connection $\Gamma^\tau_{\mu\nu}$. We take the eigenvalue of $F^\nu{}_\nu$ to be $(-l)$. Putting Eq. (2.117) into (2.120) we find

$$
\begin{aligned}
\mathfrak{D}_\mu\psi &= \partial_\mu\psi - F^\nu{}_\tau\Gamma^\tau_{\mu\nu}\psi \\
&= \partial_\mu\psi - 2F^\nu{}_\tau(\delta^\tau_\mu\phi_\nu + \delta^\tau_\nu\phi_\mu - g_{\mu\nu}\phi^\tau)\psi \\
&= \partial_\mu\psi + 2l\phi_\mu\psi + 2S_\mu{}^\tau\phi_\tau\psi,
\end{aligned}
\tag{2.122}
$$

where we have defined $S_{\mu\nu} \equiv F_{\mu\nu} - F_{\nu\mu}$. This is exactly the covariant derivative found using nonlinear realisation. To extend the covariant derivative for general coordinate transformations, we have to introduce a vielbein, i.e., we have to include the symmetric part of $F^\mu{}_\nu$. This is not possible using only the conformal group, since no appropriate connection can be defined for this purpose.

We define the Riemann tensor [10, 11]

$$R_{\mu\tau\nu}{}^\rho = \partial_\mu\Gamma^\rho_{\tau\nu} - \partial_\tau\Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\tau\nu}\Gamma^\rho_{\mu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\tau\sigma} \tag{2.123}$$

the Ricci tensor

$$R_{\mu\nu} = R_{\mu\tau\nu}{}^{\tau} \tag{2.124}$$

and the curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.125}$$

We can decompose the Ricci tensor into three linearly independent rank two tensors

$$\begin{array}{rcl} R_{(\mu\nu)} &=& \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu}) \\ &=& 2\{\partial_\nu\phi_\mu + \partial_\mu\phi_\nu - 4\phi_\mu\phi_\nu + 4g_{\mu\nu}(\phi_\tau\phi^\tau + \frac{1}{4}\partial_\tau\phi^\tau)\} \\ R_{[\mu\nu]} &=& \frac{1}{2}(R_{\mu\nu} - R_{\nu\mu}) = 2\{\partial_\nu\phi_\mu - \partial_\mu\phi_\nu\} \\ g_{\mu\nu}R &=& 12g_{\mu\nu}(2\phi_\tau\phi^\tau + \partial_\tau\phi^\tau). \end{array} \tag{2.126}$$

For a non-Riemannian geometry, i.e., $\partial_\mu\sigma \neq 2\phi_\mu$, the antisymmetric part of the Ricci tensor will not vanish. We can combine these tensors to form covariant objects, e.g.,

$$\begin{array}{rcl} \mathfrak{D}_\mu\phi_\nu &=& \frac{1}{8}\left(2R_{(\mu\nu)} - \frac{1}{3}g_{\mu\nu}R - R_{[\mu\nu]}\right) \\ &=& \partial_\mu\phi_\nu + g_{\mu\nu}\phi_\tau\phi^\tau - 2\phi_\mu\phi_\nu \end{array} \tag{2.127}$$

which is the covariant derivative of ϕ_μ.

In conclusion, we get the same covariant derivatives of both preferred fields and more general fields of degree l, despite if we use nonlinear realisation techniques or explicitly define a metric tensor and a connection based on the preferred fields.

2.3.4 Simultaneous realisation

Having now found the derivatives covariant under the G_{11} group and the conformal group separately, we want to construct quantities covariant under both these groups simultaneously [1, 12]. The role the preferred field σ played in the conformal realisation makes it natural to identify it with the diagonal elements of the metric

$$m_\mu{}^a = \bar{m}_\mu{}^a + \sigma\delta_\mu^a, \qquad \bar{m}_\mu{}^\mu = 0, \tag{2.128}$$

where $m_\mu{}^a$ is the Goldstone field defining the vielbein from the G_{11} realisation. The fields $\bar{m}_\mu{}^a$, $A_{c_1c_2c_3}$ and $A_{c_1\ldots c_6}$ transform under conformal transformations as their indices suggest.

The covariant derivative of $A_{c_1c_2c_3}$ is according to (2.75)

$$\begin{array}{rcl} \tilde{D}_a A_{c_1c_2c_3} &=& \mathsf{e}^\mu{}_a\{\partial_\mu A_{c_1c_2c_3} + 3(\mathsf{e}^{-1}\partial_\mu\mathsf{e})_{[c_1}{}^b A_{|b|c_2c_3]}\} \\ &=& \bar{\mathsf{e}}^\mu{}_a\{e^{-\sigma}\partial_\mu A_{c_1c_2c_3} + 3e^{-\sigma}[(\bar{\mathsf{e}}^{-1}\partial_\mu\bar{\mathsf{e}})_{[c_1}{}^b A_{|b|c_2c_3]} \\ && +\partial_\mu\sigma\delta^b_{[c_1}A_{|b|c_2c_3]}]\}, \end{array} \tag{2.129}$$

where $\bar{\mathsf{e}}_\mu{}^a = (e^{\bar{m}})_\mu{}^a = (e^m)_\mu{}^a e^{-\sigma}$. Knowing the covariant derivative of an arbitrary field under conformal transformations (2.111) and assuming $\partial_\mu\sigma = 2\phi_\mu$, we find

$$\begin{aligned}\tilde{D}_a A_{c_1c_2c_3} &= \bar{\mathsf{e}}^\mu{}_a\{\mathfrak{D}_\mu A_{c_1c_2c_3} + 3e^{-\sigma}[-\eta_{\mu[c_1}\partial^b\sigma A_{|b|c_2c_3]} \\ &\quad +\partial_{[c_1}\sigma A_{|\mu|c_2c_3]} + (\bar{\mathsf{e}}^{-1}\partial_\mu\bar{\mathsf{e}})_{[c_1}{}^b A_{|b|c_2c_3]} + \partial_\mu\sigma A_{c_1c_2c_3}]\},\end{aligned} \tag{2.130}$$

where $A_{c_1c_2c_3}$ has the degree $l=0$. Let us consider the above equation at order $(\bar{m})^0$

$$\begin{aligned}\tilde{D}_a A_{c_1c_2c_3} &\approx \mathfrak{D}_a A_{c_1c_2c_3} + 3e^{-\sigma}[-\eta_{a[c_1}\partial^b\sigma A_{|b|c_2c_3]} + \partial_{[c_1}\sigma A_{|a|c_2c_3]} \\ &\quad +\partial_a\sigma A_{c_1c_2c_3}].\end{aligned} \tag{2.131}$$

The contributions from the last three terms vanishes only if we completely antisymmetrise in the indices a, c_1, c_2 and c_3

$$\begin{aligned}\tilde{D}_{[a}A_{c_1c_2c_3]} &= \mathfrak{D}_{[a}A_{c_1c_2c_3]} + 3e^{-\sigma}[-\eta_{[ac_1}\partial^b\sigma A_{|b|c_2c_3]} \\ &\quad +\partial_{[c_1}\sigma A_{ac_2c_3]} + \partial_{[a}\sigma A_{c_1c_2c_3]}] \\ &= \mathfrak{D}_{[a}A_{c_1c_2c_3]}.\end{aligned} \tag{2.132}$$

The quantity simultaneously covariant under both the conformal group and the G_{11} group is

$$\begin{aligned}\tilde{F}_{c_1c_2c_3c_4} &\equiv 4\tilde{D}_{[c_1}A_{c_2c_3c_4]} \\ &= 4\{\mathsf{e}^\mu{}_{[c_1}\partial_{|\mu|}A_{c_2c_3c_4]} + 3\mathsf{e}^\mu{}_{[c_1}(\mathsf{e}^{-1}\partial_{|\mu|}\mathsf{e})_{c_2}{}^b A_{|b|c_3c_4]}\}\end{aligned} \tag{2.133}$$

A similar analysis for the field $A_{c_1\ldots c_6}$ yields a field strength with seven indices

$$\begin{aligned}\tilde{F}_{c_1\ldots c_7} &\equiv 7\tilde{D}_{[c_1}A_{c_2\ldots c_7]} \\ &= 7\{\mathsf{e}^\mu{}_{[c_1}\partial_{|\mu|}A_{c_2\ldots c_7]} + 6\mathsf{e}^\mu{}_{[c_1}(\mathsf{e}^{-1}\partial_{|\mu|}\mathsf{e})_{c_2}{}^b A_{|b|c_3\ldots c_7]} \\ &\quad +5\tilde{F}_{[c_1c_2c_3c_4}A_{c_5c_6c_7]}\}.\end{aligned} \tag{2.134}$$

For a matter field ψ of degree $l=0$ the covariant derivatives under the conformal and the G_{11} group are

$$\begin{aligned}\tilde{D}_a\psi &= \mathsf{e}^\mu{}_a\partial_\mu\psi + \tfrac{1}{2}\omega_{ab}{}^c S^b{}_c\psi \quad \text{the } \mathrm{G}_{11} \text{ group} \\ \mathfrak{D}_\mu\psi &= e^{-\sigma}(\partial_\mu + S_\mu{}^\rho\partial_\rho\sigma)\psi \quad \text{the conformal group,}\end{aligned} \tag{2.135}$$

respectively. Using $\bar{\mathsf{e}}^\mu{}_a = \mathsf{e}^\mu{}_a e^\sigma$ we find

$$\begin{aligned}\tilde{D}_a\psi &= \bar{\mathsf{e}}^\mu{}_a e^{-\sigma}\partial_\mu\psi + \tfrac{1}{2}\bar{\mathsf{e}}^\mu{}_a e^{-\sigma}\omega_{\mu b}{}^c S^b{}_c\psi \\ &= \bar{\mathsf{e}}^\mu{}_a\{\mathfrak{D}_\mu - e^{-\sigma}S_\mu{}^\rho\partial_\rho\sigma + \tfrac{1}{2}e^{-\sigma}\omega_{\mu b}{}^c S^b{}_c\}\psi,\end{aligned} \tag{2.136}$$

requiring the expression to be covariant also under the conformal group will yield a constraint, putting the part in Eq. (2.136) which is not covariant under the conformal group to zero. The consistent constraint is

$$\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0, \tag{2.137}$$

which gives the relation between the spin connection and the vielbein (2.60) if we insert the definition $\Omega_{abc} \equiv \mathsf{e}^{\mu}{}_{a}\mathsf{e}^{\rho}{}_{b}\partial_{\mu}\mathsf{e}_{\rho c} - \omega_{abc}$. We can also rewrite Ω_{abc} as

$$\begin{aligned}\Omega_{abc} &= \bar{\mathsf{e}}^{\mu}{}_{a}\bar{\mathsf{e}}^{\rho}{}_{b}\mathfrak{D}_{\mu}\bar{\mathsf{e}}_{\rho c} + e^{-\sigma}\bar{\mathsf{e}}^{\mu}{}_{a}\bar{\mathsf{e}}^{\rho}{}_{b}[-\eta_{\mu\rho}\partial^{d}\sigma\bar{\mathsf{e}}_{dc} - \eta_{\mu c}\partial^{d}\sigma\bar{\mathsf{e}}_{\rho d} \\ &\quad +\partial_{\rho}\sigma\bar{\mathsf{e}}_{\mu c} + \partial_{c}\sigma\bar{\mathsf{e}}_{\rho\mu}] + \eta_{bc}\partial_{a}\sigma - \omega_{abc},\end{aligned} \tag{2.138}$$

where $\mathfrak{D}_{\mu}\bar{\mathsf{e}}_{\rho c}$ denotes the covariant derivative of $\bar{\mathsf{e}}_{\rho c}$ under the conformal group. Putting this into Eq. (2.137) gives the relation

$$\omega_{abc} - 2\eta_{a[b}\partial_{c]}\sigma = \bar{\mathsf{e}}^{\mu}{}_{a}\bar{\mathsf{e}}^{\rho}{}_{[b}\mathfrak{D}_{|\mu}\bar{\mathsf{e}}_{\rho|c]} - \bar{\mathsf{e}}^{\mu}{}_{b}\bar{\mathsf{e}}^{\rho}{}_{(a}\mathfrak{D}_{|\mu}\bar{\mathsf{e}}_{\rho|c)} + \bar{\mathsf{e}}^{\mu}{}_{c}\bar{\mathsf{e}}^{\rho}{}_{(a}\mathfrak{D}_{|\mu}\bar{\mathsf{e}}_{\rho|b)}, \tag{2.139}$$

which we can use to write

$$\tilde{D}_{a}\psi = \bar{\mathsf{e}}^{\mu}{}_{a}\mathfrak{D}_{\mu}\psi + \frac{1}{2}\left\{\bar{\mathsf{e}}^{\mu}{}_{a}\bar{\mathsf{e}}^{\rho}{}_{b}\mathfrak{D}_{\mu}\bar{\mathsf{e}}_{\rho}{}^{c} - \bar{\mathsf{e}}^{\mu}{}_{b}\bar{\mathsf{e}}^{\rho c}\mathfrak{D}_{\mu}\bar{\mathsf{e}}_{\rho a} + \bar{\mathsf{e}}^{\mu c}\bar{\mathsf{e}}^{\rho}{}_{a}\mathfrak{D}_{\mu}\bar{\mathsf{e}}_{\rho b}\right\} S^{b}{}_{c}\psi. \tag{2.140}$$

This shows that $\tilde{D}_{a}\psi$ is simultaneously covariant under both the G_{11} and the conformal group if the constraint (2.137) is satisfied.

To identify with general relativity, we define the covariant derivative of a vector $A_{\mu} = \mathsf{e}_{\mu}{}^{a}A_{a}$ as

$$\begin{aligned}D_{\mu}A_{\nu} &\equiv \mathsf{e}_{\mu}{}^{a}\mathsf{e}_{\nu}{}^{b}\tilde{D}_{a}A_{b} \\ &= \mathsf{e}_{\mu}{}^{a}\mathsf{e}_{\nu}{}^{b}\left\{\mathsf{e}^{\tau}{}_{a}\partial_{\tau}A_{b} + \tfrac{1}{2}\omega_{ac}{}^{d}(S^{c}{}_{d})_{b}{}^{f}A_{f}\right\} \\ &= \partial_{\mu}A_{\nu} - (\mathsf{e}^{\rho}{}_{a}\partial_{\mu}\mathsf{e}_{\nu}{}^{a} - \mathsf{e}_{\mu}{}^{a}\mathsf{e}_{\nu}{}^{b}\mathsf{e}^{\rho}{}_{c}\omega_{ab}{}^{c})A_{\rho} \\ &= \partial_{\mu}A_{\nu} - \Gamma^{\rho}_{\mu\nu}A_{\rho},\end{aligned} \tag{2.141}$$

where $(S^{c}{}_{d})_{b}{}^{f} = (\eta^{c}{}_{b}\eta_{d}{}^{f} - \eta^{cf}\eta_{db})$ for vectors. Using the relation between the spin connection and the vielbein (2.60), we can show that $\Gamma^{\rho}_{\mu\nu}$ is just the Christoffel symbol

$$\Gamma^{\rho}_{\mu\nu} = \mathsf{e}^{\rho}{}_{a}\partial_{\mu}\mathsf{e}_{\nu}{}^{a} - \mathsf{e}_{\mu}{}^{a}\mathsf{e}_{\nu}{}^{b}\mathsf{e}^{\rho}{}_{c}\omega_{ab}{}^{c} = \frac{1}{2}g^{\rho\tau}(\partial_{\mu}g_{\nu\tau} + \partial_{\nu}g_{\mu\tau} - \partial_{\tau}g_{\mu\nu}) \tag{2.142}$$

with $g_{\mu\nu} = \mathsf{e}_{\mu}{}^{a}\mathsf{e}_{\nu}{}^{b}\eta_{ab}$ being the metric tensor.

The invariant equations of motion can now be formed out of the field strengths, $\tilde{F}_{c_1c_2c_3c_4}$ and $\tilde{F}_{c_1\ldots c_7}$, and the Riemann tensor

$$\begin{aligned}R_{b}{}^{c} &= d\omega_{b}{}^{c} + \omega_{b}{}^{d}\omega_{d}{}^{c} = \tfrac{1}{2}dx^{\mu}\wedge dx^{\nu}R_{\mu\nu b}{}^{c} \qquad \Rightarrow \\ R_{\mu\nu b}{}^{c} &= \partial_{\mu}\omega_{\nu b}{}^{c} - \partial_{\nu}\omega_{\mu b}{}^{c} + \omega_{\mu b}{}^{d}\omega_{\nu d}{}^{c} - \omega_{\nu b}{}^{d}\omega_{\mu d}{}^{c}.\end{aligned} \tag{2.143}$$

The spin connection will enter only through these quantities. The equations of motion have also to be covariant under local Lorentz transformations. The only possible non-trivial equations are then

$$\tilde{F}_{c_1c_2c_3c_4} = \frac{1}{7!}\varepsilon_{c_1\ldots c_{11}}\tilde{F}^{c_5\ldots c_{11}} \tag{2.144}$$

and

$$\begin{aligned} & R_{\mu\nu b}{}^{c}\mathsf{e}_c{}^{\nu}\mathsf{e}_a{}^{\mu} - \tfrac{1}{2}\eta_{ab}R_{\mu\nu d}{}^{c}\mathsf{e}_c{}^{\nu}\mathsf{e}^{d\mu} = \\ = \; & \tfrac{C}{4}\left(\tilde{F}_{ac_1c_2c_3}\tilde{F}_b{}^{c_1c_2c_3} - \tfrac{1}{6}\eta_{ab}\tilde{F}_{c_1c_2c_3c_4}\tilde{F}^{c_1c_2c_3c_4}\right), \end{aligned} \tag{2.145}$$

where C is a numerical constant. The first equation is precisely Eq. (2.50), while the second equation is the Einstein's equation of gravity.

2.3.5 Gauge symmetry

The reducing matrix in the G_{11} realisation

$$g = e^{x^f P_f} e^{m_a{}^b K^a{}_b} \exp\left(\frac{1}{3!}A_{c_1c_2c_3}R^{c_1c_2c_3} + \frac{1}{6!}A_{c_1\ldots c_6}R^{c_1\ldots c_6}\right), \tag{2.146}$$

transforms as $g \to g_0 g$ under a pure global transformation. If we take

$$g_0 = \exp\left(\frac{1}{3!}c_{\mu_1\mu_2\mu_3}\delta^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}R^{a_1a_2a_3} + \frac{1}{6!}c_{\mu_1\ldots\mu_6}\delta^{\mu_1\ldots\mu_6}_{a_1\ldots a_6}R^{a_1\ldots a_6}\right) \tag{2.147}$$

then

$$\begin{aligned} g \to g_0 g \;\; = \;\; & e^{x^f P_f} e^{m_a{}^b K^a{}_b} \exp\{\, \tfrac{1}{3!}(A_{a_1a_2a_3} + c_{a_1a_2a_3})R^{a_1a_2a_3} \\ & + \tfrac{1}{6!}(A_{a_1\ldots a_6} + c_{a_1\ldots a_6} + 20c_{[a_1a_2a_3}A_{a_4a_5a_6]})R^{a_1\ldots a_6}\,\}\,. \end{aligned} \tag{2.148}$$

The variations of the Goldstone fields are

$$\begin{aligned} \delta h_a{}^b &= 0 \\ \delta A_{a_1a_2a_3} &= c_{a_1a_2a_3} \\ \delta A_{a_1\ldots a_6} &= c_{a_1\ldots a_6} + 20c_{[a_1a_2a_3}A_{a_4a_5a_6]}, \end{aligned} \tag{2.149}$$

where $c_{\mu_1\mu_2\mu_3} = \mathsf{e}_{\mu_1}{}^{a_1}\mathsf{e}_{\mu_2}{}^{a_2}\mathsf{e}_{\mu_3}{}^{a_3}c_{a_1a_2a_3}$. Using Eq. (D.20) with $l = -p$, the fields with curved indices will transform under special conformal transformations according to

$$\begin{aligned} \delta A_{\mu_1\ldots\mu_p} \;\; = \;\; & (2x^\nu\beta_\nu x^\mu\partial_\mu - x^2\beta^\mu\partial_\mu)A_{\mu_1\ldots\mu_p} + 2p\beta^\mu x_\mu A_{\mu_1\ldots\mu_p} \\ & + 2p(\beta_{[\mu_1}x^\rho A_{|\rho|\mu_2\ldots\mu_p]} - x_{[\mu_1}\beta^\rho A_{|\rho|\mu_2\ldots\mu_p]}), \end{aligned} \tag{2.150}$$

where $p = \{3, 6\}$ and $A_{\mu_1\ldots\mu_p} \equiv (e^{\bar{m}})_{\mu_1}{}^{a_1}\ldots(e^{\bar{m}})_{\mu_p}{}^{a_p}e^{p\sigma}A_{a_1\ldots a_p}$.

We assume now that $\delta A_{\mu_1\ldots\mu_p} = c_{\mu_1\ldots\mu_p}$. Defining $\Lambda^{(1)}_{\mu_2\ldots\mu_p} = x^\rho c_{\rho\mu_2\ldots\mu_p}$, the transformations can be written as

$$\begin{aligned}
\delta_c A_{\mu_1\ldots\mu_p} &= c_{\mu_1\ldots\mu_p} = \partial_{[\mu_1}\Lambda^{(1)}_{\mu_2\ldots\mu_p]} = \partial_{[\mu_1} x^\rho c_{|\rho|\mu_2\ldots\mu_p]} \\
\delta_\beta A_{\mu_1\ldots\mu_p} &= (2x^\nu\beta_\nu x^\mu\partial_\mu - x^2\beta^\mu\partial_\mu)A_{\mu_1\ldots\mu_p} + 2p(\beta_{[\mu_1} x^\rho A_{|\rho|\mu_2\ldots\mu_p]} \\
&\quad - x_{[\mu_1}\beta^\rho A_{|\rho|\mu_2\ldots\mu_p]}) + 2p\beta^\mu x_\mu A_{\mu_1\ldots\mu_p}.
\end{aligned} \tag{2.151}$$

Taking the commutation relation between these we get

$$\begin{aligned}
[\delta_c,\delta_\beta]A_{\mu_1\ldots\mu_p} &= \delta_c(\delta_\beta A_{\mu_1\ldots\mu_p}) - \delta_\beta(\delta_c A_{\mu_1\ldots\mu_p}) \\
&= \delta_c\{(2x^\nu\beta_\nu x^\mu\partial_\mu - x^2\beta^\mu\partial_\mu)A_{\mu_1\ldots\mu_p} + 2p(\beta_{[\mu_1} x^\rho A_{|\rho|\mu_2\ldots\mu_p]} \\
&\quad - x_{[\mu_1}\beta^\rho A_{|\rho|\mu_2\ldots\mu_p]}) + 2p\beta^\mu x_\mu A_{\mu_1\ldots\mu_p}\} - \delta_\beta c_{\mu_1\ldots\mu_p} \\
&= 2p(\beta_{[\mu_1} x^\rho c_{|\rho|\mu_2\ldots\mu_p]} - x_{[\mu_1}\beta^\rho c_{|\rho|\mu_2\ldots\mu_p]} + \beta^\mu x_\mu c_{\mu_1\ldots\mu_p}) \\
&= \partial_{[\mu_1}\Lambda^{(2)}_{\mu_2\ldots\mu_p]} \equiv \delta_{\Lambda^{(2)}} A_{\mu_1\ldots\mu_p}
\end{aligned} \tag{2.152}$$

with

$$\Lambda^{(2)}_{\mu_2\ldots\mu_p} = (2px^\nu\beta_\nu x^\rho - x^2\beta^\rho)c_{\rho\mu_2\ldots\mu_p} - 2(p-1)x_{[\mu_2}\beta^\rho x^\nu c_{|\nu\rho|\mu_3\ldots\mu_p]}. \tag{2.153}$$

We have in the calculations used that $\partial_\mu c_{\mu_1\ldots\mu_p} = 0$ and $\delta_\beta c_{\mu_1\ldots\mu_p} = 0$.

Generally $\delta_\beta(\delta_{\Lambda^{(r)}} A_{\mu_1\ldots\mu_p}) = 0$, which results in

$$\begin{aligned}
[\delta_{\Lambda^{(r)}},\delta_\beta]A_{\mu_1\ldots\mu_p} &= \delta_{\Lambda^{(r)}}(\delta_\beta A_{\mu_1\ldots\mu_p}) \\
&= (2x^\nu\beta_\nu x^\mu\partial_\mu - x^2\beta^\mu\partial_\mu)\partial_{[\mu_1}\Lambda^{(r)}_{\mu_2\ldots\mu_p]} + \\
&\quad +2p(\beta_{[\mu_1} x^\rho\partial_{|\rho|}\Lambda^{(r)}_{\mu_2\ldots\mu_p]} - x_{[\mu_1}\beta^\rho\partial_{|\rho|}\Lambda^{(r)}_{\mu_2\ldots\mu_p]}) \\
&\quad +2p\beta^\mu x_\mu\partial_{[\mu_1}\Lambda^{(r)}_{\mu_2\ldots\mu_p]} \\
&= \partial_{[\mu_1}\Lambda^{(r+1)}_{\mu_2\ldots\mu_p]} \equiv \delta_{\Lambda^{(r+1)}} A_{\mu_1\ldots\mu_p},
\end{aligned} \tag{2.154}$$

where

$$\begin{aligned}
\Lambda^{(r+1)}_{\mu_2\ldots\mu_p} &= \{2x^\nu\beta_\nu x^\mu\partial_\mu - x^2\beta^\mu\partial_\mu + 2(p-1)x^\nu\beta_\nu\}\Lambda^{(r)}_{\mu_2\ldots\mu_p} \\
&\quad +2(p-1)\{\beta_{[\mu_2} x^\rho\Lambda^{(r)}_{|\rho|\mu_3\ldots\mu_p]} - x_{[\mu_2}\beta^\rho\Lambda^{(r)}_{|\rho|\mu_3\ldots\mu_p]}\}
\end{aligned} \tag{2.155}$$

is defined recursively. It is then clear that one can obtain gauge transformations $\Lambda^{(r)}_{\mu_2\ldots\mu_p}$ of $A_{\mu_1\ldots\mu_p}$ at arbitrary order r in x^μ, by taking repeated commutators with the special conformal transformations. We have thus shown that simultaneous realisation under the G_{11} and the conformal group leads to local gauge transformations of the fields.

3

Lorentzian Kac-Moody algebras

The field content of the eleven dimensional supergravity suggests that M-theory possesses some rank eleven symmetry algebra. There are also mounting evidence that Kac-Moody algebras of infinite type and generalised Kac-Moody (Borcherds) algebras might appear in the guise of duality symmetries in string and M-theory. It is therefore conjectured that M-theory includes a rank eleven Lorentzian Kac-Moody symmetry denoted E_{11}. Some fragments of this symmetry have indeed been found in the maximal supergravities in ten and eleven dimensions. Before we continue with eleven dimensional supergravity, we will in this chapter look at the general theory of Lorentzian Kac-Moody algebras mainly following the reference [13]. Some special attention will then be given to the group E_{11}. Since all the hyperbolic Kac-Moody algebras have a real principal so(2,1) subalgebra, it is interesting to see the constraints obtained by imposing the existence of an so(2,1) subalgebra on the Lorentzian Kac-Moody algebras.

3.1 General theory

A finite dimensional simple Lie algebra **g** is defined through its Cartan matrix [8]

$$A_{ij} = \frac{2}{(\alpha_i, \alpha_i)}(\alpha_i, \alpha_j), \tag{3.1}$$

where α_i are the simple roots of the algebra, this definition implies automatically that $A_{ii} = 2$. The fundamental weights are defined by

$$\frac{2}{(\alpha_i, \alpha_i)}(\alpha_i, \lambda_j) = \delta_{ij}, \tag{3.2}$$

using which we can express the inverse of the Cartan matrix as

$$A^{-1}_{ij} = \frac{2}{(\alpha_i,\alpha_i)}(\lambda_i,\lambda_j). \tag{3.3}$$

Since the simple roots constitute a basis for the root lattice, the highest root of a finite dimensional Lie algebra can be written as

$$\theta = \sum_{i=1}^{r} n_i\alpha_i, \tag{3.4}$$

where r is the rank of the algebra and n_i are called Kac labels

$$n_i = \frac{2}{(\alpha_i,\alpha_i)}(\theta,\lambda_i). \tag{3.5}$$

The Coxeter number is defined as

$$h(\mathbf{g}) \equiv 1 + \sum_{i=1}^{r} n_i \tag{3.6}$$

with $\sum_{i=1}^{r} n_i$ being the height of the highest root. In the same way as the simple roots form a basis for the root lattice, the fundamental weights span the weight lattice and we define the Weyl vector by

$$\rho \equiv \sum_{i=1}^{r} \frac{2}{(\alpha_i,\alpha_i)}\lambda_i. \tag{3.7}$$

We can now generalise the definitions to a Kac-Moody algebra, which is specified through a Cartan matrix satisfying [14]

$$\begin{aligned} A_{ii} &= 2 \\ A_{ij} &= \text{negative integers or zero, } i \neq j \\ A_{ij} = 0 &\Rightarrow A_{ji} = 0. \end{aligned} \tag{3.8}$$

In addition to these criteria, we will here consider only symmetric generalised Cartan matrices, i.e., $A_{ij} = A_{ji}$. The assumption that A_{ij} is symmetric means that we have chosen all the simple roots to be space-like with $(\alpha_i,\alpha_i) = 2$. If the Cartan matrix is positive definite, the algebra is finite dimensional. If the Cartan matrix is positive semi-definite, the algebra is of affine type. Moreover, we call the Kac-Moody algebra Lorentzian, if the Cartan matrix is non-singular and possesses precisely one negative eigenvalue. Among the Lorentzian algebras, the hyperbolic ones are characterised by the property that the entries of the associated inverse Cartan matrix have a definite sign [15, 16], $A^{-1}_{ij} \leq 0$. In the case of finite dimensional or affine algebras, the algebra is referred as simply-laced if the Cartan matrix is symmetric, and the only allowed values for the off-diagonal elements are 0 and (-1). All the roots are then of the same length, and the simple roots will span an even lattice $\Lambda_R(A)$.

The generators of the Kac-Moody algebra can be divided into the generators of the Cartan subalgebra H_i and the generators of the positive roots E_α and the negative roots F_α. The roots are defined as eigenvectors of the Cartan generators under the adjoint action. All the root generators can be obtained from those corresponding to the simple roots (E_i, F_i) using the Lie brackets. The generators are taken to obey the Chevalley-Serre relations

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, E_j] &= A_{ij} E_j \\ [H_i, F_j] &= -A_{ij} F_j \\ [E_i, F_j] &= \delta_{ij} H_i \\ (\mathrm{ad}_{X_i})^{1-A_{ij}} X_j &= 0 \qquad \text{for } i \neq j, \end{aligned} \tag{3.9}$$

where X_i is either E_i or F_i, and (ad_{X_i}) denotes the adjoint representation of the generator X_i. The Chevalley-Serre relations are invariant under the Cartan involution

$$E_\alpha \to -F_\alpha, \quad F_\alpha \to -E_\alpha, \quad H_\alpha \to -H_\alpha, \tag{3.10}$$

with α being any positive root. The generators $(E_\alpha - F_\alpha)$ will then be even under the Cartan involution and form the maximal compact subalgebra of the original Kac-Moody algebra. The remaining generators $(E_\alpha + F_\alpha)$ and H_α are odd under the involution.

The information contained in the Cartan matrix can also be presented as an unoriented graph with r nodes called a Dynkin diagram. Each simple root is represented by a node, and the number of links between the nodes i and j equals $\max\{|A_{ij}|, |A_{ji}|\}$. The Dynkin diagram of a Lorentzian Kac-Moody algebra is a connected diagram possessing at least one node whose deletion yields a diagram whose connected components are of finite type except for at most one of affine type [13]. The Dynkin diagram of a hyperbolic Kac-Moody algebra, on the other hand, is a connected diagram such that deletion of any one node leaves a (possibly mutually disconnected) set of connected Dynkin diagrams, each of which is of finite type except for at most one of affine type.

Let C denote the overall Dynkin diagram, yielding the reduced diagram C_R after having deleted one node. The node deleted is called the central node, and the reduced diagram can possibly contain mutually disconnected diagrams $C_1, C_2, \ldots, C_n$, see Fig. 3.1. The Cartan matrix

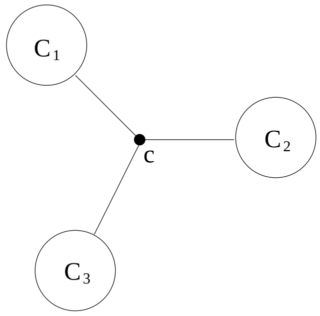

Figure 3.1: *The Dynkin diagram C of a Kac-Moody algebra. After having deleted the central node, the reduced diagram will consist of three disconnected diagrams.*

of C_R is obtained from the Cartan matrix of C by deleting the row and column corresponding to the central node. The number of links between the central node c and node i is denoted by

$$\eta_i = -A_{ci}, \tag{3.11}$$

where A_{ij} is the Cartan matrix of C and is symmetric by assumption. Since the Cartan matrix is non-singular, we can write the fundamental weights as

$$\lambda_i = \sum_{j=1}^{r} (A^{-1})_{ij}\alpha_j \tag{3.12}$$

satisfying Eq. (3.2). For hyperbolic algebras all the fundamental weights lie inside the same light-cone, since $A_{ij}^{-1} = \frac{2}{(\alpha_i,\alpha_i)}(\lambda_i, \lambda_j)$ are all negative or zero. For Lorentzian algebras which are not of hyperbolic type, some of the fundamental weights have to be space-like. We can now form a weight lattice $\Lambda_w(A)$, which is dual to the root lattice, using the fundamental weights. It is interesting to form the quotient

$$\Lambda_w(A)/\Lambda_R(A) = Z(A), \tag{3.13}$$

where $Z(A)$ is a finite abelian group containing $|Z(A)|$ elements and relates to the Cartan matrix of C via

$$\det A = \pm|Z(A)|. \tag{3.14}$$

We can also define the Weyl vector ρ

$$\rho = \sum_{i=1}^{r} \frac{2}{(\alpha_i, \alpha_i)}\lambda_i = \sum_{i,j=1}^{r} (A^{-1})_{ij} \frac{2\alpha_j}{(\alpha_j, \alpha_j)} \tag{3.15}$$

satisfying $(\rho, \alpha_i) = 1$. Whether all the coefficients $\sum_{i=1}^{r}(A^{-1})_{ij}$ have the same sign or not is crucial for the existence of a principal so(2,1) subalgebra. In the finite case the coefficients $\sum_{i=1}^{r}(A^{-1})_{ij}$ are all positive, while they are all negative in the hyperbolic case. In the Lorentzian case mixed signs are possible, but at least one of them is negative. More about real principal so(2,1) subalgebras can be found in Appendix E.

3.2 The root system

We will now construct the simple roots of a rank r Dynkin diagram C of Lorentzian type, by using the simple roots of the corresponding reduced Dynkin diagram C_R.

3.2.1 The reduced algebra is finite

Let us first assume that the reduced diagram corresponds to a finite dimensional Lie algebra. Since C_R is of finite type, the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{r-1}$ are linearly independent and span a $(r-1)$-dimensional Euclidean space. We let these roots also be the simple roots of the overall diagram C, and then add the central node to the set of simple roots. The central node can be taken as

$$\alpha_c = -\nu + x, \quad \text{with } \nu = \sum_{i=1}^{r-1} \eta_i \lambda_i = -\sum_{i=1}^{r-1} A_{ci}\lambda_i \tag{3.16}$$

satisfying the constraints $(\alpha_c, \alpha_i) = A_{ci}$. We have used λ_i to denote the fundamental weights of the reduced diagram C_R, while x is a direction orthogonal to the root space of C_R. Demanding the central node to be space-like leads to

$$2 = A_{cc} = (\alpha_c, \alpha_c) = \nu^2 + x^2, \tag{3.17}$$

which determines the sign of x^2. The root space of C is Euclidean or Lorentzian depends on whether x^2 is positive or negative, this is because we have assumed the root space of C_R to be Euclidean. If x^2 vanishes, the root space will have a positive semi-definite metric, and thus corresponds to an affine Lie algebra.

We use $l_c, l_1, l_2, \ldots, l_{r-1}$ to denote the fundamental weights of C, and similarly let $\lambda_1, \lambda_2, \ldots, \lambda_{r-1}$ denote the fundamental weights of C_R. These two sets are related to each other by

$$l_i = \lambda_i + \frac{(\nu, \lambda_i)}{x^2}x, \quad l_c = \frac{1}{x^2}x \tag{3.18}$$

providing that x^2 is non-vanishing, i.e., the overall diagram is not of affine type. The overall Weyl vector is then given by

$$R = \sum_{i=c}^{r-1} l_i = \rho + \frac{[1 + (\nu, \rho)]}{x^2} x, \tag{3.19}$$

where $\rho = \sum_{i=1}^{r-1} \lambda_i$ is the Weyl vector for the reduced diagram C_R, noting that $(\alpha_i, \alpha_i) = 2$ for symmetric Cartan matrices. Taking the scalar product of the Weyl vector with the fundamental weight l_c

$$(R, l_c) = \frac{[1 + (\nu, \rho)]}{x^2}, \tag{3.20}$$

we see that at least one of the coefficients in the expansion of the Weyl vector R in terms of simple roots is negative if x^2 is chosen negative, since (ν, ρ) is positive when the reduced diagram is of finite type.

3.2.2 The reduced algebra is finite or affine

Assume now the connected components of the reduced diagram are allowed to be of both finite and affine type. Since the Cartan matrix B for the reduced diagram C_R is obtained from the Cartan matrix A for the overall diagram C, we can relate the determinants of these two matrices by

$$\det A = x^2 \det B = (2 - \nu^2) \det B. \tag{3.21}$$

This can be shown by using the definition $A_{ij} = \frac{2}{(\alpha_i, \alpha_i)}(\alpha_i, \alpha_j)$, writing out the simple roots in an orthogonal basis and observing that x is orthogonal to all the simple roots of C_R. Noting $(B^{-1})_{ij} = \frac{2}{(\alpha_i, \alpha_i)}(\lambda_i, \lambda_j)$ we find

$$\begin{aligned} \det A &= \left(2 - \sum_{i,j=1}^{r-1} \eta_i (B^{-1})_{ij} \eta_j \right) \det B \\ &= 2 \det B - \sum_{i,j=1}^{r-1} \eta_i (\text{adj } B)_{ij} \eta_j, \end{aligned} \tag{3.22}$$

where the adjugate matrix $(\text{adj } B)_{ij} = \det B (B^{-1})_{ij}$ is the matrix of cofactors for B. If B is singular we just drop the first term on the right hand side. When C_R is disconnected with the connected components $C_1, \ldots, C_n$, the Cartan matrix B can be written in a block diagonal form

$$B = \text{diag}(B_1, B_2, \ldots, B_n), \tag{3.23}$$

where B_β is the Cartan matrix of the β:th component C_β. Adopting the notation $\Delta_\beta = \det B_\beta$, Eq. (3.22) becomes

$$\det A = \Delta_1 \dots \Delta_n \left(2 - \sum_{\beta=1}^{n} \left\{ \sum_{i,j \in C_\beta} \frac{\eta_i (\mathrm{adj}\ B_\beta)_{ij} \eta_j}{\Delta_\beta} \right\} \right), \tag{3.24}$$

allowing evaluation of determinants of Cartan matrices iteratively. For consistency we put $\Delta_n = 1$ and $\mathrm{adj} B_n = 0$ if the diagram is empty. Also, if p of the connected components of C_R correspond to affine algebras, then $\det A$ will have a $(p-1)$-fold zero. Unless $p = 1$ the overall algebra is neither affine nor Lorentzian, since it will have one negative and $(p-1)$ zero eigenvalues.

Using Eq. (3.24) on the Dynkin diagram for the exceptional algebras e_N and choosing the tip of the shortest leg to be the central node, one finds

$$\det A(\mathrm{e}_N) = 9 - N. \tag{3.25}$$

This indicates that e_N is Lorentzian for $N \geq 10$.

We construct now the simple roots of C by using the simple roots of each connected components of C_R. If C_β is of finite type, the simple roots α_i belonging to this diagram are linearly independent, whereas in the case of an affine diagram the simple roots are linearly dependent satisfying

$$\sum_{i \in C_\beta} n_i \alpha_i = 0. \tag{3.26}$$

The positive integers n_i are the Coxeter labels for the affine diagram C_β. The simple roots of the overall diagram are given by

$$\begin{array}{rcl} a_i &=& \alpha_i + \eta_i k = \alpha_i - A_{ci} k, \quad i \in C_R \\ a_c &=& -(k + \bar{k}), \end{array} \tag{3.27}$$

where the vectors k and $\bar{k}$ span the even self-dual Lorentzian lattice $\Pi_{1;1}$ satisfying $k^2 = \bar{k}^2 = 0$ and $(k, \bar{k}) = 1$, see Appendix F.

Let us concentrate on the Lorentzian Lie algebras, i.e., algebras which when deleting the central node yields only one affine component and the rest of finite type. Let also the affine component of the reduced diagram be C_1, this means that $\Delta_1 = \det B_1 = 0$ and Eq. (3.24) becomes

$$\det A = -\Delta_2 \dots \Delta_n \sum_{i,j \in C_1} \eta_i (\mathrm{adj}\ B_1)_{ij} \eta_j. \tag{3.28}$$

Note that by doing so the dependence on $\eta_i = -A_{ci}$ for $i \notin C_1$ has disappeared. Since C_1 is a connected diagram corresponding to a simply-laced affine algebra $\mathbf{g}^{(1)}$, its adjugate matrix can be written as

$$(\mathrm{adj}\ B_1)_{ij} = |Z(A(\mathbf{g}))| n_i n_j, \tag{3.29}$$

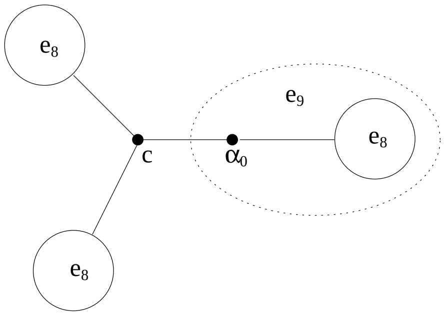

Figure 3.2: *The Dynkin diagram of a Kac-Moody algebra with self-dual Lorentzian root lattice. The root lattice in this example will be* $\Pi_{25;1}$.

where $\mathbf{g}^{(1)}$ is the affine extension of $\mathbf{g}$. Eq. (3.29) is motivated by the fact that $\sum_j (B_1)_{ij}(\mathrm{adj}\ B_1)_{jk} = \delta_{ik}\det B_1 = 0$ and $\sum_j B_{ij}n_j = 0$ for an affine algebra. The specific form of Eq. (3.29) comes about due to that $(\mathrm{adj}\ B_1)_{ij}$ is symmetric, and $(\mathrm{adj}\ B_1)_{00} = \det A(\mathbf{g}) = |Z(A(\mathbf{g}))|$ according to Eq. (3.14). Inserting Eq. (3.29) into (3.28) gives

$$\det A = -\Delta_2 \ldots \Delta_n |Z(A(\mathbf{g}))| \left(\sum_{i\in C_1} n_i\eta_i\right)^2, \tag{3.30}$$

and thus $\det A$ is explicitly negative.

It is of particular interest to study root lattices that are even, integral and self-dual Lorentzian. All the Cartan matrices considered so far guarantee an even, integral Lorentzian lattice. For the self-duality, the determinant of the Cartan matrix has to equal (-1). From Eq. (3.30) we see that imposing this leads to the condition that all the connected parts $C_2, \ldots, C_n$ have to give rise to self-dual lattices, i.e., $\Delta_i = 1$ for $i = 2, \ldots, n$, while C_1 must be the affine extension of an algebra with self-dual lattice. Since the only finite dimensional algebra having self-dual lattice is e_8, each diagram C_i is corresponding to an e_8 algebra except for C_1, which corresponds to the affine algebra e_9. Finally the sum $\sum_{i\in C_1} n_i\eta_i$ must equal unity also, meaning that all the η_i in C_1 must vanish except the one belonging to the node with Coxeter label equals one, i.e., the affine node. These unique Lorentzian self-dual lattices are denoted by $\Pi_{8n+1;1}$ and have the dimensions $(8n+2)$, see Fig. 3.2. When $n \geq 2$ there exist many Cartan matrices corresponding to the same lattice, since we are still free to choose the numbers $\eta_i, i \notin C_1$.

To simplify notations we include all the diagrams of finite type in C_2, which now is allowed to be disconnected. The fundamental weights of the overall diagram C are

$$\begin{array}{rcl} l_\beta & = & \lambda_\beta, \quad \beta \in C_2 \\ l_c & = & -k \\ l_i & = & \lambda_i - \frac{n_i}{\eta}(k - \bar{k} + \nu), \quad i \in C_1, \end{array} \tag{3.31}$$

where the vectors k and $\bar{k}$ again span the lattice $\Pi_{1;1}$, and λ_β are the fundamental weights of C_2. The algebra corresponding to C_1 is the untwisted affine extension of $\mathbf{g}$, for which λ_i are the fundamental weights. We also define

$$\begin{array}{rcl} \nu & = & \nu(\mathbf{g}) + \nu(C_2) \equiv \sum_{i=1}^{r(\mathbf{g})} \eta_i \lambda_i + \sum_{\beta \in C_2} \eta_\beta \lambda_\beta \\ \eta & \equiv & \sum_{i \in C_1} n_i \eta_i, \end{array} \tag{3.32}$$

with $r(\mathbf{g})$ being the rank of $\mathbf{g}$. The Weyl vector will be

$$R(C) = \rho(C_2) + \rho(\mathbf{g}) - \frac{h(\mathbf{g})}{\eta}(k - \bar{k} + \nu) - k, \tag{3.33}$$

with $\rho(C_2)$ and $\rho(\mathbf{g})$ being the Weyl vectors of C_2 and $\mathbf{g}$ respectively, and the Coxeter number is given by

$$h(\mathbf{g}) = \sum_{j=0}^{r(\mathbf{g})} n_j. \tag{3.34}$$

3.2.3 Real principal so(2,1) subalgebra

Taking the scalar product

$$(R, l_c) = -\frac{h(\mathbf{g})}{\eta} < 0, \tag{3.35}$$

we see that the only three dimensional principal subalgebra allowed is so(2,1), but for the algebra to really contain a real principal so(2,1) subalgebra the Weyl vector squared

$$R^2 = [\rho(C_2) - \frac{h(\mathbf{g})}{\eta}\nu(C_2)]^2 + [\rho(\mathbf{g}) - \frac{h(\mathbf{g})}{\eta}\nu(\mathbf{g})]^2 - \frac{2h(\mathbf{g})[h(\mathbf{g}) + \eta]}{\eta^2} \tag{3.36}$$

must be negative [13, 16].

Let us first consider the case where the central node is linked to the affine node by a single link, i.e., $\eta_i = \delta_{i0}$ for $i \in C_1$. Eq. (3.36) is then simplified to

$$R^2 = \rho(\mathbf{g})^2 - 2h(\mathbf{g})[h(\mathbf{g}) + 1] + [\rho(C_2) - h(\mathbf{g})\nu(C_2)]^2, \tag{3.37}$$

since $\eta = 1$ and $\nu(\mathbf{g}) = 0$. Using the Freudenthal-de Vries strange formula [8] on the simply-laced finite algebra $\mathbf{g}$

$$\rho(\mathbf{g})^2 = \frac{h(\mathbf{g})[h(\mathbf{g})+1]r(\mathbf{g})}{12}, \tag{3.38}$$

we finally arrive at

$$R^2 = \frac{h(\mathbf{g})[h(g)+1][r(\mathbf{g})-24]}{12} + [\rho(C_2) - h(\mathbf{g})\nu(C_2)]^2. \tag{3.39}$$

A minimal condition for the existence of principal so(2,1) subalgebra is thus given by that the rank of $\mathbf{g}$ has to be less than 24, allowing only a finite number of possibilities for $\mathbf{g}$. If C_2 is empty, that is if C is an over-extension of $\mathbf{g}$, then the condition $R^2 < 0$ reduces to $r(\mathbf{g}) < 24$.

Consider now the remaining configurations where the central node is linked by a single link to the node $\star$ in C_1 such that $n_\star \geq 2$, i.e., $\eta_i = \delta_{i\star}$ for $i, \star \in C_1$. Note that all the nodes with $n_i = 1$ are related by the symmetry of the affine Dynkin diagram, hence these nodes are equivalent. The only simply-laced semi-simple finite algebras containing Coxeter labels greater than one are e_6, e_7, e_8 and d_r, so C_1 has to be the affine extension of one of these algebras. Calculations with e_6, e_7 and e_8 show that the sum of the last two terms in Eq. (3.36) is negative, thus the sign of R^2 is depending on the choice of C_2. Similarly, one finds for d_r that the sum of the last two terms in Eq. (3.36) is negative only if $r < 26$. The conclusion is then that whenever C_1 is linked to the central node by a single link, imposing the overall algebra to contain a principal so(2,1) subalgebra, i.e., $R^2 < 0$, the number of possibilities to choose C_1 is finite.

Having found the restrictions on C_1 for the overall algebra to contain a principal so(2,1) subalgebra, we then try to find corresponding upper bounds for the rank of C_2 assuming the central node is linked to the node $\star$ in C_1 via a single link.

First we observe the condition $R^2 < 0$ can be written as

$$[\rho(C_2) - h_0\nu(C_2)]^2 \leq M_0(\mathbf{g}), \tag{3.40}$$

where $h_0 \equiv \frac{h(\mathbf{g})}{\eta}$ and $M_0(\mathbf{g}) \equiv \frac{2h(\mathbf{g})[h(\mathbf{g})+\eta]}{\eta^2} - [\rho(\mathbf{g}) - h_0\nu(\mathbf{g})]^2$ only depend on $\mathbf{g}$. It can be verified explicitly that $h_0 = \frac{1}{n_\star}\sum_{j=0}^{r(\mathbf{g})} n_j \geq 2$ for all the simply-laced Lie algebras $\mathbf{g}$. If C_2 is not connected, we can split the left hand side of Eq. (3.40) into a sum over the connected simple components, where for each component the factor $[\rho(C_2^i) - h_0\nu(C_2^i)]^2$ is strictly positive. It is then clear that the number of simple components in C_2 is finite for a given $\mathbf{g}$. What remains now is to show that the rank of each simple component must be bounded, and we do this for the special cases where C_2 is a_r or d_r. In the rest of this section we use the notations $\rho = \rho(C_2)$ and $\nu = \nu(C_2)$.

The case of $\mathbf{a}_r$

For the Lie algebras a_r, the vector $\nu = \sum_{\beta \in \mathrm{a}_r} \eta_\beta \lambda_\beta$ can be written in an orthogonal basis $\{e_i\}$ as

$$\nu = \sum_{i=1}^{r+1} l_i e_i, \tag{3.41}$$

where the coefficients obey $l_{i+1} - l_i = -\eta_i$. Writing $(\rho - h_0\nu)$ in the same basis we get

$$(\rho - h_0\nu) = \sum_{i=1}^{r+1} \left(\frac{r+2-2i}{2} - h_0 l_i \right) e_i = \sum_{i=1}^{r+1} m_i e_i, \tag{3.42}$$

since $\rho = \sum_{i=1}^{r+1} \frac{r+2-2i}{2} e_i$ for a_r [8]. Observing that $m_{i+1} - m_i = h_0\eta_i - 1$, with $h_0 \geq 2$ and $\eta_i \in \mathbb{N}_0$ we find $|m_{i+2}| \geq |m_i| \geq \frac{1}{2}$ by induction. This can then be used to give

$$(\rho - h_0\nu)^2 = \sum_{i=1}^{r+1} |m_i|^2 \geq \frac{r}{8}, \tag{3.43}$$

putting which into Eq. (3.40) yields an upper bound $\frac{r}{8} \leq M_0(\mathbf{g})$ for the rank of allowed a_r.

The case of $\mathbf{d}_r$

Assume first that ν is not a spinor weight. By analogy with a_r, we use $\nu = \sum_{i=1}^{r} l_i e_i$ and $\rho = \sum_{i=1}^{r} (r-i) e_i$ for d_r [8] to obtain

$$(\rho - h_0\nu) = \sum_{i=1}^{r} (r - i - h_0 l_i) e_i. \tag{3.44}$$

Since all the coefficients of e_i are integers and at most $\frac{r}{h_0} + 1$ of them vanish, we get

$$(\rho - h_0\nu)^2 = \sum_{i=1}^{r} (r - i - h_0 l_i)^2 \geq \sum_{k=1}^{r - (\frac{r}{h_0} + 1)} 1 = r(1 - \frac{1}{h_0}) - 1. \tag{3.45}$$

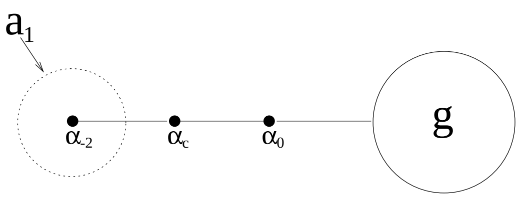

Figure 3.3: *The Dynkin diagram of a very extended Kac-Moody algebra.*

If ν is a spinor weight, each l_i is then half-odd-integer. The calculations will remain the same, but with h_0 replaced by $h_0/2$ when h_0 is even. The rank of C_2 is thus again bounded.

For the case where the central node is linked to the affine node of C_1 only, and is connected to one node of each simple component of C_2 by a single link, one can show that the rank of the overall algebra with principal so(2,1) subalgebra can never exceed 42, regardless the choices of C_1 and C_2 [13].

3.3 Very extended Lie algebras

In this section we will look at the special class of Lorentzian Kac-Moody algebras, obtained by joining an affine Kac-Moody algebra $\mathbf{g}^{(1)}$ via the affine node to a central node that links in turn to the single finite dimensional Lie algebra a_1. The resulting algebra is called very extended, see Fig. 3.3.

We start with a finite dimensional semi-simple Lie algebra $\mathbf{g}$ of rank r, and from now on we do not require that $\mathbf{g}$ has a symmetric Cartan matrix. The simple roots of $\mathbf{g}$ are denoted α_i, $i = 1, \ldots, r$. These are normalised such that $\theta^2 = 2$, where θ denotes the highest root. The lattice spanned by the simple roots is called Λ_g and the corresponding Cartan matrix is defined as in Eq. (3.1). The only exception to our analysis is when $\mathbf{g}$ is a_1, and we do not consider that case. The very extension of $\mathbf{g}$ is obtained by adding three new simple roots.

i. Add the simple root

$$\alpha_0 = k - \theta, \quad (\alpha_0, \alpha_0) = 2, \tag{3.46}$$

where $k \in \Pi_{1;1}$. The corresponding Lie algebra is just the affine extension of $\mathbf{g}$ and we denote it by $\mathbf{g}_0$. The root lattice of $\mathbf{g}_0$ is a part of $\Lambda_g \oplus \Pi_{1;1}$, more precisely those vectors x in this lattice which are orthogonal to k, i.e., $(x, k) = 0$ for all $x \in \Lambda_{g_0}$. The

Cartan matrix is

$$\mathbf{A_{g_0}} = \begin{pmatrix} & \mathbf{A_g} & & \begin{matrix} q_1 \\ \vdots \\ q_r \end{matrix} \\ q'_1 & \dots & q'_r & 2 \end{pmatrix}, \tag{3.47}$$

where we have introduced $q'_i \equiv (\alpha_0, \alpha_i)$ and $q_i \equiv \frac{2}{(\alpha_i,\alpha_i)}(\alpha_i, \alpha_0)$. Since $\mathbf{g}_0$ is an affine Lie algebra, the determinant of its Cartan matrix vanishes, i.e., $\det A_{g_0} = 0$.

ii. Add yet another simple root

$$\alpha_{-1} = -(k + \bar{k}), \quad (\alpha_{-1}, \alpha_{-1}) = 2. \tag{3.48}$$

The resulting algebra is an over-extended Lie algebra denoted by $\mathbf{g}_{-1}$. The root lattice $\Lambda_{g_{-1}}$ is now the whole $\Lambda_g \oplus \Pi_{1;1}$, indicating that the algebra is Lorentzian. The other scalar products involving the new simple root α_{-1} are $(\alpha_{-1}, \alpha_0) = -1$ and $(\alpha_{-1}, \alpha_i) = 0$, using which we find the Cartan matrix

$$\mathbf{A_{g_{-1}}} = \begin{pmatrix} & \mathbf{A_g} & & \begin{matrix} q_1 \\ \vdots \\ q_r \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ q'_1 & \dots & q'_r & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}. \tag{3.49}$$

The determinant of the Cartan matrix is then

$$\det A_{g_{-1}} = 2 \det A_{g_0} - \det A_g = - \det A_g, \tag{3.50}$$

verifying that $\mathbf{g}_{-1}$ is Lorentzian.

iii. The even further enlarged algebra $\mathbf{g}_{-2}$ is obtained by adding the simple root

$$\alpha_{-2} = k - (l + \bar{l}), \quad (\alpha_{-2}, \alpha_{-2}) = 2, \tag{3.51}$$

where l and $\bar{l}$ span a new $\Pi_{1;1}$ lattice. The whole root lattice consists of vectors in $\Lambda_g \oplus \Pi_{1;1} \oplus \Pi_{1;1} = \Lambda_{g_{-1}} \oplus \Pi_{1;1}$ which are orthogonal to the time-like vector $s = l - \bar{l} = (1, 1)$. The Cartan matrix of $\mathbf{g}_{-2}$ has the form

$$\mathbf{A_{g_{-2}}} = \begin{pmatrix} & \mathbf{A_g} & & \begin{matrix} q_1 \\ \vdots \\ q_r \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ q'_1 & \dots & q'_r & 2 & -1 & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.52}$$

due to that $(\alpha_{-2}, \alpha_{-1}) = -1$ and $(\alpha_{-2}, \alpha_0) = (\alpha_{-2}, \alpha_i) = 0$. Clearly this is again a Lorentzian Kac-Moody algebra

$$\det A_{g_{-2}} = 2 \det A_{g_{-1}} - \det A_{g_0} = -2 \det A_g. \tag{3.53}$$

The same value for the determinant is obtained using Eq. (3.30).

The fundamental weights of the over-extended algebra are

$$\begin{aligned} \lambda_i &= \lambda_i^f - (\lambda_i^f, \theta)(k - \bar{k}), \quad i = 1, \ldots, r \\ \lambda_0 &= -(k - \bar{k}) \\ \lambda_{-1} &= -k, \end{aligned} \tag{3.54}$$

with λ_i^f being the fundamental weights of $\mathbf{g}$, whereas the fundamental weights of the very extended algebra are

$$\begin{aligned} \lambda_i &= \lambda_i^f - (\lambda_i^f, \theta)[k - \bar{k} - \tfrac{1}{2}(l + \bar{l})], \quad i = 1, \ldots, r \\ \lambda_0 &= -[k - \bar{k} - \tfrac{1}{2}(l + \bar{l})] \\ \lambda_{-1} &= -k \\ \lambda_{-2} &= -\tfrac{1}{2}(l + \bar{l}). \end{aligned} \tag{3.55}$$

These together with the corresponding simple roots satisfy Eq. (3.2) for the respective algebra. For the over-extended algebra $\mathbf{g}_{-1}$ the Weyl vector is given by

$$\rho = \rho^f + h\bar{k} - (h+1)k, \tag{3.56}$$

where ρ^f is the Weyl vector and $h = 1 + \sum_{i=1}^r n_i$ is the Coxeter number of $\mathbf{g}$, respectively. Similarly,

$$\rho = \rho^f + h\bar{k} - (h+1)k - \frac{1}{2}(1-h)(l + \bar{l}) \tag{3.57}$$

is the Weyl vector of the very extended Kac-Moody algebra.

The weight lattice spanned by the fundamental weights is just the lattice Λ_w dual to the root lattice Λ_R, $\Lambda_w = \Lambda_R^*$. Generally we have

$$(\Lambda_1 \oplus \Lambda_2)^* = \Lambda_1^* \oplus \Lambda_2^* \tag{3.58}$$

for two arbitrary lattices, and this is sufficient for us to find the weight lattices of these extended algebras. The weight lattice of $\mathbf{g}_{-1}$ is

$$\Lambda_{g_{-1}}^* = \Lambda_g^* \oplus \Pi_{1;1}^* = \Lambda_g^* \oplus \Pi_{1;1}, \tag{3.59}$$

the last equality is due to the self-duality of $\Pi_{1;1}$. Inserting this into Eq. (3.13) we find

$$Z_{g_{-1}} = \frac{\Lambda_{g_{-1}}^*}{\Lambda_{g_{-1}}} = \frac{\Lambda_g^* \oplus \Pi_{1;1}}{\Lambda_g \oplus \Pi_{1;1}} = \frac{\Lambda_g^*}{\Lambda_g} = Z_g \tag{3.60}$$

in consistency with the relation between the determinants in (3.50). The weight lattice of $\mathbf{g}_{-2}$, on the other hand, is given by

$$\Lambda^*_{g_{-2}} = \Lambda^*_g \oplus \Pi_{1;1} \oplus \{(r, -r) : 2r \in \mathbb{Z}\}, \tag{3.61}$$

where the convention in Appendix F has been used. Since the root lattice of $\mathbf{g}_{-2}$ is $\Lambda_{g_{-2}} = \Lambda_g \oplus \Pi_{1;1} \oplus \{(t, -t) : t \in \mathbb{Z}\}$, we obtain

$$Z_{g_{-2}} = \frac{\Lambda^*_{g_{-2}}}{\Lambda_{g_{-2}}} = \frac{\Lambda^*_g \oplus \Pi_{1;1} \oplus \{(r, -r) : 2r \in \mathbb{Z}\}}{\Lambda_g \oplus \Pi_{1;1} \oplus \{(t, -t) : t \in \mathbb{Z}\}} = Z_g \times \mathbb{Z}_2. \tag{3.62}$$

Again, the result is consistent with the determinant relation in Eq. (3.53).

3.3.1 Self-dual lattices

Let us study the even self-dual lattices a little more in detail. We start with a finite dimensional semi-simple rank r Lie algebra $\mathbf{g}$, for which the root lattice Λ_g is an even self-dual lattice of dimension r or a sublattice of a such lattice. Even self-dual Euclidean lattices exist only in dimensions $D = 8n$, $n \in \mathbb{N}$.

The root lattice of e_{11}

In eight dimensions, i.e., for $n = 1$, the only lattice fulfilling these criteria is the root lattice of e_8. The set of simple roots for e_8 written in an orthogonal basis is

$$\begin{array}{rcl}
\alpha_1 &=& (0,0,0,0,0,1,-1,0)\\
\alpha_2 &=& (0,0,0,0,1,-1,0,0)\\
\alpha_3 &=& (0,0,0,1,-1,0,0,0)\\
\alpha_4 &=& (0,0,1,-1,0,0,0,0)\\
\alpha_5 &=& (0,1,-1,0,0,0,0,0)\\
\alpha_6 &=& (-1,-1,0,0,0,0,0,0)\\
\alpha_7 &=& (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\\
\alpha_8 &=& (1,-1,0,0,0,0,0,0),
\end{array} \tag{3.63}$$

combining these with the Coxeter numbers as coefficients yields the highest root

$$\theta = (0,0,0,0,0,0,-1,1). \tag{3.64}$$

We denote the corresponding affine, over-extended and very extended algebras by e_9, e_{10} and e_{11}, respectively. The affine root is

$$\alpha_0 = k - \theta = (-\theta; (1,0)), \tag{3.65}$$

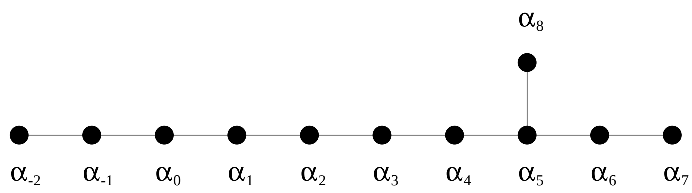

Figure 3.4: *The Dynkin diagram of the Kac-Moody algebra e_{11}.*

whereas the over-extended root is given by

$$\alpha_{-1} = -(k + \bar{k}) = (\mathbf{0}; (-1, 1)). \tag{3.66}$$

These together with the simple roots of e_8 span the root lattice of e_{10}, the resulting lattice $\Lambda_{e_8} \oplus \Pi_{1;1}$ is even, self-dual and Lorentzian. Such lattices occur only in dimensions $D = 8n + 2$, $n \in \mathbb{N}$, and are unique in each dimension, hence we denote them by $\Pi_{8n+1;1}$. It is then obvious that the root lattice of e_{10} is $\Pi_{9;1} = \Lambda_{e_8} \oplus \Pi_{1;1}$. Adding the very extended root

$$\alpha_{-2} = k - (l + \bar{l}) = (\mathbf{0}; (1, 0); (-1, 1)) \tag{3.67}$$

extends the span of the simple roots to being part of the lattice

$$\Pi_{10;2} = \Pi_{9;1} \oplus \Pi_{1;1} = \Lambda_{e_8} \oplus \Pi_{1;1} \oplus \Pi_{1;1}, \tag{3.68}$$

which is the unique even self-dual lattice with signature $(10, 2)$. The Dynkin diagram of the algebra e_{11} can be seen in Fig. 3.4. To find the number of conjugacy classes for the e_{11} weight lattice we use Eq. (3.13)

$$Z_{e_{11}} = \frac{\Lambda^*_{e_{11}}}{\Lambda_{e_{11}}} = \mathbb{Z}_2. \tag{3.69}$$

The e_{11} algebra is of great importance, since it is conjectured to be a symmetry of M-theory.

The root lattice of k_{27}

Another algebra important in string theory is the very extension of the algebra d_{24}. The root lattice of d_{24} is by itself not self-dual, but it is a sublattice of one of the Niemeier lattices [17]. Niemeier lattices are the even self-dual Euclidean lattices in dimension 24, and there are totally 24 of them. The simple roots of d_{24} in an orthogonal basis are

$$\begin{aligned} \alpha_i &= (\underbrace{0, \ldots, 0}_{i-1}, 1, -1, \underbrace{0, \ldots, 0}_{23-i}), \qquad i = 1, \ldots, 23 \\ \alpha_{24} &= (\underbrace{0, \ldots, 0}_{22}, 1, 1). \end{aligned} \tag{3.70}$$

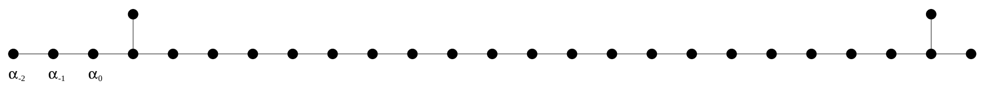

Figure 3.5: *The Dynkin diagram of the Kac-Moody algebra k_{27}.*

As mentioned before, the root lattice of d_{24} is not self-dual and

$$Z_{d_{24}} = \frac{\Lambda^*_{d_{24}}}{\Lambda_{d_{24}}} = \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{3.71}$$

which is consistent with the fact that $\det A_{d_{24}} = 4$. We can construct a 24-dimensional self-dual lattice by adding the point

$$g = \frac{1}{2}(\underbrace{1, \ldots, 1}_{24}) = \lambda_{24} \qquad \in \Lambda^*_{d_{24}} \tag{3.72}$$

to the lattice $\Lambda_{d_{24}}$, the resulting lattice is then $\Lambda^N_{d_{24}} = \Lambda^*_{d_{24}}/\mathbb{Z}_2$. This new point satisfies $(g, g) = 6$ and $2g \in \Lambda_{d_{24}}$.

Following the general procedure we add an affine and an over-extended root

$$\begin{aligned} \alpha_0 &= k - \theta = ((-1, -1, \underbrace{0, \ldots, 0}_{22}); (1, 0)) \\ \alpha_{-1} &= -(k + \bar{k}) = (\mathbf{0}; (-1, 1)) \end{aligned} \tag{3.73}$$

to the collection of simple roots for d_{24}, respectively. The root lattice of the corresponding algebra k_{26} is thus part of the even self-dual lattice $\Pi_{25;1} = \Lambda^N_{d_{24}} \oplus \Pi_{1;1} = \Lambda^*_{k_{26}}/\mathbb{Z}_2$ in 26 dimensions. Adding also the very extended root

$$\alpha_{-2} = k - (l + \bar{l}) = (\mathbf{0}; (1, 0); (-1, 1)), \tag{3.74}$$

we obtain the algebra k_{27} with its root lattice being contained in

$$\Lambda_{d_{24}} \oplus \Pi_{1;1} \oplus \Pi_{1;1} \tag{3.75}$$

and

$$Z_{k_{27}} = \frac{\Lambda^*_{k_{27}}}{\Lambda_{k_{27}}} = \frac{\Lambda^*_{d_{24}}}{\Lambda_{d_{24}}} \times \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3.76}$$

The algebra k_{27}, with a Dynkin diagram as in Fig. 3.5, has been thought to be a symmetry of the 26-dimensional closed bosonic string.

The root lattice of m_{19}

Let us also look at the very extension of the rank 16 algebra d_{16}, the procedure is almost identical to the case of k_{27}. The root lattice of d_{16} satisfy again

$$Z_{d_{16}} = \frac{\Lambda^*_{d_{16}}}{\Lambda_{d_{16}}} = \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{3.77}$$

and it can be made self-dual by adding the point

$$g = \frac{1}{2}(\underbrace{1, \ldots, 1}_{16}), \qquad (g, g) = 4. \tag{3.78}$$

The resulting even self-dual lattice is $\Lambda^s_{d_{16}} = \Lambda^*_{d_{16}}/\mathbb{Z}_2$.

The root lattice of the over-extended algebra m_{18} is a sublattice of the 18-dimensional even self-dual Lorentzian lattice $\Pi_{17;1} = \Lambda^s_{d_{16}} \oplus \Pi_{1;1} = \Lambda^*_{m_{18}}/\mathbb{Z}_2$. Finally, adding the very extended root

$$\alpha_{-2} = (\mathbf{0}; (1, 0); (-1, 1)) \tag{3.79}$$

yields an algebra denoted by m_{19}, whose root lattice is part of

$$\Lambda_{d_{16}} \oplus \Pi_{1;1} \oplus \Pi_{1;1}. \tag{3.80}$$

It is remarkable that construction of the very extended Lorentzian Kac-Moody algebras based on Euclidean self-dual lattices leads to algebras with fundamental importance in physics. The obtained algebras, e_{11} and k_{27}, are conjectured to be symmetries of M-theory and the 26-dimensional bosonic string, respectively. The symmetries of these two theories are thus deeply related to the unique even self-dual Lorentzian lattices in ten and 26 dimensions, respectively. Studying the Dynkin diagrams of e_{11} and k_{27}, Figs. 3.4 and 3.5, we conclude that e_{11} is a subalgebra of k_{27}, indicating that the whole M-theory is possibly described by the bosonic string in 26 dimensions. Also, it seems natural that there should exist an 18-dimensional string with m_{19} as its symmetry.

3.3.2 Principal so(2,1) subalgebra

The condition for the existence of principal so(2,1) subalgebra is

$$\sum_a A^{-1}_{ab} = \sum_a \frac{2}{(\alpha_a, \alpha_a)}(l_a, l_b) \leq 0, \tag{3.81}$$

where a and $b = -2, -1, 0, 1, \ldots, r$. Putting in the fundamental weights for the over-extended Kac-Moody algebras given in Eq. (3.54) we get

$$\begin{aligned}\sum_a A^{-1}_{a,-1} &= -h\\ \sum_a A^{-1}_{a,0} &= -(2h+1)\\ \sum_a A^{-1}_{a,j} &= -\tfrac{1}{2}n_j(\alpha_j,\alpha_j)(2h+1) + \sum_i (A^f)^{-1}_{ij},\end{aligned} \tag{3.82}$$

where n_j, $h = 1 + \sum_{i=1}^{r} n_i$, $(A^f)^{-1}_{ij}$ are the Coxeter labels, the Coxeter number and the inverse Cartan matrix of the finite dimensional Lie algebra $\mathbf{g}$, respectively. The corresponding expressions for the very extended Kac-Moody algebras are

$$\begin{aligned}\sum_a A^{-1}_{a,-2} &= -\tfrac{1}{2}(h-1)\\ \sum_a A^{-1}_{a,-1} &= -h\\ \sum_a A^{-1}_{a,0} &= -\tfrac{3}{2}(h+1)\\ \sum_a A^{-1}_{a,j} &= -\tfrac{3}{4}n_j(\alpha_j,\alpha_j)(h+1) + \sum_i (A^f)^{-1}_{ij}.\end{aligned} \tag{3.83}$$

The only sums not satisfying the condition (3.81) automatically are $\sum_a A^{-1}_{a,j}$, where information about the specific finite dimensional Lie algebra $\mathbf{g}$ is required.

As an illustrative example we take $\mathbf{g}$ to be a_r, which satisfy $(\alpha_j, \alpha_j) = 2$, $n_j = 1$ and $h = r+1$, see reference [8]. Combining the fundamental weights $\lambda_i = \sum_{j=1}^{i} e_j - \frac{i}{r+1}\sum_{j=1}^{r+1} e_j$ gives

$$\begin{aligned}\sum_i (A^f)^{-1}_{ij} &= \sum_i \frac{2}{(\alpha_i,\alpha_i)}(\lambda_i,\lambda_j) = \sum_{i=1}^{j} i\frac{r+1-j}{r+1} + \sum_{i=j+1}^{r} j\frac{r+1-i}{r+1}\\ &= \tfrac{j}{2}(r+1-j),\end{aligned} \tag{3.84}$$

which in turn can be used in Eq. (3.83) to yield

$$\sum_a A^{-1}_{a,j} = -\frac{3}{2}(r+2) + \frac{j}{2}(r+1-j) \tag{3.85}$$

for the very extension of a_r. Maximising the above equation

$$(\sum_a A^{-1}_{a,j})_{\max} = \begin{cases} \frac{1}{8}(r^2-10r-23) & , \quad j = \frac{r+1}{2} \quad \text{when } r \text{ is odd}\\ \frac{1}{8}(r^2-10r-24) & , \quad j = \frac{r}{2} \quad \text{when } r \text{ is even}\end{cases} \tag{3.86}$$

shows that $\sum_a A^{-1}_{a,j}$ is negative only if $r \leq 12$. Hence, the rank of a_r cannot exceed 12 for the corresponding very extension to contain a principal so(2,1) subalgebra.

Performing similar calculations for all the simple finite dimensional Lie algebras, we find the constraints summarised in Table 3.1. Note that

$\mathbf{g}$	Over-extension	Very extension
a_r	$r \leq 16$	$r \leq 12$
d_r	$r \leq 16$	$r \leq 12$
e_r	$r = 6, 7, 8$	$r = 6, 7, 8$
b_r	$r \leq 15$	$r \leq 11$
c_r	$r \leq 8$	$r \leq 6$
f_4	yes	yes
g_2	yes	yes

Table 3.1: *The algebras with principal so(2,1) subalgebras.*

e_{11} contains a principal so(2,1) subalgebra, while neither k_{27} nor m_{19} does since these are the very extensions of the Lie algebras d_{24} and d_{16}, respectively.

4

Symmetries in M-theory

When eleven dimensional supergravity is dimensionally reduced, the obtained scalars have coset space symmetries (Table 2.1). This coset construction for the the scalar sectors has been extended to include also the gauge fields for the maximal supergravity theories [18, 19]. For spacetime dimensions larger than three, the coset space $\mathrm{E}_n/\mathrm{F}_n$ is based on the normal real forms of the finite dimensional semi-simple Lie algebras from the E-series, with the subgroup invariant under the Cartan involution taken as the local subgroup F_n. These subgroups are precisely the maximal compact subgroups of the non-compact groups E_n. There is also some evidence [20, 21] that dimensional reduction of eleven dimensional supergravity to two dimensions and one dimension will yield the affine extension of e_8 (called e_9) and the hyperbolic algebra e_{10}, respectively. Continuing in the same spirit, it has now been conjectured that the eleven dimensional supergravity itself has a coset symmetry based on the very extended Kac-Moody algebra e_{11}. The linearly realised local symmetry is assumed to be the subalgebra invariant under the Cartan involution. The generators in this subalgebra have the form $(E_\alpha - F_\alpha)$ for an arbitrary positive root α, E_α and F_α are the positive and negative root generators, respectively. It then follows that $E_\alpha \sim F_\alpha$, and the coset representatives can all be taken to belong to the Borel subalgebra of e_{11}. Note that whenever we use the word algebra in this chapter, we actually mean the normal real form of the complex algebra, except for the local subalgebras where we mean their compact real forms.

It was shown in Chapter 2 that the field equations of the bosonic sector of eleven dimensional supergravity arise from simultaneous nonlinear realisation under the conformal group and the group G_{11}. In this coset formulation all the bosonic fields of the theory, i.e., the graviton and the rank three gauge field are treated as Goldstone bosons. For the final symmetry to be the Kac-Moody algebra e_{11}, we have to enlarge the algebra g_{11} defined in Eqs. (2.51) and (2.52). There are two ways to go [22]. The first one is to add more generators to g_{11}, which also belong to the local subalgebra. The Cartan form will now transform under a larger local subgroup, see Eq. (2.64). Since the number of the preferred fields remains the same, most of the calculations are unchanged. The only constraint for the local subalgebra is that the field equations obtained must be invariant under the corresponding larger local subgroup. The second method is to perform the nonlinear realisation with a larger algebra than g_{11} and some suitable local subalgebra, the difference now is that we do not demand the field content to remain the same. Instead, we have to show that this gives an alternative description of eleven dimensional supergravity, with the same on-shell degrees of freedom. To find the total symmetry, we let the theory in both cases be simultaneously nonlinearly realised under the conformal group. The hope is that the resulting theory contains a coset symmetry G/H, with G being a group corresponding to a Kac-Moody algebra and H its subgroup invariant under the Cartan involution. It seems also reasonable that the symmetries E_n and F_n found in the dimensional reductions of eleven dimensional supergravity are subgroups of G and H, respectively. Again, the natural candidate for G is E_{11}.

4.1 The Borel subalgebras

The starting point for finding the final global symmetry algebra g is the algebra g_{11} from Chapter 2, whose generators can be decomposed as

$$\begin{aligned} g_{11}^{+} &= \{K^a{}_b\ (a<b), a=1,\ldots,11;\ R^{c_1c_2c_3};\ R^{c_1\ldots c_6}\} \\ g_{11}^{0} &= \left\{H_a \equiv K^a{}_a - K^{a+1}{}_{a+1}, a=1,\ldots,10;\ D=\sum_{a=1}^{11} K^a{}_a\right\} \end{aligned} \tag{4.1}$$

$g_{11}^{-} = K^a{}_b\ (a>b)$ and P_a with $a=1,\ldots,11$. The role the generators P_a play is quite different from the other ones and it should be noted that there are no preferred fields associated with them. Let us thus for now concentrate on the generators $K^a{}_b$, $R^{c_1c_2c_3}$ and $R^{c_1\cdots c_6}$. The basic requirement on the final Kac-Moody algebra g is that it contains all the generators in g_{11}, except P_a which will be treated separately. We demand the set of positive root generators of g to contain g_{11}^{+}. Also, the Cartan subalgebra of g must contain g_{11}^{0}. For notational simplicity we introduce $g_{11}^{0+} = g_{11}^{0} \oplus g_{11}^{+}$. Note that the generators $K^a{}_b$ form a $gl(11) \cong A_{10} \oplus u(1)$ algebra, with H_a, $K^a{}_b$ $(a<b)$ and $K^a{}_b$ $(a>b)$ being the Cartan generators, positive and negative root generators of A_{10}, respectively[1]. Since the final algebra g is assumed to include the finite dimensional exceptional algebras, we analyse this aspect a little more in detail.

4.1.1 The Borel subalgebra of e_7

Consider the simple Lie algebra e_7. Restricting the indices to only take the values $\{5, \ldots, 11\}$, the positive root generators of e_7 can be identified as

$$\hat{K}^i{}_j = K^i{}_j - \frac{1}{7}\delta^i_j \sum_{l=5}^{11} K^l{}_l \quad (i<j), \qquad S_i = \frac{1}{6!}\varepsilon_{ii_1\ldots i_6} R^{i_1\ldots i_6} \tag{4.2}$$

and $R^{i_1i_2i_3}$. Similarly, the Cartan generators of e_7 can be found as

$$H_i = K^i{}_i - K^{i+1}{}_{i+1} \quad \text{and} \quad \hat{D} = \sum_{l=5}^{11} K^l{}_l. \tag{4.3}$$

These generators together with $\hat{K}^i{}_j = K^i{}_j$ $(i>j)$ obey the commutation relations

$$\begin{array}{rcl} [\hat{K}^i{}_j, \hat{K}^k{}_l] &=& \delta^k_j \hat{K}^i{}_l - \delta^i_l \hat{K}^k{}_j \\ [\hat{K}^i{}_j, R^{k_1k_2k_3}] &=& 3\delta^{[k_1}_j R^{|i|k_2k_3]} - \frac{3}{7}\delta^i_j R^{k_1k_2k_3} \\ [\hat{K}^i{}_j, S_k] &=& -\delta^i_k S_j + \frac{1}{7}\delta^i_j S_k \\ [R^{i_1i_2i_3}, R^{i_4i_5i_6}] &=& 2\varepsilon^{i_1\ldots i_6 j} S_j \\ [R^{i_1i_2i_3}, S_k] &=& 0 \\ [\hat{D}, \hat{K}^i{}_j] &=& 0 \\ [\hat{D}, R^{i_1i_2i_3}] &=& 3R^{i_1i_2i_3} \\ [\hat{D}, S_i] &=& 6S_i \\ [S_i, S_j] &=& 0, \end{array} \tag{4.4}$$

which are precisely the commutation relations of the Borel subalgebra of e_7 written with respect to its A_6 subalgebra [22]. Note that the generators $\hat{K}^i{}_j$ $(i>j)$ are the negative root generators of A_6 and thus not included in the Borel subalgebra of e_7. This shows that the Borel subalgebra of e_7 is also a symmetry of the eleven dimensional supergravity, just as the A_6 subalgebra of e_7. The whole e_7 can be found as a symmetry if we enlarge the set of negative roots generators $\{\hat{K}^i{}_j, (i>j)\}$ to also include the generators S^i and $R_{i_1i_2i_3}$. If these generators are also contained in

[1]In this chapter all the classical simple Lie algebras are denoted by capital letters, e.g. A_r.

the local subalgebra, the equations of motion will then remain the same. The whole e_7 is generated by the simple root generators

$$E_i = K^i{}_{i+1}, i = 5, \ldots, 10; \quad E_{11} = R^{9,10,11} \tag{4.5}$$

and the Cartan generators

$$H_i = K^i{}_i - K^{i+1}{}_{i+1}, i = 5, \ldots, 10; \quad H_{11} = K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11} - \frac{1}{3}\hat{D}. \tag{4.6}$$

The new local subalgebra is A_7.

4.1.2 The Borel subalgebra of e_8

Having found the Borel subalgebra of e_7 as a symmetry algebra of eleven dimensional supergravity in the previous subsection, we also suggested how to incorporate the whole e_7 as a symmetry. Perform now a similar analysis for the algebra e_8. Introduce the generators

$$\begin{aligned} H_i = K^i{}_i - K^{i+1}{}_{i+1}; \quad \hat{D} = \sum_{l=4}^{11} K^l{}_l \\ \hat{K}^i{}_j = K^i{}_j - \tfrac{1}{8}\delta^i_j \sum_{l=4}^{11} K^l{}_l \quad (i<j); \quad S_{k_1k_2} = \tfrac{1}{6!}\varepsilon_{k_1k_2i_1\ldots i_6} R^{i_1\ldots i_6} \\ \hat{K}^i{}_j = K^i{}_j \quad (i>j); \quad R^{i_1i_2i_3}, \end{aligned} \tag{4.7}$$

where the indices take the values $\{4, \ldots, 11\}$. Using the commutation relations of g_{11} we find the Lie brackets

$$\begin{aligned} [\hat{K}^i{}_j, \hat{K}^k{}_l] &= \delta^k_j \hat{K}^i{}_l - \delta^i_l \hat{K}^k{}_j \\ [\hat{K}^i{}_j, R^{k_1k_2k_3}] &= 3\delta^{[k_1}_j R^{|i|k_2k_3]} - \tfrac{3}{8}\delta^i_j R^{k_1k_2k_3} \\ [\hat{K}^i{}_j, S_{k_1k_2}] &= -2\delta^i_{[k_1} S_{|j|k_2]} + \tfrac{2}{8}\delta^i_j S_{k_1k_2} \\ [R^{i_1i_2i_3}, R^{i_4i_5i_6}] &= \varepsilon^{i_1\ldots i_6jk} S_{jk} \\ [\hat{D}, \hat{K}^i{}_j] &= 0 \\ [\hat{D}, R^{i_1i_2i_3}] &= 3R^{i_1i_2i_3} \\ [\hat{D}, S_{ij}] &= 6S_{ij} \\ [S_{i_1i_2}, S_{j_1j_2}] &= 0 \\ [S_{i_1i_2}, R^{j_1j_2j_3}] &= 0. \end{aligned} \tag{4.8}$$

Comparing Eq. (4.8) with the Lie brackets of e_8 written with respect to its regular A_7 subalgebra, we can identify $\hat{K}^i{}_j$ with the generators of A_7. $\hat{D}$ spans the eighth dimension in the Cartan subalgebra for e_8, $R^{i_1i_2i_3}$ and

$S_{i_1 i_2}$ are positive root generators [22]. The adjoint representation of e_8 decomposes as $248 = 1 + 63 + (56 + \bar{28} + \bar{8}) + (\bar{56} + 28 + 8)$ with

$$\begin{array}{rcl} 1: & \hat{D} & \text{Cartan subalgebra} \\ 63: & \hat{K}^i{}_j & \text{Regular } \mathrm{A}_7 \text{ subalgebra} \\ & & \text{Cartan, positive and negative root generators} \\ (56+\bar{28}+\bar{8}): & & \text{Positive root generators} \\ & & R^{i_1 i_2 i_3}\ (56) \text{ and } S_{i_i i_2}\ (\bar{28}) \\ (\bar{56}+28+8): & & \text{Negative root generators.} \end{array} \tag{4.9}$$

As can be seen, there is a missing $\bar{8}$ of the Borel subalgebra of e_8. The way to solve this is to assume these missing generators form an ideal, which is trivially realised. The full non-trivial realisation of the Borel subalgebra can be obtained by adding the generators $R^{c_1\ldots c_8,d}$ to the algebra g_{11}. These new generators are antisymmetric in the indices $c_1, \ldots, c_8$. The commutation relations of g_{11} modify to

$$\begin{array}{rcl} [K^a{}_b, P_c] &=& -\delta^a_c P_b; \;\; [P_a, P_b] = 0 \\ [K^a{}_b, K^c{}_d] &=& \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \\ [K^a{}_b, R^{c_1 c_2 c_3}] &=& 3\delta^{[c_1}_b R^{|a|c_2 c_3]} = \delta^{c_1}_b R^{a c_2 c_3} + \delta^{c_2}_b R^{a c_3 c_1} + \delta^{c_3}_b R^{a c_1 c_2} \\ [K^a{}_b, R^{c_1\ldots c_6}] &=& 6\delta^{[c_1}_b R^{|a|c_2\ldots c_6]} \\ [R^{c_1 c_2 c_3}, R^{c_4 c_5 c_6}] &=& 2R^{c_1\ldots c_6} \\ [R^{c_1\ldots c_6}, R^{d_1\ldots d_6}] &=& 0 \\ [R^{c_1\ldots c_6}, R^{d_1 d_2 d_3}] &=& 3R^{c_1\ldots c_6[d_1 d_2, d_3]} \\ [R^{c_1\ldots c_8,a}, R^{d_1 d_2 d_3}] &=& 0; \;\; [R^{c_1\ldots c_8,a}, R^{d_1\ldots d_6}] = 0; \;\; [R^{c_1\ldots c_8,a}, R^{d_1\ldots d_8,b}] = 0 \\ [K^a{}_b, R^{c_1\ldots c_8,d}] &=& 8\delta^{[c_1}_b R^{|a|c_2\ldots c_8],d} + \delta^d_b R^{c_1\ldots c_8,a}, \end{array} \tag{4.10}$$

with $a, b, \ldots = 1, \ldots, 11$. Compared to Eq. (2.52), the commutator $[R^{c_1\ldots c_6}, R^{d_1 d_2 d_3}]$ is now non-vanishing. To satisfy the Jacobi identities, the new generators must in addition obey

$$R^{[c_1\ldots c_8,d]} = 0. \tag{4.11}$$

Restricting the indices to the values $\{4, \ldots, 11\}$ yields the generators in Eq. (4.7) as well as $R^{i_1\ldots i_8,j} = \varepsilon^{i_1\ldots i_8} S^j$. The corresponding Lie brackets are given by the first eight rows in Eq. (4.8) as well as the relations

$$\begin{array}{rcl} [\hat{K}^i{}_j, S^k] = \delta^k_j S^i - \frac{1}{8}\delta^i_j S^k & ; & [S_{i_1 i_2}, R^{j_1 j_2 j_3}] = 3\delta^{[j_1 j_2}_{i_1 i_2} S^{j_3]} \\ [R^{i_1 i_2 i_3}, S^j] = 0 & ; & [S_{i_1 i_2}, S^j] = 0 \\ [\hat{D}, S^i] & = & 9S^i. \end{array} \tag{4.12}$$

Excluding the generators $\hat{K}^i{}_j$ $(i > j)$, these define precisely the Borel subalgebra of e_8.

4.1.3 The Borel subalgebra of e_{11}

Suppose the final symmetry of M-theory is a Kac-Moody algebra g. Since the coset formulation of the bosonic sector of eleven dimensional supergravity uses only the g_{11} algebra and the conformal algebra, one should find the Cartan and simple root generators of the searched for Kac-Moody algebra inside g_{11}. The generators needed to generate the whole g_{11}^+ are

$$E_a = K^a{}_{a+1}, a = 1, \ldots, 10; \;\; E_{11} = R^{9,10,11}, \tag{4.13}$$

where E_a, $a = 1, \ldots, 10$ are also the simple root generators of the finite dimensional algebra A_{10}. Obviously, the generators in (4.13) are natural choices for the simple root generators of the final Kac-Moody algebra. Similarly, the set g_{11}^0 can be identified with the Cartan subalgebra of g, but the basis choice

$$H_a = K^a{}_a - K^{a+1}{}_{a+1}, a = 1, \ldots, 10; \;\; H_{11} = K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11} - \frac{1}{3}D, \tag{4.14}$$

with $D = \sum_{a=1}^{11} K^a{}_a$, makes the embeddings of e_7 and e_8 more explicit. The generators in (4.13) and (4.14) form the Chevalley-Serre basis of g. Applying the Chevalley-Serre relations, we then find the Cartan matrix of the very extended Kac-Moody algebra e_{11}. The Dynkin diagram of e_{11} shows that it indeed contains e_7, e_8 and A_{10} as subalgebras. In fact, e_{11} also contains all the algebras e_n listed in Table 2.1. Using the Lie brackets of g_{11} it is straightforward to verify that the Chevalley-Serre relations (3.9) involving the positive root generators are satisfied. However, we have not yet found any symmetries in eleven dimensional supergravity that correspond to the negative root generators, which must exist and obey the remaining Chevalley-Serre relations for the theory to be invariant under a Kac-Moody algebra. Moreover, it is insufficient with only the g_{11} commutation relations to generate the whole Borel subalgebra of g, because some of the positive root generators form a trivially realised ideal in eleven dimensional supergravity. We have already encounted this problem when trying to identify the Borel subalgebra of e_8. Later we will show that this leads to an alternative description of eleven dimensional supergravity.

The local subalgebra h in eleven dimensions must include all the local subalgebras f_n obtained by dimensional reduction in Table 2.1. In addition, it must contain so(10,1), i.e., the local subalgebra we used in Chapter 2. The proposed subalgebra, which is the one invariant under the Cartan involution, fulfills all the criteria. The real form chosen for h should be non-compact, since it must contain the non-compact so(10,1) as a subalgebra. The non-zero components in the Killing form are [8]

$$\begin{cases} \kappa(H_a, H_b) &= I_{\mathrm{adj}} A^{ab} \\ \kappa(E_\alpha, F_\beta) &\sim \delta_{\alpha,\beta}, \end{cases} \tag{4.15}$$

where A^{ab} is the Cartan matrix of e_{11}.

Taking e_{11} and its subalgebra invariant under the Cartan involution to be the final global symmetry algebra and the local subalgebra, respectively, will yield an infinite number of preferred fields. This is due to the fact that e_{11} is infinite dimensional. The problem may be solved by the inverse Higgs mechanism [23], where the Goldstone fields superfluous for the content of the theory can be eliminated by expressing them in terms of other preferred fields. The remaining Goldstone bosons will then correspond to the generators in (4.10).

The addition of $R^{c_1\dots c_8,b}$ gives some new insight on the relation between eleven dimensional and IIA supergravities. Separating off the eleventh index we define

$$\begin{gathered} \tilde{K}^a{}_b = K^a{}_b;\ \ \tilde{R}^a = K^a{}_{11};\ \ \tilde{R}^{a_1a_2} = R^{a_1a_2 11};\ \ \tilde{R}^{a_1a_2a_3} = R^{a_1a_2a_3} \\ \tilde{R}^{a_1\dots a_5} = R^{a_1\dots a_5 11};\ \ \tilde{R}^{a_1\dots a_6} = -R^{a_1\dots a_6};\ \ \tilde{R}^{a_1\dots a_7} = \tfrac{1}{2} R^{a_1\dots a_7 11,11} \\ \tilde{R}^{a_1\dots a_8} = \tfrac{3}{8} R^{a_1\dots a_8,11};\ \ \tilde{R} = \tfrac{1}{12}\left(-\sum_{a=1}^{10} K^a{}_a + 8K^{11}{}_{11}\right), \end{gathered} \tag{4.16}$$

with the indices $a, \dots = 1, \dots, 10$. The algebra made of these generators is exactly the one, on which nonlinear realisation of IIA supergravity is based [1, 22, 24]. This indicates that the final symmetry algebra of IIA is also e_{11}. Observe that we can make the identification $R^{a_1\dots a_8,11} = -8R^{11a_1\dots a_7,a_8}$ due to Eq. (4.11).

4.2 Levels and representations

To understand the Kac-Moody algebra e_{11} a little better, we do some systematic analysis. Consider the Lorentzian Kac-Moody algebras from Chapter 3, i.e., algebras whose Dynkin diagram C possesses at least one node α_c such that deleting it leads to finite dimensional semi-simple Lie algebras and at most one affine algebra. For simplicity we assume that the remaining Lie algebra is an irreducible finite dimensional algebra with the corresponding reduced Dynkin diagram C_R. The simple roots of C_R are denoted by α_i, $i = 1, \dots, r-1$, while the central node is defined as

$$\alpha_c = -\nu + x, \qquad \nu = -\sum_{i=1}^{r-1} A_{ci}\lambda_i, \tag{4.17}$$

where x is a vector in a space orthogonal to α_i, and λ_i are the fundamental weights of C_R. We will also assume that $(\alpha_c, \alpha_c) = 2$, and hence $x^2 = 2 - \nu^2$. Any positive root can be written as

$$\alpha = l\alpha_c + \sum_{i=1}^{r-1} m_i \alpha_i = lx - \Lambda, \tag{4.18}$$

where

$$\Lambda = -l \sum_{i=1}^{r-1} A_{ci}\lambda_i - \sum_{i,j=1}^{r-1} m_i A^f_{ij}\lambda_j \tag{4.19}$$

and A^f_{jk} is the Cartan matrix for C_R. The integers l and m_i are non-negative. Moreover, l is called the level and counts the number of times the root α_c occurs. For a fixed l, the root space of C can be described by its representation content with respect to C_R. This can be done due to the fact that Λ is a positive linear combination of the fundamental weights of C_R.

An irreducible representation is completely specified by its highest weight. Thus, at a certain level, whether or not a representation of C_R is contained in C depends on the existence of positive integers p_j satisfying

$$p_j \equiv (\Lambda, \alpha_j^\vee) = -lA_{cj} - \sum_{k=1}^{r-1} m_k A^f_{kj} \geq 0, \tag{4.20}$$

with l and m_k being non-negative integers. We can then write the constraint for the allowed C_R representations in C as

$$\sum_{j=1}^{r-1} (A^f)^{-1}_{jk} p_j = -l \sum_{j=1}^{r-1} A_{cj} (A^f)^{-1}_{jk} - m_k, \tag{4.21}$$

where p_j are the components of the highest weight in the Dynkin basis that uniquely defines a representation of C_R. Note that the left hand side and the first term on the right hand side are both non-negative, while the numbers m_k take the values $m_k = 0, 1, 2, \ldots$ [14]. Assuming that we are only dealing with symmetric Cartan matrices, the length of the roots are bounded by $\alpha^2 = 2, 1, 0, \ldots$. Generally we have

$$\alpha^2 = l^2 x^2 + \sum_{i,j=1}^{r-1} p_i p_j (\lambda_i, \lambda_j) = l^2 \frac{\det A_C}{\det A_{C_R}} + \sum_{i,j=1}^{r-1} p_i (A^f)^{-1}_{ij} p_j = 2, 1, 0, \ldots \tag{4.22}$$

where Eq. (3.21) has been used. For fixed l the first term in (4.22) is also fixed. It then automatically follows that in the root space of C, we can never take a positive root to a negative one by acting with raising and lowering generators of C_R. Any representation of C_R must be contained entirely in the positive root space of C, but with a copy also in the negative root space due to the symmetry.

The Eqs. (4.21) and (4.22) are all we need to find potential highest weight vectors of representations of the reduced subalgebra C_R, among the vectors in the positive root lattice of C which have length squared $2, 1, 0, \ldots$. For a certain C_R-representation to actually belong to the root generators of C, the Serre relations and the Jacobi identities must be satisfied. Those vectors leading to generators which do not satisfy these additional requirements belong in fact to the weight lattice of C. Furthermore, the root multiplicity of the Kac-Moody algebras is an unsolved issue, except for a few special cases.

4.2.1 Low level representations of e_{11}

Using the concept of levels, we can systematically study the representation content of e_{11}. More specifically, we can express all the root generators of e_{11} in terms of representations of A_{10}. The Dynkin diagram of e_{11} is given in Fig. 3.4. We consider the node above the horizontal line as the preferred node $\alpha_c = \alpha_{11}$, deleting which yields the Lie algebra A_{10}. The nodes on the horizontal line are denoted by α_i, $i = 1, \ldots, 10$ from left to right. We use D when referring to the dimension of the very extended Kac-Moody algebra, in this case $D = 11$. Let us now level by level go through the positive root lattice.

The level zero contains the adjoint representation of A_{10} and the generator D. The inverse Cartan matrix of A_{D-1} is [8, 13]

$$(A^f)^{-1}_{jk} = \left\{ \begin{array}{ccc} \frac{j(r+1-k)}{r+1} = \frac{j(D-k)}{D} & , & j \leq k \\ \frac{k(r+1-j)}{r+1} = \frac{k(D-j)}{D} & , & j \geq k \end{array} \right. , \tag{4.23}$$

inserting which into Eq. (4.21) with $D = 11$ will give

$$\sum_{j \leq k} j(11-k)p_j + \sum_{j>k} k(11-j)p_j = -11n_k + l \left\{ \begin{array}{ccc} 3k & , & k \leq 8 \\ 8(11-k) & , & k = 9, 10 \end{array} \right. , \tag{4.24}$$

$n_k = 0, 1, 2, \ldots$. Also, Eq. (4.22) becomes

$$\alpha^2 = -\frac{2}{11}l^2 + \frac{1}{11}\sum_{i \leq j} p_i i(11-j)p_j + \frac{1}{11}\sum_{i>j} p_i j(11-i)p_j \leq 2, 1, 0, \ldots, \tag{4.25}$$

where $\det A_{e_n} = 9 - n$ and $\det A_{A_n} = n + 1$ have been used. It is then straightforward to work out the A_{10} representations for every fixed value

Level	Highest weight	A_{10} representation
$l=0$	$(1,0,0,0,0,0,0,0,0,1)$	$K^a{}_b$ $(A_{10} \oplus D)$
$l=1$	$(0,0,0,0,0,0,0,1,0,0)$	$R^{c_1c_2c_3}$
$l=2$	$(0,0,0,0,1,0,0,0,0,0)$	$R^{c_1\ldots c_6}$
$l=3$	$(0,0,1,0,0,0,0,0,0,1)$	$R^{c_1\ldots c_8,d}$ $(R^{[c_1\ldots c_8,d]}=0)$
	$(0,1,0,0,0,0,0,0,0,0)$	$\star$ $R^{c_1\ldots c_9}$
$l=4$	$(0,1,0,0,0,0,0,1,0,0)$	
	$(1,0,0,0,0,0,0,0,0,2)$	
	$(1,0,0,0,0,0,0,0,1,0)$	$\star$
	$(0,0,0,0,0,0,0,0,0,1)$	

Table 4.1: *The A_{10} representations occurring at the lowest levels of e_{11} [25, 26, 27]. The representations marked with $\star$ do not belong to the root space of e_{11}, but they do belong to the weight space.*

of l. At the lowest levels these are given in Table 4.1. Note that the highest weight of each representation is written in the Dynkin basis. All these generators appear in the nonlinear realisation of eleven dimensional supergravity, except $R^{c_1\ldots c_9}$ which are omitted from the e_{11} algebra since they do not fulfill the Jacobi identities and have also vanishing outer multiplicities [25, 27]. To obtain the whole e_{11} algebra up to level three, we must also add the negative root generators $R_{c_1c_2c_3}$, $R_{c_1\ldots c_6}$ and $R_{c_1\ldots c_8,d}$. The commutation relations are those given in (4.10), omitting the first row, together with

$$\begin{aligned}
[K^a{}_b, R_{c_1c_2c_3}] &= -3\delta^a_{[c_1}R_{|b|c_2c_3]} \\
[K^a{}_b, R_{c_1\ldots c_6}] &= -6\delta^a_{[c_1}R_{|b|c_2\ldots c_6]} \\
[K^a{}_b, R_{c_1\ldots c_8,d}] &= -8\delta^a_{[c_1}R_{|b|c_2\ldots c_8],d} - \delta^a_d R_{c_1\ldots c_8,b} \\
[R_{c_1c_2c_3}, R_{c_4c_5c_6}] &= 2R_{c_1\ldots c_6} \\
[R_{a_1\ldots a_6}, R_{b_1b_2b_3}] &= 3R_{a_1\ldots a_6[b_1b_2,b_3]} \\
[R^{a_1a_2a_3}, R_{b_1b_2b_3}] &= 36\delta^{[a_1a_2}_{[b_1b_2}K^{a_3]}{}_{b_3]} - 4\delta^{a_1a_2a_3}_{b_1b_2b_3}D \\
[R_{a_1a_2a_3}, R^{b_1\ldots b_6}] &= -\tfrac{6!}{3!}\delta^{[b_1b_2b_3}_{a_1a_2a_3}R^{b_4b_5b_6]},
\end{aligned} \tag{4.26}$$

where $\delta^{a_1a_2}_{b_1b_2} \equiv \delta^{[a_1}_{b_1}\delta^{a_2]}_{b_2}$. The Cartan and simple root generators are given in (4.14) and (4.13), respectively.

The fundamental weights of e_{11} are

$$l_i = \lambda_i + (\nu, \lambda_i)\frac{x}{x^2}, \quad l_c = \frac{x}{x^2}, \tag{4.27}$$

where λ_i are the fundamental weights of A_{10} and $\nu = -\sum_i A_{ci}\lambda_i = \lambda_8$. The representation $R^{c_1\ldots c_9}$ we omitted corresponds to the vector $(3x-\lambda_2)$,

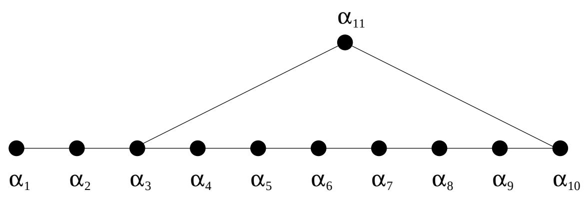

Figure 4.1: *The Dynkin diagram of the Kac-Moody algebra A_8^{+++}.*

which we now recognise as a negative fundamental weight $-l_2$ of e_{11}. Considering $-l_2$ as the lowest weight of an e_{11} representation, we find in the root string the vector

$$-l_2 + \alpha_2 + \ldots + \alpha_8 + \alpha_{11} = 4x - \lambda_1 - \lambda_9, \tag{4.28}$$

this vector appears also at level four in Table 4.1. The A_{10} representation corresponding to this vector must as well be discarded from the root generators of e_{11}, since they are included in a basic representation of e_{11}.

The subalgebra invariant under the Cartan involution is generated by $(E_\alpha - F_\alpha)$ and at low levels it includes the generators

$$\begin{array}{rcl} J_{ab} & = & K^c{}_b\eta_{ac} - K^c{}_a\eta_{bc} \\ S_{a_1a_2a_3} & = & R^{b_1b_2b_3}\eta_{b_1a_1}\eta_{b_2a_2}\eta_{b_3a_3} - R_{a_1a_2a_3} \\ S_{a_1\ldots a_6} & = & R^{b_1\ldots b_6}\eta_{b_1a_1}\ldots\eta_{b_6a_6} + R_{a_1\ldots a_6} \\ S_{a_1\ldots a_8,c} & = & R^{b_1\ldots b_8,d}\eta_{b_1a_1}\ldots\eta_{b_8a_8}\eta_{dc} - R_{a_1\ldots a_8,c}. \end{array} \tag{4.29}$$

These generators belong to the maximal compact subalgebra of the normal real form of e_{11}. The generators J_{ab} constitute a representation for the Lorentz algebra, while $S_{a_1a_2a_3}$ and $S_{a_1\ldots a_6}$ obey

$$\begin{array}{rcl} [S^{a_1a_2a_3}, S_{b_1b_2b_3}] & = & -36\delta^{[a_1a_2}_{[b_1b_2}J^{a_3]}{}_{b_3]} + 2S^{a_1a_2a_3}{}_{b_1b_2b_3} \\ [S_{a_1a_2a_3}, S^{b_1\ldots b_6}] & = & -\frac{6!}{3!}\delta^{[b_1b_2b_3}_{a_1a_2a_3}S^{b_4b_5b_6]} - 3S^{b_1\ldots b_6}{}_{[a_1a_2,a_3]}. \end{array} \tag{4.30}$$

4.2.2 Low level representations of $\mathbf{A}_{D-3}^{+++}$

Let us first consider the very extension of A_8, see Fig. 4.1. We take α_{11} to be the preferred node, deleting which gives the finite dimensional A_{10}. The subalgebra A_{10} is used to decompose the root system of A_8^{+++}. Eq. (4.21) becomes

$$\sum_{j\le k} j(11-k)p_j + \sum_{j>k} k(11-j)p_j = -11m_k + l\left\{\begin{array}{ccc} 9k & , & k=1,2 \\ 33-2k & , & k\ge 3 \end{array}\right. , \tag{4.31}$$

Level	Highest weight	A_{10} representation
$l=0$	$(1,0,0,0,0,0,0,0,0,1)$	$K^a{}_b$ $(\mathrm{A}_{10}\oplus D)$
$l=1$	$(0,0,1,0,0,0,0,0,0,1)$	$R^{c_1\dots c_8,d}$ $(R^{[c_1\dots c_8,d]}=0)$
	$(0,1,0,0,0,0,0,0,0,0)$	$\star\ R^{c_1\dots c_9}$
$l=2$	$(0,0,0,0,1,0,0,0,0,1)$	
	$(0,0,0,0,0,1,0,0,1,0)$	
	$(1,0,1,0,0,0,0,0,0,0)$	
	$(1,0,0,1,0,0,0,0,0,1)$	$\star$

Table 4.2: *The A_{10} representations occurring at the lowest levels of A_8^{+++} [26, 27]. The representations marked with $\star$ belong only to the weight space.*

where $m_k=0,1,2,\dots$. Using also Eq. (4.22) we find the highest weights of A_{10} representations in the root lattice of A_8^{+++}, the lowest levels are given in Table 4.2. Since A_8 is a subalgebra of e_8, the very extended A_8 must as well be a subalgebra of e_{11}. This means that all the A_{10} representations occurring in the adjoint representation of A_8^{+++} have to appear in the adjoint representation of e_{11}. Comparing the levels $3l$ and l in (4.24) and (4.31), respectively, we find these coincide if we at the same time make the identifications

$$n_k=\begin{cases} m_k & ,\quad k=1,2\\ m_k+l(k-3) & ,\quad k=3,\dots,8\\ m_k+l(21-2k) & ,\quad k=9,10.\end{cases}\tag{4.32}$$

It then follows that those A_{10} representations found at level l for A_8^{+++} will also be found at level $3l$ for e_{11}, this is verified by comparing Tables 4.1 and 4.2.

We generalise now the results to A_{D-3}^{+++} of arbitrary dimension D, the Dynkin diagram in Fig. 4.1 can readily be generalised. Omitting the preferred node, we find the finite dimensional Lie algebra A_{D-1}. At level zero we have again the generators $K^a{}_b$ forming an adjoint representation for the algebra $\mathrm{A}_{D-1}\oplus D$. The A_{D-1} representations at all the other levels are obtained by using Eqs. (4.21) and (4.22). In particular, at level one we have

$$\begin{array}{lll} l=1: & p_3=1, p_{D-1}=1 & R^{c_1\dots c_{(D-3)},d}\quad (R^{[c_1\dots c_{(D-3)},d]}=0)\\ & p_2=1 & R^{c_1\dots c_{(D-2)}}.\end{array}\tag{4.33}$$

The solution $p_2=1$ does not belong to the root space of A_{D-3}^{+++}, instead, it is the negative fundamental weight $-l_2=x-\lambda_2$. Observing that the root string of $-l_2$ includes the representation $(p_1=1,p_4=1,p_{D-1}=1)$

at level two, we also must exclude this A_{D-1} representation from the root space. This explains the exclusion of $R^{c_1\ldots c_9}$ from the A_8^{+++} algebra.

The level zero and one generators of A_{D-3}^{+++} obey the Lie brackets

$$\begin{aligned}[][K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \\ [K^a{}_b, R^{c_1\ldots c_{(D-3)},d}] &= (D-3)\delta^{[c_1}_b R^{|a|c_2\ldots c_{(D-3)}],d} + \delta^d_b R^{c_1\ldots c_{(D-3)},a}\end{aligned} \quad (4.34)$$

plus those involving the negative root generators $R_{c_1\ldots c_{(D-3)},d}$.

4.3 Dual formulation of gravity

The preferred fields in the G_{11} nonlinear realisation are $m_a{}^b$, $A_{c_1c_2c_3}$ and $A_{c_1\ldots c_6}$. The gauge fields are related through the equations of motion (2.50), while $m_a{}^b$ describes gravity. Since including the Borel subalgebra of e_8 requires the addition of the generators $R^{c_1\ldots c_8,d}$, there must exist a preferred field of the form $A_{c_1\ldots c_8,d}$. The relation between the gauge fields $A_{c_1c_2c_3}$ and $A_{c_1\ldots c_6}$ indicates the existence of some first order formulation of gravity using the fields $m_a{}^b$ and $A_{c_1\ldots c_8,d}$. Regarding the lower index on $m_a{}^b$ as some internal index, the field strength is found to be $f_{a_1a_2}{}^b$, with $f_{a_1\ldots a_9}{}^b$ being its "dual gauge field". We will now construct this dual formulation of gravity explicitly following the references [22, 28].

4.3.1 The Einstein-Hilbert action

In a D dimensional space-time the Einstein-Hilbert action is given by [11]

$$S = \int d^D x\, e R(\mathsf{e}_\mu{}^a, \omega_{\mu a}{}^b(\mathsf{e})) \quad (4.35)$$

with $e = \sqrt{|\det g_{\mu\nu}|}$. Defining $C_{\mu\nu}{}^a \equiv \partial_\mu \mathsf{e}_\nu{}^a - \partial_\nu \mathsf{e}_\mu{}^a$, the action can be written as

$$S = \int d^D x\, e \left(C_{ca,}{}^a C^{cb,}{}_b - \frac{1}{2} C_{ab,c} C^{ac,b} - \frac{1}{4} C_{ab,c} C^{ab,c} \right), \quad (4.36)$$

where (2.60) has been used. There is an equivalent way to write the action

$$S = \frac{1}{2} \int d^D x\, e \left(Y^{ab,c} C_{ab,c} + \frac{1}{2} Y^{ab,c} Y_{ac,b} - \frac{1}{2(D-2)} \eta_{bc} Y^{ab,c} Y_{af,}{}^f \right) \quad (4.37)$$

by introducing some field $Y^{ab,c}$. Varying Eq. (4.37)

$$\delta S = \frac{1}{2} \int d^D x\, e \left\{ C_{ab,c} + Y_{ac,b} - \frac{1}{D-2} \eta_{bc} Y_{af,}{}^f \right\} \delta Y^{ab,c} = 0 \quad (4.38)$$

yields

$$\begin{array}{rcl} Y_{[a|c|,b]} &=& -C_{ab,c} + \frac{1}{D-2}\eta_{c[b}Y_{a]f,}{}^{f} \qquad \Rightarrow \\ Y_{ab,c} &=& C_{bc,a} + C_{ca,b} - C_{ab,c} + 2(\eta_{bc}C_{af,}{}^{f} - \eta_{ac}C_{bf,}{}^{f}). \end{array} \tag{4.39}$$

Putting (4.39) into (4.37) gives (4.36), and thus shows the equivalence. Introduce the field $Y_{c_1\ldots c_{(D-2)},}{}^{d}$ dual to $Y^{ab,c}$

$$Y^{ab,d} \equiv \frac{1}{(D-2)!}\varepsilon^{abc_1\ldots c_{(D-2)}}Y_{c_1\ldots c_{(D-2)},}{}^{d}. \tag{4.40}$$

Note that the indices denoted by Roman letters are flat. Expressed in terms of the dual field the action becomes

$$\begin{array}{rcl} S &=& \frac{1}{2(D-2)!}\int d^Dx\,\Big\{\varepsilon^{\mu\nu\tau_1\ldots\tau_{(D-2)}}Y_{\tau_1\ldots\tau_{(D-2)},}{}^{d}C_{\mu\nu,d} \\ && +e\Big[-\frac{D-3}{2(D-2)}Y^{c_1\ldots c_{(D-2)},a}Y_{c_1\ldots c_{(D-2)},a} \\ && +\frac{D-2}{2}Y^{d_1\ldots d_{(D-3)}c,}{}_{c}Y_{d_1\ldots d_{(D-3)}d,}{}^{d} - \frac{1}{2}Y^{c_1\ldots c_{(D-3)}a,d}Y_{c_1\ldots c_{(D-3)}d,a}\Big]\Big\}, \end{array} \tag{4.41}$$

observe that $\varepsilon^{a_1\ldots a_D} \equiv e^{-1}\mathsf{e}_{\mu_1}{}^{a_1}\ldots\mathsf{e}_{\mu_D}{}^{a_D}\varepsilon^{\mu_1\ldots\mu_D}$ to emphasise the fact that $\varepsilon^{1\ldots D} = 1$ despite the type of indices.

Changing the indices a and b in Eq. (4.39) to curved space-time indices

$$Y_{\mu\nu,c} = C_{\nu c,\mu} + C_{c\mu,\nu} - C_{\mu\nu,c} + 2(\mathsf{e}_{\nu c}C_{\mu f,}{}^{f} - \mathsf{e}_{\mu c}C_{\nu f,}{}^{f}) \tag{4.42}$$

and using the definition (4.40), we get

$$\begin{array}{rl} \frac{e^{-1}}{(D-2)!} & \varepsilon_{\mu\nu}{}^{\tau_1\ldots\tau_{(D-2)}}Y_{\tau_1\ldots\tau_{(D-2)},c} \\ & = -C_{\mu\nu,c} + C_{c\mu,\nu} - C_{c\nu,\mu} + 2(\mathsf{e}_{\nu c}C_{\mu f,}{}^{f} - \mathsf{e}_{\mu c}C_{\nu f,}{}^{f}). \end{array} \tag{4.43}$$

On the other hand, varying the action (4.41) with respect to the vielbein $\mathsf{e}_{\mu a}$, while regarding $Y_{\tau_1\ldots\tau_{(D-2)},}{}^{a}$ as some independent field gives

$$\begin{array}{rcl} \delta S &=& \frac{1}{2(D-2)!}\int d^Dx\{2\varepsilon^{\mu\nu\tau_1\ldots\tau_{(D-2)}}\partial_\nu Y_{\tau_1\ldots\tau_{(D-2)},}{}^{a} \\ && +[\text{terms of order } (Y_{\tau_1\ldots\tau_{(D-2)},}{}^{a})^2]\}\delta\mathsf{e}_{\mu a} = 0 \end{array} \tag{4.44}$$

or

$$\varepsilon^{\mu\tau_1\ldots\tau_{(D-1)}}\partial_{\tau_1}Y_{\tau_2\ldots\tau_{(D-1)},}{}^{a} = [\text{terms of order } (Y_{\tau_1\ldots\tau_{(D-2)},}{}^{a})^2]. \tag{4.45}$$

To lowest order in $Y_{\tau_1\ldots\tau_{(D-2)},}{}^{a}$ the equations of motion for gravity become

$$\varepsilon^{\mu\tau_1\ldots\tau_{(D-1)}}\partial_{\tau_1}Y_{\tau_2\ldots\tau_{(D-1)},}{}^{a} = 0, \tag{4.46}$$

leading to the conclusion

$$Y_{\tau_1 \ldots \tau_{(D-2)},a} = \partial_{[\tau_1} m_{\tau_2 \ldots \tau_{(D-2)}],a}. \tag{4.47}$$

We have thus shown that there exists a dual formulation of gravity based on the fields $\mathsf{e}_\mu{}^a$ and $m_{\tau_1 \ldots \tau_{(D-3)}}{}^a$. In particular when $D = 11$, the preferred field corresponding to the generators $R^{c_1 \ldots c_8,d}$ does really belong to the field content of eleven dimensional supergravity. One finds that the local Lorentz transformation $\delta \mathsf{e}_{\mu\nu} = \lambda_{\mu\nu}$ is a symmetry only if $\delta m_{\tau_2 \ldots \tau_{(D-2)},a} = -\varepsilon_{\tau_2 \ldots \tau_{(D-2)} a \rho \kappa} \lambda^{\rho\kappa}$ [28].

Based on these considerations, it is conjectured that pure gravity in D dimensions could be described as nonlinear realisation based on the very extended A_{D-3}. This is also consistent with the observation that A_{D-3}^{+++} is a subalgebra of e_{11}, since eleven dimensional supergravity contains gravity. The preferred fields $m_a{}^b$ and $m_{c_1 \ldots c_{(D-3)},d}$ used in the dual formulation of gravity correspond precisely to the generators $K^a{}_b$ and $R^{c_1 \ldots c_{(D-3)},d}$ at levels zero and one for A_{D-3}^{+++}. The major difference is that $R^{[c_1 \ldots c_{(D-3)},d]} = 0$, while $m_{c_1 \ldots c_{(D-3)},d}$ does not have a such constraint.

4.3.2 The field equations

Instead of starting from the Einstein-Hilbert action, we now want to derive the Einstein's equation from field equations [28]. Define a $(D-2)$-form

$$Y_a \equiv \varepsilon_{a b_1 \ldots b_{(D-1)}} \mathsf{e}^{b_1} \wedge \ldots \wedge \mathsf{e}^{b_{(D-3)}} \wedge \omega^{b_{(D-2)} b_{(D-1)}} \tag{4.48}$$

with $\mathsf{e}^a = dx^\mu \mathsf{e}_\mu{}^a$ and $\omega^{ab} = dx^\mu \omega_\mu{}^{ab}$. Written as an antisymmetric tensor we get

$$Y_{\rho_1 \ldots \rho_{(D-2)},a} = (D-2)!\, \varepsilon_{a b_1 \ldots b_{(D-1)}} \mathsf{e}_{[\rho_i}{}^{b_1} \ldots \mathsf{e}_{\rho_{(D-3)}}{}^{b_{(D-3)}} \omega_{\rho_{(D-2)}]}{}^{b_{(D-2)} b_{(D-1)}}. \tag{4.49}$$

Define also a $(D-1)$-form

$$\begin{aligned} M_a \;\equiv\; & -\varepsilon_{a b_1 \ldots b_{(D-1)}} \{ (D-3) \omega^{b_1}{}_f \wedge \mathsf{e}^f \wedge \mathsf{e}^{b_2} \wedge \ldots \wedge \mathsf{e}^{b_{(D-3)}} \wedge \omega^{b_{(D-2)} b_{(D-1)}} \\ & + (-1)^{D-3} \mathsf{e}^{b_1} \wedge \ldots \wedge \mathsf{e}^{b_{(D-3)}} \wedge \omega^{b_{(D-2)} f} \wedge \omega_f{}^{b_{(D-1)}} \} \end{aligned} \tag{4.50}$$

or as a tensor

$$\begin{aligned} M_{\rho_1 \ldots \rho_{(D-1)},a} &= -(D-1)! \varepsilon_{a b_1 \ldots b_{(D-1)}} \times \\ & \{ (D-3) \omega_{[\rho_1}{}^{b_1}{}_{|f|} \mathsf{e}_{\rho_2}{}^f \mathsf{e}_{\rho_3}{}^{b_2} \ldots \mathsf{e}_{\rho_{(D-2)}}{}^{b_{(D-3)}} \omega_{\rho_{(D-1)}]}{}^{b_{(D-2)} b_{(D-1)}} \\ & + (-1)^{D-3} \mathsf{e}_{[\rho_1}{}^{b_1} \ldots \mathsf{e}_{\rho_{(D-3)}}{}^{b_{(D-3)}} \omega_{\rho_{(D-2)}}{}^{b_{(D-2)} f} \omega_{\rho_{(D-1)}]f}{}^{b_{(D-1)}} \}. \end{aligned} \tag{4.51}$$

The equations of motion are then

$$\begin{aligned} dY_a - M_a &= 0 \qquad \text{or} \\ \varepsilon^{\mu\nu\rho_1\ldots\rho_{(D-2)}}\{(D-1)\partial_\nu Y_{\rho_1\ldots\rho_{(D-2)},a} - M_{\nu\rho_1\ldots\rho_{(D-2)},a}\} &= 0. \end{aligned} \tag{4.52}$$

The torsion and the curvature tensor are defined as

$$\begin{aligned} T^a = d\mathsf{e}^a + \omega^a{}_b \wedge \mathsf{e}^b \quad &\Rightarrow \quad T_{\mu\nu}{}^a = 2(\partial_{[\mu}\mathsf{e}_{\nu]}{}^a + \omega_{[\mu}{}^a{}_{|b|}\mathsf{e}_{\nu]}{}^b) \\ R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad &\Rightarrow \quad R_{\mu\nu}{}^a{}_b = 2(\partial_{[\mu}\omega_{\nu]}{}^a{}_b + \omega_{[\mu}{}^a{}_{|c|}\omega_{\nu]}{}^c{}_b), \end{aligned} \tag{4.53}$$

respectively, with $R_{\nu b} = \mathsf{e}^\mu{}_a R_{\mu\nu}{}^a{}_b$ and $R = \mathsf{e}^{\nu b}\mathsf{e}^\mu{}_a R_{\mu\nu}{}^a{}_b$. Using Eqs. (4.49), (4.51) and (4.52), we find

$$\begin{aligned} &\varepsilon_{ab_1\ldots b_{(D-1)}}\varepsilon^{\mu\nu\rho_1\ldots\rho_{(D-2)}}\{\tfrac{D-3}{2}T_{\nu\rho_1}{}^{b_1}\mathsf{e}_{\rho_2}{}^{b_2}\ldots\mathsf{e}_{\rho_{(D-3)}}{}^{b_{(D-3)}}\omega_{\rho_{(D-2)}}{}^{b_{(D-2)}b_{(D-1)}} \\ &\quad +(-1)^{D-3}\mathsf{e}_\nu{}^{b_1}\mathsf{e}_{\rho_1}{}^{b_2}\ldots\mathsf{e}_{\rho_{(D-4)}}{}^{b_{(D-3)}}\omega_{\rho_{(D-3)}}{}^{b_{(D-2)}f}\omega_{\rho_{(D-2)}f}{}^{b_{(D-1)}} \\ &\quad +\mathsf{e}_{\rho_1}{}^{b_1}\ldots\mathsf{e}_{\rho_{(D-3)}}{}^{b_{(D-3)}}\partial_\nu\omega_{\rho_{(D-2)}}{}^{b_{(D-2)}b_{(D-1)}}\} = 0. \end{aligned} \tag{4.54}$$

Noticing the sum of the last two terms as the Einstein tensor, we finally get

$$\begin{aligned} &\varepsilon_{ab_1\ldots b_{(D-1)}}\varepsilon^{\mu\nu\rho_1\ldots\rho_{(D-2)}}\{\tfrac{D-3}{2}T_{\nu\rho_1}{}^{b_1}\mathsf{e}_{\rho_2}{}^{b_2}\ldots\mathsf{e}_{\rho_{(D-3)}}{}^{b_{(D-3)}}\omega_{\rho_{(D-2)}}{}^{b_{(D-2)}b_{(D-1)}}\} \\ &\quad -2(-1)^{D-2}e(D-3)!(R^\mu{}_a - \tfrac{1}{2}\mathsf{e}^\mu{}_a R) = 0. \end{aligned} \tag{4.55}$$

If we use the Levi-Civita connection the torsion will vanish, then the Einstein's equation in vacuum is obtained. Eq. (4.52) is second order in space-time derivatives, to find an equation system with only first order derivatives we introduce the fields

$$\mathsf{e}_\mu{}^a \equiv (e^m)_\mu{}^a, \qquad m_{c_1\ldots c_{(D-3)},d}, \qquad k_{c_1\ldots c_{(D-2)},d}. \tag{4.56}$$

The equation system

$$\begin{cases} Y_a - k_a = dm_a + \Omega_a{}^b \wedge m_b \\ dk_a + \Omega_a{}^b \wedge k_b - \Omega_a{}^b \wedge Y_b - M_a = 0 \end{cases} \tag{4.57}$$

with $\Omega_a{}^b = dx^\mu \Omega_{\mu a}{}^b$, $m_a = dx^{\rho_1} \wedge \ldots \wedge dx^{\rho_{(D-3)}} m_{\rho_1\ldots\rho_{(D-3)},a}$ and $k_a = dx^{\rho_1} \wedge \ldots \wedge dx^{\rho_{(D-2)}} k_{\rho_1\ldots\rho_{(D-2)},a}$, is just what we are looking for, since putting the first line into the second yields

$$dY_a - (d\Omega_a{}^b + \Omega_a{}^c \wedge \Omega_c{}^b) \wedge m_b - M_a = 0 \tag{4.58}$$

or

$$\begin{cases} dY_a - M_a = 0 \\ d\Omega_a{}^b + \Omega_a{}^c \wedge \Omega_c{}^b = 0 \end{cases} . \tag{4.59}$$

Note that the second line in Eq. (4.59) is satisfied automatically if $\Omega_{\mu a}{}^b$ is defined as $\Omega_{\mu a}{}^b = (\mathsf{e}^{-1}\partial_\mu \mathsf{e})_a{}^b$. Eq. (4.57) in terms of tensors is

$$\begin{cases} \varepsilon^{\mu\nu\rho_1\ldots\rho_{(D-2)}}\{Y_{\rho_1\ldots\rho_{(D-2)},a} - k_{\rho_1\ldots\rho_{(D-2)},a} - (D-2)\hat{D}_{\rho_1} m_{\rho_2\ldots\rho_{(D-2)},a}\} = 0 \\ \varepsilon^{\mu\nu\rho_1\ldots\rho_{(D-2)}}\{\hat{D}_\nu k_{\rho_1\ldots\rho_{(D-2)},a} - \Omega_{\nu a}{}^b Y_{\rho_1\ldots\rho_{(D-2)},b} - \frac{1}{D-1} M_{\nu\rho_1\ldots\rho_{(D-2)},a}\} = 0 \end{cases} \tag{4.60}$$

with $\hat{D}_\mu m_{\rho_1\ldots\rho_{(D-3)},a} = \partial_\mu m_{\rho_1\ldots\rho_{(D-3)},a} + \Omega_{\mu a}{}^b m_{\rho_1\ldots\rho_{(D-3)},b}$.

4.3.3 Nonlinear realisation of gravity

The reformulation of gravity contains fields with the same index structures as the low level generators of A_{D-3}^{+++}, written with respect to A_{D-1} representations. We can start from the other end and try to find covariant equations, that are globally invariant under A_{D-3}^{+++} and locally invariant under its subgroup which is invariant under the Cartan involution. We consider only the generators at levels zero and one, namely $K^a{}_b$, $R^{c_1\ldots c_{(D-3)},d}$ and $R_{c_1\ldots c_{(D-3)},d}$. We take

$$g = e^{x^f P_f} e^{m_a{}^b K^a{}_b} e^{\hat{m}_{c_1\ldots c_{(D-3)},d} R^{c_1\ldots c_{(D-3)},d}} \tag{4.61}$$

to be the coset representative, where $\hat{m}_{[c_1\ldots c_{(D-3)},d]} = 0$ is also satisfied. The Cartan forms is then

$$\mathcal{V} \equiv g^{-1}dg - \omega = dx^\mu\{\mathsf{e}_\mu{}^a P_a + \Omega_{\mu a}{}^b K^a{}_b + \tilde{D}_\mu \hat{m}_{c_1\ldots c_{(D-3)},d} R^{c_1\ldots c_{(D-3)},d}\} - \omega \tag{4.62}$$

with

$$\begin{array}{rcl} \mathsf{e}_\mu{}^a & \equiv & (e^m)_\mu{}^a \\ \Omega_{\mu a}{}^b & \equiv & \partial_\mu m_a{}^b = (\mathsf{e}^{-1}\partial_\mu \mathsf{e})_a{}^b \\ \omega & = & dx^\mu \omega_{\mu a}{}^b K^a{}_b \\ \tilde{D}_\mu \hat{m}_{c_1\ldots c_{(D-3)},d} & \equiv & \partial_\mu \hat{m}_{c_1\ldots c_{(D-3)},d} + (D-3)\Omega_{\mu[c_1}{}^a \hat{m}_{|a|c_2\ldots c_{(D-3)}],d} \\ & & +\Omega_{\mu d}{}^a \hat{m}_{c_1\ldots c_{(D-3)},a}. \end{array} \tag{4.63}$$

The Einstein's equation (4.57) written in tangent indices is indeed built up using the fields $\mathsf{e}_\mu{}^a$, $\Omega_{ab}{}^c$, $\omega_{ab}{}^c$ and

$$\begin{array}{l} \tilde{D}_{c_1}\hat{m}_{c_2\ldots c_{(D-2)},d} \\ \quad = \mathsf{e}_{c_1}{}^{\mu_1}\ldots \mathsf{e}_{c_{(D-2)}}{}^{\mu_{(D-2)}}(\partial_{\mu_1}\hat{m}_{\mu_2\ldots\mu_{(D-2)},d} + \Omega_{\mu_1 d}{}^a \hat{m}_{\mu_2\ldots\mu_{(D-2)},a}), \end{array} \tag{4.64}$$

but the appearance of $k_{c_1\ldots c_{(D-2)},d}$ and $m_{[c_1\ldots c_{(D-3)},d]}$ indicate that we must include higher level generators of A^{+++}_{D-3} in the nonlinear realisation. The fact that A^{+++}_{D-3} is infinite dimensional may be solved by the inverse Higgs mechanism, leaving only a finite number of Goldstone fields as result [28]. The final covariant objects are obtained by simultaneous nonlinear realisation with the conformal group. Carrying out this procedure should give (4.57) as the unique equations invariant under both groups up to the considered level.

4.3.4 Some more features

The generalisation to include matter fields is done by introducing a rank p gauge field $A_{(p)} = \frac{1}{p!}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}A_{\mu_1\ldots\mu_p}$ with field strength $F_{(p+1)} = dA_{(p)}$ or

$$F_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}. \tag{4.65}$$

The dual field strength is given by

$$G_{(D-p-1)} = *F_{(p+1)}. \tag{4.66}$$

Introduce also

$$\begin{cases} F_{(p)a} = \frac{1}{p!}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}F_{\mu_1\ldots\mu_p,a} \\ G_{(D-p-2)a} = \frac{1}{(D-p-2)!}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{(D-p-2)}}G_{\mu_1\ldots\mu_{(D-p-2)},a} \end{cases}, \tag{4.67}$$

using which we define

$$\hat{M}_a = M_a + \{G_{(D-p-1)}\wedge F_{(p)a} - (-1)^p G_{(D-p-2)a}\wedge F_{(p+1)}\}. \tag{4.68}$$

Replacing M_a in Eq. (4.52) or (4.57) by $\hat{M}_a$ we obtain

$$R^\mu{}_a - \frac{1}{2}\mathsf{e}^\mu{}_a R = \frac{(D-p)!}{(D-3)!p!}\{F^{\mu\rho_1\ldots\rho_p}F_{a\rho_1\ldots\rho_p} - \frac{1}{2(p+1)}\mathsf{e}^\mu{}_a F^{\rho_1\ldots\rho_{p+1}}F_{\rho_1\ldots\rho_{p+1}}\} \tag{4.69}$$

with the connection chosen so that the torsion vanishes. The right hand side is precisely the desired contribution from the energy momentum tensor. It can be verified explicitly that $d\hat{M}_a = 0$, which is consistent with the fact that the equations of motion $\hat{M}_a = dY_a$ must conserve the energy and momentum.

The equation system with the linearised Einstein's equation involves the unconstrained field $m_{c_1\ldots c_{(D-3)},d}$. However, one can have a dual formulation of gravity which at the linearised level is given by [29, 30, 31]

$$R_{\mu\nu ab} = \varepsilon_{\mu\nu}{}^{\rho_1\ldots\rho_{(D-2)}}(\partial_a\partial_{[\rho_1}l_{\rho_2\ldots\rho_{(D-2)}],b} - \partial_b\partial_{[\rho_1}l_{\rho_2\ldots\rho_{(D-2)}],a}), \tag{4.70}$$

where $l_{[c_1\ldots c_{(D-3)},d]} = 0$. Since the equation is second order in space-time derivatives, it has an additional gauge symmetry related to the local Lorentz transformations discussed below Eq. (4.47). This gauge symmetry can then be used to gauge $l_{[c_1\ldots c_{(D-3)},d]}$ away, and hence from this point of view $m_{[c_1\ldots c_{(D-3)},d]}$ is not really present. Trying to treat gravity in the same way as the gauge fields, we want to have a dual formulation of gravity, expressed by equations with only first order space-time derivatives. To achieve this, we need only to introduce $A_{c_1\ldots c_6}$ for the gauge field $A_{c_1c_2c_3}$, but the non-linearity of the Einstein's equations requires an infinite number of fields, explaining the infinite dimension of the conjectured A_8^{+++} symmetry.

4.4 Nonlinear realisation based on e_{11}

We want now to perform nonlinear realisation based on the generators from the levels zero, one and two of e_{11}. Since the calculations are almost identical to those in Chapter 2, it will be done in less detail here.

We use the subalgebra invariant under the Cartan involution as the local subalgebra, a representative for the coset G/H is given by

$$g = e^{x^f P_f} e^{m_a{}^b K^a{}_b} e^{\frac{1}{3!}A_{a_1a_2a_3}R^{a_1a_2a_3}} e^{\frac{1}{6!}A_{b_1\ldots b_6}R^{b_1\ldots b_6}} e^{\hat{m}_{c_1\ldots c_8,d}R^{c_1\ldots c_8,d}} \tag{4.71}$$

with $\hat{m}_{[c_1\ldots c_8,d]} = 0$. Calculating the Cartan form

$$\begin{aligned}
\mathcal{V} &\equiv g^{-1}dg - \omega \\
&= dx^\mu\{\mathsf{e}_\mu{}^a P_a + \Omega_{\mu a}{}^b K^a{}_b + \tfrac{1}{3!}\tilde{D}_\mu A_{c_1c_2c_3}R^{c_1c_2c_3} \\
&\quad +\tfrac{1}{6!}\tilde{D}_\mu A_{c_1\ldots c_6}R^{c_1\ldots c_6} + \tilde{D}_\mu \hat{m}_{c_1\ldots c_8,d}R^{c_1\ldots c_8,d}\} - \omega
\end{aligned} \tag{4.72}$$

we find

$$\begin{aligned}
\mathsf{e}_\mu{}^a &\equiv (e^m)_\mu{}^a \\
\Omega_{\mu a}{}^b &\equiv (\mathsf{e}^{-1}\partial_\mu \mathsf{e})_a{}^b \\
\tilde{D}_\mu A_{c_1c_2c_3} &\equiv \partial_\mu A_{c_1c_2c_3} + 3(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_{[c_1}{}^b A_{|b|c_2c_3]} \\
\tilde{D}_\mu A_{c_1\ldots c_6} &\equiv \partial_\mu A_{c_1\ldots c_6} + 6(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_{[c_1}{}^b A_{|b|c_2\ldots c_6]} - 20 A_{[c_1c_2c_3}\tilde{D}_{|\mu|}A_{c_4c_5c_6]} \\
\tilde{D}_\mu \hat{m}_{c_1\ldots c_8,d} &\equiv \partial_\mu \hat{m}_{c_1\ldots c_8,d} + 8(\mathsf{e}^{-1}\partial_\mu \mathsf{e})_{[c_1}{}^b \hat{m}_{|b|c_2\ldots c_8],d} - \mathrm{Tr} \\
&\quad -\tfrac{1}{(3!)^3}A_{[c_1c_2c_3}(\tilde{D}_{|\mu|}A_{c_4c_5c_6})A_{c_7c_8]d} - \tfrac{1}{3!6!}A_{[c_1\ldots c_6}\tilde{D}_{|\mu|}A_{c_7c_8]d}
\end{aligned} \tag{4.73}$$

with $(-\mathrm{Tr})$ denoting a term which ensures $\tilde{D}_\mu \hat{m}_{[c_1\ldots c_8,d]} = 0$.

Simultaneous nonlinear realisation with the conformal group yields

$$\begin{aligned}
\tilde{F}_{c_1\ldots c_4} &\equiv 4(\mathsf{e}_{[c_1}{}^\mu \partial_{|\mu|}A_{c_2c_3c_4]} + \mathsf{e}_{[c_1}{}^\mu(\mathsf{e}^{-1}\partial_{|\mu|}\mathsf{e})_{c_2}{}^b A_{|b|c_3c_4]} + \ldots) \\
\tilde{F}_{c_1\ldots c_7} &\equiv 7(\mathsf{e}_{[c_1}{}^\mu \partial_{|\mu|}A_{c_2\ldots c_7]} + \mathsf{e}_{[c_1}{}^\mu(\mathsf{e}^{-1}\partial_{|\mu|}\mathsf{e})_{c_2}{}^b A_{|b|c_3\ldots c_7]} + \ldots \\
&\quad +5\tilde{F}_{[c_1\ldots c_4}A_{c_5c_6c_7]})
\end{aligned} \tag{4.74}$$

as the objects covariant under the both groups. Combining these we find the equations of motion for the gauge fields as

$$\tilde{F}^{c_1\dots c_4} = \frac{1}{7!}\varepsilon_{c_1\dots c_{11}}\tilde{F}^{c_5\dots c_{11}}. \tag{4.75}$$

The equations of motion for gravity should be given by Eq. (4.60), but with $\hat{D}_\mu m_{c_1\dots c_8,d}$ replaced by $\tilde{D}_\mu \hat{m}_{c_1\dots c_8,d}$ from (4.73) and the addition of the field $m_{[c_1\dots c_8,d]}$.

The generator $D \equiv \sum_{a=1}^{11} K^a{}_a$ can be used to measure the level of a generator, since

$$[D, K^a{}_b] = 0 \qquad \text{and} \qquad [D, R^{9,10,11}] = R^{9,10,11} \tag{4.76}$$

with $R^{9,10,11}$ corresponding to the eleventh simple root. Acting on the coset representative with e^{rD} gives

$$\begin{aligned} g \to g' = e^{rD}g \;&=\; e^{e^{-r}x^f P_f} e^{(m_a{}^b + r\delta_a^b)K^a{}_b} e^{\frac{1}{3!}A_{a_1a_2a_3}R^{a_1a_2a_3}} \\ &\quad \times e^{\frac{1}{6!}A_{b_1\dots b_6}R^{b_1\dots b_6}} e^{\hat{m}_{c_1\dots c_8,d}R^{c_1\dots c_8,d}} \end{aligned} \tag{4.77}$$

i.e.,

$$\begin{cases} x^\mu \to x'^\mu = e^{-r}x^\mu \\ m_a{}^b \to m'_a{}^b = m_a{}^b + r\delta_a^b \end{cases}. \tag{4.78}$$

Thus, D corresponds to a symmetry which leaves the equations of motion (4.60) and (4.75) invariant.

4.5 The translation generators

The translation generators in g_{11} carry one lower index and transform under A_{10}, but we did not find any such generators in e_{11}. The role of the translation generators is to introduce space-time into the theory. Due to its transformation properties, P_a is equivalent to a tensor with ten upper antisymmetric indices, and corresponds to the A_{10} representation with λ_1 as the highest weight. Consider thus the possibility that the translation generators are contained in a representation of e_{11}. From Eq. (4.27) we see that the e_{11} representation with $-l_1 = \frac{3}{2}x - \lambda_1$ at level $l = \frac{3}{2}$ as the lowest weight contains the A_{10} representation $\Lambda = \lambda_1$. Calculating the root string to this lowest weight $-l_1$, we find the first weight vector where the eleventh root α_{11} enters to be

$$-l_1 + \alpha_1 + \ldots + \alpha_8 + \alpha_{11} = \frac{5}{2}x - \lambda_9. \tag{4.79}$$

A new representation for A_{10} as a tensor with two upper antisymmetric indices then appears. Continuing in this way, we can find all the A_{10}

Level	Highest weight	A_{10} representation
$l = 3/2$	$(1,0,0,0,0,0,0,0,0,0)$	$P^{a_1\dots a_{10}}$
$l = 5/2$	$(0,0,0,0,0,0,0,0,1,0)$	$W^{a_1a_2}$
$l = 7/2$	$(0,0,0,0,0,1,0,0,0,0)$	$W^{a_1\dots a_5}$
$l = 9/2$	$(0,0,0,1,0,0,0,0,0,1)$	$W^{a_1\dots a_7,b}$
	$(0,0,1,0,0,0,0,0,0,0)$	$W^{a_1\dots a_8}$

Table 4.3: *The A_{10} representations occurred at the lowest levels of the e_{11} representation with $-l_1$ as the lowest weight [32].*

representations contained in the e_{11} representation $-l_1$, at the lowest levels these are given in Table 4.3.

A linear representation $U(A)$ of a Lie algebra g must be a homomorphism, i.e., $U(A_1A_2) = U(A_1)U(A_2)$ for all $A_1, A_2 \in \mathrm{g}$. Let $|\chi_s\rangle$ denote the states spanning a module for g, and define the matrices

$$U(A)|\chi_s\rangle = (c(A))_s{}^t|\chi_t\rangle. \tag{4.80}$$

The matrices $c(A)$ are not the matrix representations of g since $c(A_1A_2) = c(A_2)c(A_1)$, instead, we can use these to define $d(A) \equiv c(A^\ddagger)$ which preserve the algebra multiplication. The operation $A^\ddagger$ is defined as $A^\ddagger \equiv I^CI^I(A)$, where I^C is a Cartan involution, while $I^I(A) = -A$ and inverts the order of the operators at the same time. The matrices $d(A)$ form a representation since

$$d(A_1A_2) = c(I^CI^I(A_1A_2)) = c(I^C(-A_1))c(I^C(-A_2)) = d(A_1)d(A_2). \tag{4.81}$$

We can associate a generator X_s to each state $|\chi_s\rangle$ in the module, obeying the commutation relations

$$[X_s, X_t] = 0 \qquad \text{and} \qquad [X_s, A] = d(A)_s{}^rX_r, \;\; A \in \mathrm{g}. \tag{4.82}$$

Thus, $\{X_s\}$ and g form a semi-direct product algebra [32]. The Jacobi identities can be checked explicitly.

Consider now the semi-direct product of e_{11} and its $-l_1$ representation denoted by $\mathrm{e}_{11}\oplus_s\mathrm{L}_1$. The lowest level generators of L_1 are given in Table 4.3. Instead of the generators $P^{a_1\dots a_{10}}$, we use the translation generators

$$P_a = \frac{1}{10!}\varepsilon_{ab_1\dots b_{10}}P^{b_1\dots b_{10}}. \tag{4.83}$$

Following Eq. (4.82) we have

$$\begin{aligned}
[K^a{}_b, P_c] &= -\delta^a_c P_b + \tfrac{1}{2}\delta^a_b P_c \\
[R^{a_1a_2a_3}, P_b] &= 3\delta^{[a_1}_b W^{a_2a_3]} \\
[R^{a_1a_2a_3}, W^{b_1b_2}] &= W^{a_1a_2a_3b_1b_2} \\
[R^{a_1a_2a_3}, W^{b_1\ldots b_5}] &= W^{b_1\ldots b_5[a_1a_2,a_3]}.
\end{aligned} \tag{4.84}$$

Using the Jacobi identities and Eq. (4.84) we also find

$$\begin{aligned}
[R^{a_1\ldots a_6}, P_b] &= -3\delta^{[a_1}_b W^{a_2\ldots a_6]} \\
[R_{a_1a_2a_3}, P_b] &= 0 \\
[R_{a_1a_2a_3}, W^{b_1b_2}] &= 12\delta^{b_1b_2}_{[a_1a_2} P_{a_3]} \\
[R_{a_1a_2a_3}, W^{b_1\ldots b_5}] &= 120\delta^{[b_1b_2b_3}_{a_1a_2a_3} W^{b_4b_5]}
\end{aligned} \tag{4.85}$$

Adding the vectors from the $-l_1$ representation to the root lattice will extend the root lattice to the full weight lattice of e_{11}.

5

Conclusions

In this thesis work some symmetry aspects of M-theory have been studied. Here the relevant symmetry groups of a theory are the ones used when the theory in question is reformulated as nonlinear realisation. The technique of nonlinear realisations is used to describe the low energy dynamics of a theory, in which a symmetry is spontaneously broken, but at high energies the same symmetry is expected to be linearly realised. In this way we found that the bosonic part of eleven dimensional supergravity was globally invariant under both the conformal group and an enlarged affine group called G_{11}. In the calculations, the Lorentz group was taken as the only local symmetry group. We then argued that M-theory itself should contain a coset symmetry, with the very extended group E_{11} and its Cartan involution invariant subgroup being the global and the local symmetry group, respectively. At least we showed that the Lorentzian group E_{11} satisfies some necessary conditions, e.g., it contains all the symmetry groups occurring in the dimensionally reduced supergravity theories. Also, the generators of G_{11} can be identified with the lowest level generators of E_{11}. The Goldstone bosons introduced in the nonlinear realisation will correspond to the gravity degrees of freedom and the gauge fields. If the infinite dimensional e_{11} is the symmetry algebra, new symmetries that mix gravity and the tensor gauge fields non-trivially would appear at higher levels. During the identification of the Goldstone bosons we had to reformulate the theory of gravity to include a field, which is dual to the vielbein. It was then also conjectured that pure gravity in D dimensions should have a hidden Kac-Moody symmetry, namely the very extended version of A_{D-3} usually denoted as A_{D-3}^{+++}. The way the space-time generators and the conformal algebra relate to e_{11} nonlinear realisation is still not very well understood.

The existence of fermions in the eleven dimensional supergravity implies an osp(1|64) symmetry, which is the only algebra containing both the supersymmetry group and the conformal group. This algebra contains gl(32) as a subalgebra, which rotates the spinor index on the supercharges. Being a brane rotating symmetry [33], sl(32) is the most natural generalisation of the Lorentz algebra when branes are present. Consequently, sl(32) must also be a symmetry of M-theory and somehow be related to e_{11}. Indeed, the commutation relations in (4.30) are part of the commutation relations defining the sl(32) automorphism algebra [32]. The rank two and rank five antisymmetric tensors appeared in the e_{11} representation with the lowest weight $-l_1$ (Table 4.3) can be identified with the central charges of the eleven dimensional supersymmetry algebra. Thus, sl(32) is contained in the Cartan involution invariant subalgebra of e_{11}, not as a subalgebra, but as a truncation and hence to get the entire osp(1|64) the $-l_1$ representation must also be added. Nonlinear realisation of $e_{11} \oplus_s L_1$ should be able to describe point particles and branes on an equal footing.

In the familiar nonlinear realisations used in particle physics, space-time is introduced in an ad hoc way and the reducing matrix does not depend on space-time. As a result, a group transformation does not change the space-time dependence of any solution. The nonlinear realisation considered here involves space-time in a more complicated manner, and thus one can expect that the possible group transformations that one can carry out on any solution are much more extensive. In this thesis we have interpreted the tensors found in the $-l_1$ representation of e_{11} as space-time translations, central charges of the supersymmetry algebra as well as an infinite number of other states. However, there is an alternative approach to the translation generators. The theory is still thought to be a nonlinear realisation of e_{11}, but the fields are now assumed to depend on an auxiliary parameter and all the space-time dependence was assumed to arise in the same way from the equations of motion of the nonlinear realisation [34, 35]. At level seven there is an A_{10} representation which also may be identified with the translation generators, and hence providing another way to introduce space-time into the theory. It was then conjectured that the adjoint representation of all G^{+++} contains a generator, which belongs to the correct A representation to be identified as space-time translations [35]. But it was soon found that this is not true for e.g., D_{n-3}^{+++} for odd n and A_{n-3}^{+++} for even n. Thus this mechanism cannot be applied to IIB theory and the bosonic string D_{24}^{+++} [27]. Furthermore, even if a generator with the correct A representation properties does occur, it is usually not unique. For e_{11} such a generator occurs at levels $(7 + 11s)$ for any non-negative integer s. The question is then which of these is the real space-time translation generator. Moreover, one finds that some of these operators occur with outer multiplicity greater than one, i.e., even at a given level there is often more than one choice. Even though the level seven candidate does have the correct A properties, it has the problem being non-commutative. Our usual picture of space-time translations as commuting quantities must then change drastically. Having these considerations in mind, it is at the present stage more plausible to let the translation generators come from a basic representation of e_{11}.

In Chapter 3 we have studied the simple roots of the Lorentzian Kac-Moody algebras, but for a general Kac-Moody algebra not very much is known. The real roots have single multiplicity, while no closed expression exists for the imaginary roots, only the recursive Peterson and Freudenthal formulas [14] can be applied. Calculating the outer multiplicities for the vectors in Table 4.1, one finds that the vectors we omitted from the root space of e_{11} were the only ones having non-positive multiplicities as they should [25, 26, 27]. The real roots are those related to the simple roots by a Weyl transformation, and the imaginary roots are Weyl conjugate to elements in the fundamental Weyl chamber which have connected support on the Dynkin diagram [14, 26]. It is also known that the Weyl transformations are related to U-duality [26]. The representation theory of a Kac-Moody algebra is somewhat different from the one of a finite dimensional algebra. Consider the level decomposition of a Kac-Moody algebra in Chapter 4, at positive levels highest weight representations are obtained, whereas at negative levels lowest weight representations are obtained. The adjoint representation of a very extended algebra is neither a highest nor a lowest weight representation, instead, it can be obtained by taking tensor products of the highest and lowest weight representations [27].

Given the Dynkin diagram of e_{11}, starting from the very extended node one can embed an A_9 sub-Dynkin diagram in two ways. These are the ten-dimensional type IIA and IIB superstring theories [24, 36]. The A_9 subalgebra describes the gravity sector of the theory. Generally, different ways to embed a gravity subalgebra, i.e., an A_{D-1} subalgebra with D denoting the dimension of the theory obtained, inside the Dynkin diagram correspond to generalised T-dualities [26].

During the calculations we have used the subalgebra invariant under the standard Cartan involution as the local symmetry, but when considering the vielbein defining gravity line sl(n) one finds that the corresponding local subalgebra is so(n) and not so(n-1,1).To get a space-time with Lorentzian signature a Wick rotation is thus needed for the actual real form of the local subalgebra of e_{11} [35, 37]. Instead, one can define a general involution [38]

$$E_a \to \eta_a F_a, \quad F_a \to \eta_a E_a, \quad H_a \to -H_a, \tag{5.1}$$

where $\eta_a = \pm 1$ and a is numbering the simple roots. For any combination of signs η_a, the generators invariant under the corresponding involution will constitute a real form of the local subalgebra. Different real forms of the local subalgebra will give rise to different space-time signatures, and all the possibilities are connected under the Weyl group of e_{11}.

A theory containing e_{11} as symmetry should contain an infinite number of BPS solutions. Each of these solutions must have a field in the theory as its source, and the objects occurring in the $-l_1$ representation are the corresponding charges. The fields of the nonlinear realisation are contained in the group elements of E_{11} when modded out by the action of the local subgroup. Consequently, given any solution we can write down the corresponding E_{11} group element. One finds that the group elements corresponding to the common half BPS solutions of eleven dimensional supergravity have the form [38, 39]

$$g = \exp\left(-\frac{1}{(\beta,\beta)}\beta \cdot H \ln N\right) \exp\{(1-N)E_\beta\}, \tag{5.2}$$

where N is a harmonic function, β is the e_{11} root corresponding to the generator E_β and H are the Cartan generators. This elegant form of the group elements for the usual half BPS branes leads to the idea that E_{11} can be useful when classifying solutions with a given amount of supersymmetry.

Except the eleven dimensional e_{11} there are other very extended Kac-Moody algebras with great importance, e.g., the conjectured gravity symmetry A_{D-3}^{+++} in D dimensions. It turned out that the low level content of the adjoint representation of a general very extended algebra G^{+++} predicted a field content, which under nonlinearly realisation was in agreement with the oxidation theory associated with algebra G [26, 40]. Another interesting example is given by k_{27}, which is thought to be the symmetry of the bosonic string [1, 22]. The local subalgebra is again taken to be the subalgebra invariant under the Cartan involution. Nonlinear realisation based on the lowest level generators yields the right field equations. One interesting property of k_{27} is that its Dynkin diagram contains e_{11}, indicating that the closed bosonic string contains the superstrings in ten dimensions [41] and maybe even M-theory itself. The closed bosonic string on a torus is invariant under the fake monster Lie algebra [42, 43], one can then ask how the fake monster Lie algebra is related to k_{27}.

We have pointed out some of the physical aspects of $\mathrm{e}_{11}\oplus_s\mathrm{L}_1$, but maybe the semi-direct product is just part of some larger algebra. When adding also the $-l_2$ representation to obtain $\mathrm{e}_{11}\oplus_s\mathrm{L}_1\oplus_s\mathrm{L}_2$, the translation generators no longer need to commute. Perhaps the very extended Kac-Moody algebras are embedded inside the so called Borcherds algebras. M-theory is itself just one step on the way and the very extended algebras play an important role in fixing the structure of that theory, similar to the supersymmetry. The final algebraic structure may include Borcherds' fake monster algebra [44], which is the vertex algebra on the unique even self-dual Lorentzian lattice in 26 dimensions.

A

Differential forms

In this appendix we will review the basics of the differential forms following the references [10, 45, 46].

We define a p-form on an n-dimensional manifold M as

$$t \equiv \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \, t_{\mu_1 \mu_2 \ldots \mu_p}, \tag{A.1}$$

where $t_{\mu_1 \mu_2 \ldots \mu_p}$ is a totally antisymmetric tensor. The wedge operator on the coordinate differentials satisfies the properties

$$\begin{aligned} (dx^\lambda \wedge dx^\mu) \wedge dx^\nu &= dx^\lambda \wedge (dx^\mu \wedge dx^\nu) & \text{(A.2)} \\ dx^\mu \wedge dx^\nu &= -dx^\nu \wedge dx^\mu. & \text{(A.3)} \end{aligned}$$

We see that when putting the index μ equal to ν, Eq. (A.3) vanishes identically. This results in that there cannot exit any forms with $p > n$ in a n-dimensional manifold. The antisymmetricity of the wedge operator is made obvious through the identification

$$\int dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \int d^n x \, \varepsilon^{\mu_1 \ldots \mu_n}, \tag{A.4}$$

where $\varepsilon^{\mu_1 \ldots \mu_n}$ is the total antisymmetric tensor density with $\varepsilon^{12 \ldots n} = +1$. The number of independent components of a p-form is $\binom{n}{p}$.

The exterior product, or the wedge product, between a p-form and a q-form is a $(p+q)$-form and it is defined as

$$t \wedge w \equiv \frac{1}{p!q!} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_{p+q}} \, t_{\mu_1 \ldots \mu_p} w_{\mu_{p+1} \ldots \mu_{p+q}}. \tag{A.5}$$

This exterior product is linear

$$(at_1 + bt_2) \wedge w = at_1 \wedge w + bt_2 \wedge w, \tag{A.6}$$

associative

$$t \wedge (w \wedge u) = (t \wedge w) \wedge u \tag{A.7}$$

and satisfy the relation

$$t \wedge w = (-1)^{pq}\, w \wedge t. \tag{A.8}$$

We define an exterior derivative

$$d \equiv dx^{\mu}\partial_{\mu}, \tag{A.9}$$

which acts on a p-form to give a $(p+1)$-form according to

$$\begin{aligned} dt \equiv d \wedge t &= \tfrac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_{p+1}}\, \partial_{\mu_1} t_{\mu_2 \ldots \mu_{p+1}} \\ &= \tfrac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_{p+1}}\, \partial_{[\mu_1} t_{\mu_2 \ldots \mu_{p+1}]}\,. \end{aligned} \tag{A.10}$$

Moreover, the exterior derivative satisfies

$$d(t \wedge w) = dt \wedge w + (-1)^p\, t \wedge dw, \tag{A.11}$$

and the Poincaré lemma

$$d(dt) = \frac{1}{p!} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\, \partial_{[\mu} \partial_{\nu} t_{\mu_1 \ldots \mu_p]} \equiv 0. \tag{A.12}$$

The so called Hodge dual of a p-form is defined as

$$(*t) \equiv \frac{1}{p!(n-p)!} \frac{1}{\sqrt{|g|}} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}\, \varepsilon_{\mu_{p+1} \ldots \mu_n \nu_1 \ldots \nu_p} t^{\nu_1 \ldots \nu_p}, \tag{A.13}$$

where $g = \det g_{\mu\nu}$. In terms of antisymmetric tensors Eq. (A.13) becomes

$$(*t)_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!} \frac{1}{\sqrt{|g|}}\, \varepsilon_{\mu_1 \ldots \mu_{n-p} \nu_1 \ldots \nu_p} t^{\nu_1 \ldots \nu_p}. \tag{A.14}$$

The Levi-Civita epsilon has the property

$$\varepsilon^{\rho_1 \ldots \rho_p \mu_{p+1} \ldots \mu_n} \varepsilon_{\rho_1 \ldots \rho_p \nu_{p+1} \ldots \nu_n} = \pm |g| p! (n-p)!\, \delta^{\mu_{p+1}}_{[\nu_{p+1}} \ldots\, \delta^{\mu_n}_{\nu_n]}, \tag{A.15}$$

where the $-(+)$-sign is chosen whenever the signature is Lorentzian (Riemannian). The Levi-Civita epsilon with lower indices is defined as $\varepsilon_{\mu_1 \ldots \mu_n} \equiv g_{\mu_1 \nu_1} \ldots g_{\mu_n \nu_n} \varepsilon^{\nu_1 \ldots \nu_n}$. Using the definition (A.13) we can also deduce that

$$*\,(*t) = \left\{ \begin{array}{ll} (-1)^{1+p(n-p)}\, t & \text{Lorentzian space} \\ (-1)^{p(n-p)}\, t & \text{Riemannian space} \end{array} \right. . \tag{A.16}$$

Having introduced the Hodge dual we can define the adjoint of the exterior derivative, which takes a p-form to a $(p-1)$-form according to

$$d^\dagger t \equiv \begin{cases} (-1)^{np+n} * d * t & \text{Lorentzian space} \\ (-1)^{np+n+1} * d * t & \text{Riemannian space} \end{cases} . \tag{A.17}$$

The Laplacian is then defined by

$$\Delta \equiv (d + d^\dagger)^2 = d \cdot d^\dagger + d^\dagger \cdot d, \tag{A.18}$$

taking a p-form to another p-form. According to the Hodge decomposition theorem an arbitrary p-form t_p can always be written as

$$t_p = d\alpha_{p-1} + d^\dagger \beta_{p+1} + \gamma_p, \tag{A.19}$$

where α_{p-1} is a $(p-1)$-form, β_{p+1} is a $(p+1)$-form and γ_p is a p-form obeying $\Delta\gamma_p = 0$. The Hodge dual is also used to define a symmetric inner product of two p-forms in a n-dimensional manifold M

$$(t, w) \equiv \int_M t \wedge *w. \tag{A.20}$$

Generally, the integral of some p-form t over a p-dimensional submanifold $\Sigma \subset M$ is

$$\int_\Sigma t = \int_\Sigma d\sigma^{\alpha_1} \wedge \ldots \wedge d\sigma^{\alpha_p}\, t^*_{\alpha_1 \ldots \alpha_p}, \tag{A.21}$$

where

$$t^* = \frac{1}{p!} d\sigma^{\alpha_1} \wedge \ldots \wedge d\sigma^{\alpha_p} \frac{\partial x^{\mu_1}}{\partial \sigma^{\alpha_1}} \cdots \frac{\partial x^{\mu_p}}{\partial \sigma^{\alpha_p}} t_{\mu_1 \ldots \mu_p} \tag{A.22}$$

is the pull back of t on Σ. The coordinates σ^α are used to parametrise the submanifold Σ. After having mapped injectively between the coordinates x and σ, we find the integral to be

$$\int_\Sigma t = \int_\Sigma d\sigma^1 \ldots d\sigma^p \frac{\partial x^{\mu_1}}{\partial \sigma^1} \cdots \frac{\partial x^{\mu_p}}{\partial \sigma^p} t_{\mu_1 \ldots \mu_p}. \tag{A.23}$$

As the end of this short review we state the Stokes theorem

$$\int_\Sigma dt = \int_{\partial\Sigma} t, \tag{A.24}$$

where $\partial\Sigma$ is the boundary of some manifold Σ.

B

The conformal group

A conformal transformation of the coordinates is by definition an invertible coordinate mapping, which leaves the metric tensor invariant up to a scale

$$g'_{\mu\nu}(x') = e^{\Lambda(x)} g_{\mu\nu}(x). \tag{B.1}$$

The space-time coordinates in d dimensions are denoted by

$$x^{\mu}, \qquad \mu = 0, 1, \ldots, d-1 \tag{B.2}$$

where the zeroth index is referring to the time direction. We will throughout this appendix use the signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1)$.

The conformal transformations do not affect the angle between two arbitrary curves crossing each other at some point, explaining the use of the word conformal. This property can easily be seen by considering the angle θ between two vectors u and v

$$\cos\theta = \frac{u \cdot v}{\sqrt{(u \cdot u)(v \cdot v)}}. \tag{B.3}$$

Each scalar product involves a metric, thus under a conformal transformation the scale factor $e^{\Lambda(x)}$ on the numerator and the denominator will cancel out.

To find the conformal generators we examine the infinitesimal coordinate transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$. The metric will then transform as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\mu\nu}(x) - (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) \tag{B.4}$$

to first order in ϵ. For conformal transformations the condition (B.1) has to be satisfied, which results in

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = (1 - e^{\Lambda(x)}) g_{\mu\nu}(x). \tag{B.5}$$

Taking the trace on both sides, i.e., contracting with $g^{\mu\nu}$ we find

$$d(1 - e^{\Lambda(x)}) = 2\partial_\mu \epsilon^\mu \equiv 2\partial \cdot \epsilon \tag{B.6}$$

where d denotes the dimension. Inserting this back into Eq. (B.5) we get

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial \cdot \epsilon, \tag{B.7}$$

to which we apply a derivative ∂^ν and obtain

$$\partial_\mu(\partial \cdot \epsilon) + \partial^\nu \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\mu(\partial \cdot \epsilon). \tag{B.8}$$

To solve the differential equation (B.8) we first act on it with ∂^μ yielding

$$\left(2 - \frac{2}{d}\right) \partial^\mu \partial_\mu(\partial \cdot \epsilon) = 0. \tag{B.9}$$

If we instead act on Eq. (B.8) with ∂_ν, we get

$$\partial^\sigma \partial_\sigma(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \left(2 - \frac{4}{d}\right) \partial_\mu \partial_\nu(\partial \cdot \epsilon) = 0, \tag{B.10}$$

where we also have symmetrised over μ and ν. After making use of Eqs. (B.7) and (B.9) we arrive at

$$\left(2 - \frac{4}{d}\right) \partial_\mu \partial_\nu(\partial \cdot \epsilon) = 0. \tag{B.11}$$

The general solution of this equation is (for $d \neq 2$) [47, 48]

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \tag{B.12}$$

Putting this solution into the original differential equation (B.8), we find

$$\epsilon^\mu = \alpha^\mu + \lambda x^\mu + \omega^\mu{}_\nu x^\nu + b^\mu x^2 - 2x^\mu b \cdot x, \tag{B.13}$$

with $\omega_{\mu\nu}$ being an antisymmetric tensor. From Eq. (B.13) we can read off the finite conformal transformations on the coordinates, these are

i. translations

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu \tag{B.14}$$

ii. Lorentz transformations (spatial rotations and Lorentz boosts)

$$x^\mu \to x'^\mu = \omega^\mu{}_\nu x^\nu \tag{B.15}$$

iii. dilations

$$x^\mu \to x'^\mu = \lambda x^\mu \tag{B.16}$$

iv. special conformal transformations

$$x^\mu \to x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}. \tag{B.17}$$

The specific form of the special conformal transformations is motivated by expanding Eq. (B.17) and keeping the lowest order terms only

$$\frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \approx (x^\mu + b^\mu x^2)(1 - 2b \cdot x) \approx x^\mu + b^\mu x^2 - 2x^\mu b \cdot x, \tag{B.18}$$

which are precisely the last two terms in Eq. (B.13).

We define the generators G_α of the conformal group by

$$x \to x' = x + \epsilon^\alpha G_\alpha x, \tag{B.19}$$

where ϵ^α is an infinitesimal tensor. Comparing with Eq. (B.13) we find the generators of the conformal algebra to be

$$\begin{aligned} P_\mu &= \partial_\mu \\ M_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu \\ D &= x^\nu \partial_\nu \\ K_\mu &= 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, \end{aligned} \tag{B.20}$$

corresponding to translations, Lorentz transformations, dilations and special conformal transformations, respectively. The commutation relations of the conformal algebra will thus be

$$\begin{aligned} [D, P_\mu] &= -P_\mu \\ [D, M_{\mu\nu}] &= 0 \\ [D, K_\mu] &= K_\mu \\ [K_\mu, K_\nu] &= 0 \\ [K_\mu, P_\nu] &= -2(\eta_{\mu\nu} D + M_{\mu\nu}) \\ [K_\rho, M_{\mu\nu}] &= (\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\ [P_\mu, P_\nu] &= 0 \\ [P_\rho, M_{\mu\nu}] &= (\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= (\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}). \end{aligned} \tag{B.21}$$

One can also show that there is an isomorphism between the conformal group in $(d-1, 1)$ Minkowski dimensions and the Lorentz group SO(d,2).

The conformal generators given in (B.20) were above applied directly on the coordinates, and to obtain the action of the conformal generators on a field we have to take the intrinsic degrees of freedom into account. We define the generators $\Sigma_{\mu\nu}$ (Lorentz transformations), $\bar{\Delta}$ (dilations) and κ_μ (special conformal transformations) of the stability subgroup of $x = 0$. These generators have the property that they leave the point $x = 0$ invariant. The commutation relations of the stability subgroup are

$$\begin{aligned} [\bar{\Delta}, \Sigma_{\mu\nu}] &= 0 \\ [\bar{\Delta}, \kappa_\mu] &= \kappa_\mu \\ [\kappa_\mu, \kappa_\nu] &= 0 \\ [\kappa_\rho, \Sigma_{\mu\nu}] &= (\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu) \\ [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] &= (\eta_{\mu\sigma}\Sigma_{\nu\rho} + \eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho}) \end{aligned} \tag{B.22}$$

in analogy with Eq. (B.21). The generators of the stability subgroup will also commute with P_μ, $M_{\mu\nu}$, D, K_μ and the coordinates x^μ.

For an arbitrary field $\phi_a(x)$, the basis is chosen such that

$$P_\mu \phi_a(x) = \frac{\partial}{\partial x^\mu}\phi_a(x). \tag{B.23}$$

Note that the translation generators do not act on the indices of $\phi_a(x)$. The actions of the other generators at $x = 0$ are given by

$$\begin{aligned} M_{\mu\nu}\phi_a(0) &= \Sigma_{\mu\nu}\phi_a(0) \\ D\phi_a(0) &= \bar{\Delta}\phi_a(0) \\ K_\mu\phi_a(0) &= \kappa_\mu\phi_a(0). \end{aligned} \tag{B.24}$$

Using Eqs. (B.23) and (B.24) together with the fact that

$$\phi_a(x) = \exp(x^\mu P_\mu)\phi_a(0) \tag{B.25}$$

we can find the actions of the conformal generators on arbitrary fields

$$\begin{aligned} P_\mu\phi(x) &= \partial_\mu\phi(x) \\ M_{\mu\nu}\phi(x) &= [(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}]\,\phi(x) \\ D\phi(x) &= \left[x^\nu\partial_\nu + \bar{\Delta}\right]\phi(x) \\ K_\mu\phi(x) &= \left[(2x_\mu x_\nu\partial_\nu - x^2\partial_\mu) + 2x^\nu\left(\eta_{\mu\nu}\bar{\Delta} + \Sigma_{\mu\nu}\right) + \kappa_\mu\right]\phi(x) \end{aligned} \tag{B.26}$$

The commutation relations in Eq. (B.21) are still satisfied.

C

General coordinate transformations

An infinitesimal general coordinate transformation is given by

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{C.1}$$

where ϵ^μ is an arbitrary infinitesimal parameter. We can expand ϵ^μ as an infinite series in powers of x, and the general form of the infinite number of generators is

$$T^{(n-1)\mu_1 \ldots \mu_n}{}_\nu = x^{\mu_1} \ldots x^{\mu_n} \frac{\partial}{\partial x^\nu}. \tag{C.2}$$

The index $(n-1)$ indicates the length dimension of the generator, i.e., the number of x^{μ_i} minus the number of derivatives present.

We will show that all the generators in this infinite dimensional algebra can be obtained by commuting the generators of two finite dimensional algebras, namely the conformal algebra and the affine algebra. This observation was first made by the authors of [5] and [6].

The generators of the **conformal** group are P_μ, $M_{\mu\nu}$, D and K_μ. The realisation of these generators using the coordinates can be found in Eq. (B.20), while the commutation relations of the conformal algebra are given in Eq. (B.21). On the other hand, the generators of the **affine** group are P_μ and $R_{\mu\nu}$

$$\begin{aligned} P_\mu &= \partial_\mu = T^{(-1)}{}_\mu \\ R_{\mu\nu} &= x_\mu \partial_\nu = T^{(0)}{}_{\mu\nu}. \end{aligned} \tag{C.3}$$

Note that a metric has been implicitly introduced to lower the indices in

Eq. (C.2). The Lie brackets of the affine algebra will then be

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\rho, R_{\mu\nu}] &= \eta_{\rho\mu} P_\nu \\ [R_{\mu\nu}, R_{\rho\sigma}] &= (\eta_{\nu\rho} R_{\mu\sigma} - \eta_{\mu\sigma} R_{\rho\nu}). \end{aligned} \tag{C.4}$$

Consider now the closure between the conformal and the affine group. Taking the commutator between K_ρ and $R_{\mu\nu}$ gives

$$\begin{aligned} [K_\rho, R_{\mu\nu}] &= [(2x_\rho x^\sigma \partial_\sigma - x^2 \partial_\rho), x_\mu \partial_\nu] \\ &= 2x_\mu x_\nu \partial_\rho - 2x_\mu x^\sigma \eta_{\rho\nu} \partial_\sigma - x^\sigma x_\sigma \eta_{\mu\rho} \partial_\nu, \end{aligned} \tag{C.5}$$

which contains generators quadratic in x^μ, in particular the generator ${T^{(1)}}_{\mu\nu\rho} = x_\mu x_\nu \partial_\rho$. From Eq. (C.5) we see that since ${T^{(1)}}_{\mu\nu\rho}$ is symmetric in the indices μ and ν, it is the symmetric part of the generator $R_{\mu\nu}$ that is important. Generally, we can commute $T^{(n)}$ with K_μ to obtain generators of one order higher. Thus any generator of the form given by Eq. (C.2) can be obtained as a repeated commutator of generators from the conformal and the affine algebra.

The reason an infinite dimensional algebra is generated when taking the closure of two finite dimensional algebras, is the fact that there is an overlap between the conformal group and the affine group. And more important, there must exist generators outside the overlap in both groups, and the commutation relations involving these generators are non-trivial.

$$\underbrace{K_\mu}_{conformal\ algebra} \quad , \quad \overbrace{D \quad , \quad M_{\mu\nu} \quad , \quad P_\mu}^{overlap} \quad , \quad \underbrace{S_{\mu\nu}}_{affine\ algebra} \tag{C.6}$$

The generator $S_{\mu\nu}$ denotes the symmetric and traceless part of $R_{\mu\nu}$, whereas $M_{\mu\nu}$ denotes the antisymmetric part.

The generators in the conformal algebra which are not included in the affine algebra are the special conformal ones. The geometric interpretation of the special conformal transformations is not very obvious, in comparison with the other transformations. It is essentially an inversion followed by a translation and yet another inversion

$$x^\mu \to \frac{x^\mu}{x^2} \to y^\mu = \frac{x^\mu}{x^2} + b^\mu \to \frac{y^\mu}{y^2} = \frac{x^\mu + b^\mu x^2}{1 + 2b^\nu x_\nu + b^2 x^2}. \tag{C.7}$$

We can also consider the special conformal generators directly, by first defining the coordinate inversion operator I

$$I[\phi(x^\mu)] = \phi\left(\frac{x^\mu}{x^2}\right). \tag{C.8}$$

The corresponding sequence of transformations for an arbitrary field will then be

$$\begin{aligned} \phi(x^\nu) &\rightarrow I[\phi(x^\nu)] \rightarrow P_\mu(I[\phi(x^\nu)]) \\ &\rightarrow I[P_\mu(I[\phi(x^\nu)])] = I[P_\mu](I[I[\phi(x^\nu)]]) = I[P_\mu]\phi(x^\nu). \end{aligned} \tag{C.9}$$

The operator $I[P_\mu]$ is then found to be

$$\begin{aligned} I[P_\mu] &= I[\tfrac{\partial}{\partial x^\mu}] = \frac{\partial x^\sigma}{\partial\left(\frac{x^\mu}{x^\nu x_\nu}\right)}\frac{\partial}{\partial x^\sigma} \\ &= (g^\sigma{}_\mu x^\nu x_\nu - 2x^\sigma x_\mu)\frac{\partial}{\partial x^\sigma} \\ &= -K_\mu. \end{aligned} \tag{C.10}$$

The conclusion is that the conformal transformations consist of translations, Lorentz transformations, dilations and inversions. The inversion itself cannot be written as a generator, since the operation inversion is discrete and thus contradicts the infinitesimal nature of the generators for a continuous Lie group. Therefore, inversions appear in a disguised form as special conformal transformations.

The analysis of the closure between the conformal and the affine group can easily be generalised to include supersymmetry [6].

D

Transformations under the conformal group

In this appendix we calculate the transformations of the preferred fields and an arbitrary field of degree l under a coset transformation, having the conformal group as global symmetry and the Lorentz group as local symmetry [9]. The reducing matrix

$$g = e^{-x^a P_a} e^{\phi^b K_b} e^{\sigma D} \tag{D.1}$$

transforms according to

$$g(\zeta) \to g'(\zeta') = \Lambda g(\Lambda^{-1}\zeta) h^{-1}(\Lambda^{-1}\zeta, \Lambda) \tag{D.2}$$

under the coset, Λ is a global conformal transformation and h is a local Lorentz transformation. The coordinate vector in (11,2) dimensions is given by ζ_A. Conformal transformations in (10,1) dimensions can be written as Lorentz transformations in 13 dimensions, and we assume that the local 11-dimensional Lorentz subgroup only acts on the 11-dimensional subspace spanned by the indices μ. We can thus form a true 13-vector by

$$\Phi_A = \frac{1}{2}\left({g_A}^{12} - {g_A}^{13}\right), \tag{D.3}$$

which transforms as

$$\Phi_A(\zeta) \to \Phi'_A(\zeta) = {\Lambda_A}^B \Phi_B(\Lambda^{-1}\zeta). \tag{D.4}$$

Inverting Eq. (2.92) gives

$$\begin{aligned} \phi_\mu e^{-\sigma} &= \Phi_\mu - x_\mu(\Phi_5 + \Phi_6) \\ \phi^2 e^{-\sigma} &= (\Phi_5 + \Phi_6). \end{aligned} \tag{D.5}$$

Knowing the transformation laws for $\Phi_A(\zeta)$ under different global conformal transformations, we can deduce the transformations of the preferred fields using Eq. (D.5)

i. inhomogeneous Lorentz transformations

$$\begin{array}{rcccl} x_\mu & \to & x'_\mu & = & \Lambda_\mu{}^\nu x_\nu + \alpha_\mu \\ \sigma(x) & \to & \sigma'(x') & = & \sigma(x) \\ \phi_\mu(x) & \to & \phi'_\mu(x') & = & \Lambda_\mu{}^\nu \phi_\nu(x) \end{array} \tag{D.6}$$

ii. dilations

$$\begin{array}{rcccl} x_\mu & \to & x'_\mu & = & e^{-\lambda} x_\mu \\ \sigma(x) & \to & \sigma'(x') & = & \sigma(x) + \lambda \\ \phi_\mu(x) & \to & \phi'_\mu(x') & = & e^{\lambda} \phi_\mu(x) \end{array} \tag{D.7}$$

iii. special conformal transformations

$$\begin{array}{rcccl} x_\mu & \to & x'_\mu & = & \frac{x_\mu + b_\mu x^2}{1+2b\cdot x + b^2 x^2} \\ \sigma(x) & \to & \sigma'(x') & = & \sigma(x) + \ln|1+2b\cdot x + b^2x^2| \\ \phi_\mu(x) & \to & \phi'_\mu(x') & = & (1+2b\cdot x + b^2 x^2)\phi_\mu(x) \\ & & & & +[(1+2x\cdot\phi)(1+\beta\cdot x) - 2x^2\beta\cdot\phi]\beta_\mu \\ & & & & -[2\beta\cdot\phi + \beta^2(1+2x\cdot\phi)]x_\mu . \end{array} \tag{D.8}$$

The local Lorentz group elements in 11 dimensions are obtained as

$$h(\zeta,\Lambda) = g'^{-1}\Lambda g = e^{-\sigma' D} e^{-\phi'\cdot K} e^{x'\cdot P} \Lambda e^{-x\cdot P} e^{\phi\cdot K} e^{\sigma D}, \tag{D.9}$$

into which we substitute Eqs. (D.6), (D.7) and (D.8) to get

i. inhomogeneous Lorentz transformations

$$\begin{array}{rcl} x'_\mu & = & \Lambda_\mu{}^\nu x_\nu + \alpha_\mu \\ h_{\mu\nu}(\zeta,\Lambda) & = & \Lambda_{\mu\nu} \end{array} \tag{D.10}$$

ii. dilations

$$\begin{array}{rcl} x'_\mu & = & e^{-\lambda} x_\mu \\ h_{\mu\nu}(\zeta,\Lambda) & = & g_{\mu\nu} \end{array} \tag{D.11}$$

iii. special conformal transformations

$$\begin{array}{rcl} x'_\mu & = & x_\mu + (x^2\delta^\nu_\mu - 2x_\mu x^\nu)\beta_\nu + \dots \\ h_{\mu\nu}(\zeta,\Lambda) & = & g_{\mu\nu} + 2(\beta_\mu x_\nu - \beta_\nu x_\mu) + \dots , \end{array} \tag{D.12}$$

for different types of Λ. Note that for pure translations, i.e., $x'_\mu = x_\mu + \alpha_\mu$, the Lorentz group element will be $h_{\mu\nu} = \eta_{\mu\nu}$.

We can also compute the infinitesimal variations in an arbitrary field ψ, induced by the conformal transformations. Assume $\psi(\zeta)$ is homogeneous of degree l, it will then have the transformation

$$\begin{aligned} \psi(\zeta) \to \psi'(\zeta') &= D(h)\psi(\Lambda^{-1}\zeta) \approx (1 + D(\delta h))\psi(\zeta) \\ &= \left(1 + \tfrac{1}{2}(\delta h)_\mu{}^\nu S^\mu{}_\nu\right)\psi(\zeta). \end{aligned} \tag{D.13}$$

We define a function depending on the coordinates in 11 dimensions as

$$\psi(x) = \kappa^{-l}\psi(\zeta), \tag{D.14}$$

where $l \equiv \kappa\partial/\partial\kappa$ is the degree of homogeneity and $\kappa = \zeta_{12} + \zeta_{13}$. The infinitesimal transformation of a such field is

$$\begin{aligned} \psi(x) \to \psi'(x') &= \kappa'^{-l}\psi'(\zeta') \approx \left(\tfrac{\kappa}{\kappa'}\right)^l \left(1 + \tfrac{1}{2}(\delta h)_\mu{}^\nu S^\mu{}_\nu\right)\psi(x) \\ &\approx \left(1 - l\tfrac{\delta\kappa}{\kappa} + \tfrac{1}{2}(\delta h)_\mu{}^\nu S^\mu{}_\nu\right)\psi(x) \\ &\approx \psi'(x) + \delta x^\mu \partial_\mu \psi(x), \end{aligned} \tag{D.15}$$

resulting in the infinitesimal variation

$$\delta\psi(x) = \psi'(x) - \psi(x) \approx -\left(\delta x^\mu \partial_\mu + l\frac{\delta\kappa}{\kappa} - \frac{1}{2}(\delta h)_\mu{}^\nu S^\mu{}_\nu\right)\psi(x). \tag{D.16}$$

$S^\mu{}_\nu$ are the field specified representations of the generators in the local Lorentz group. The quantity $\delta\kappa/\kappa$ is given by

$$\frac{\delta\kappa}{\kappa} = \begin{cases} 0 & \text{inhomogeneous Lorentz transformations} \\ \lambda & \text{dilations} \\ 2\beta\cdot x & \text{special conformal transformations} \end{cases}. \tag{D.17}$$

Inserting Eqs. (D.10), (D.11), (D.12) and (D.17) into Eq. (D.16) yields

i. inhomogeneous Lorentz transformations

$$\begin{aligned} \delta x_\mu &= \epsilon_\mu{}^\nu x_\nu + \alpha_\mu \\ \delta h_{\mu\nu}(\zeta,\Lambda) &= \epsilon_{\mu\nu}, \qquad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \\ \delta\sigma &= -(\epsilon^{\mu\nu}x_\nu + \alpha^\mu)\partial_\mu\sigma \\ \delta\phi_\mu &= -(\epsilon^{\nu\tau}x_\tau + \alpha^\nu)\partial_\nu\phi_\mu + \epsilon_\mu{}^\nu\phi_\nu \\ \delta\psi &= -(\epsilon^{\mu\nu}x_\nu + \alpha^\mu)\partial_\mu\psi + \tfrac{1}{2}\epsilon_\mu{}^\nu S^\mu{}_\nu\psi \end{aligned} \tag{D.18}$$

ii. dilations

$$\begin{aligned} \delta x_\mu &= -\lambda x_\mu \\ \delta h_{\mu\nu}(\zeta,\Lambda) &= 0 \\ \delta\sigma &= \lambda(x^\mu\partial_\mu\sigma + 1) \\ \delta\phi_\mu &= \lambda(x^\nu\partial_\nu + 1)\phi_\mu \\ \delta\psi &= \lambda(x^\nu\partial_\nu - l)\psi \end{aligned} \tag{D.19}$$

iii. special conformal transformations

$$
\begin{array}{rcl}
\delta x_\mu &=& (x^2\delta^\nu_\mu - 2x_\mu x^\nu)\beta_\nu \\
\delta h_{\mu\nu}(\zeta,\Lambda) &=& 2(\beta_\mu x_\nu - \beta_\nu x_\mu) \\
\delta\sigma &=& -\beta^\nu(x^2\delta^\mu_\nu - 2x_\nu x^\mu)\partial_\mu\sigma + 2\beta^\mu x_\mu \\
\delta\phi_\mu &=& -\beta_\tau(x^2 g^{\tau\nu} - 2x^\tau x^\nu)\partial_\nu\phi_\mu + \beta_\mu \\
&& +2\beta^\nu x_\nu\phi_\mu - 2\beta^\nu\phi_\nu x_\mu + 2x^\nu\phi_\nu\beta_\mu \\
\delta\psi &=& -\beta_\nu(x^2 g^{\nu\mu} - 2x^\nu x^\mu)\partial_\mu\psi - 2l\beta^\mu x_\mu\psi + 2\beta_\mu x^\nu S^\mu{}_\nu\psi,
\end{array}
\tag{D.20}
$$

where we also have expanded Eqs. (D.6), (D.7) and (D.8). The non-vanishing of the vacuum expectation values at the origin, $\delta\sigma(0) = \lambda$ and $\delta\phi_\mu(0) = \beta_\mu$, implies the asymmetry of the vacuum.

E

Real principal subalgebras

A rank r Lie algebra can always be represented in the form

$$\mathbf{g} = N_- \oplus H \oplus N_+, \tag{E.1}$$

where H is the Cartan subalgebra, N_- and N_+ are the triangular subalgebras consisting of the multiple commutators of the generators F_i and E_i, respectively. We will now find the conditions for a Lie algebra to contain a rank three real principal subalgebra [13, 16].

Principal so(3) subalgebras

Every finite dimensional semi-simple Lie algebra $\mathbf{g}$ has a principal so(3) subalgebra, due to the positive definiteness of the inverse Cartan matrix. The generators of the principal so(3) subalgebra are constructed using the Weyl vector $\rho = \sum_{i=1}^{r} \frac{2}{(\alpha_i,\alpha_i)}\lambda_i$, with α_i and λ_i being the simple roots and the fundamental weights of $\mathbf{g}$, respectively. The Cartan generator of the principal so(3) is defined by

$$J_3 = \sum_{i=1}^{r} \rho^{(i)} H_{\alpha_i}^{Cartan} \quad \Rightarrow \quad [J_3, E_\alpha] = (\rho,\alpha)E_\alpha, \tag{E.2}$$

where $H_{\alpha_i}^{Cartan}$ and E_α are the Cartan generators and the generator associated with the root α, respectively, given in the Cartan-Weyl basis. Also, the combinations of the simple root generators

$$J^+ = \sum_{i=1}^{r} k_i E_{\alpha_i} \quad , \quad J^- = \sum_{i=1}^{r} k_i E_{-\alpha_i} \tag{E.3}$$

satisfy

$$[J_3, J^\pm] = \pm J^\pm \quad , \quad [J^+, J^-] = J_3 \tag{E.4}$$

with $k_i^2 = p^{(i)}$. The Lie algebra can then be decomposed with respect to the principal so(3) subalgebra into r irreducible representations of spin s_j

$$\mathbf{g} = \bigoplus_{j=1}^{r} \mathbf{g}^{(s_j)}, \tag{E.5}$$

where $\mathbf{g}^{(s_j)}$ carries the $(2s_j+1)$-dimensional irreducible representation of so(3), and the r spins s_j are called the exponents of $\mathbf{g}$. Generally $\mathbf{g}^{(0)}$ is empty, whereas $\mathbf{g}^{(1)}$ is the adjoint representation of the principal so(3) subalgebra. The exponents s_j contain important informations about the Lie algebra $\mathbf{g}$. For instance, the orders of the invariant Casimir operators are given by the numbers $s_j + 1$, and the quadratic Casimir invariant is thus always associated to the representation $s_1 = 1$. Using the Chevalley-Serre basis $E_i = E_{\alpha_i}$, $F_i = E_{-\alpha_i}$ and $H_i = H_{\alpha_i}^{Cartan}$ we can rewrite the generators J_3 and $J^{\pm}$ according to

$$J_3 = \sum_{i=1}^{r} p_i H_i \quad , \quad J^+ = \sum_{i=1}^{r} k_i E_i \quad , \quad J^- = \sum_{i=1}^{r} k_i F_i, \tag{E.6}$$

with

$$p_i \equiv \rho^{(i)} = \sum_{i=1}^{r} A_{ij}^{-1} > 0 \quad \text{and} \quad k_i \equiv \sqrt{p_i}. \tag{E.7}$$

The relations $p_i > 0$ are due to that $A_{ij}^{-1} \geq 0$ for a finite dimensional Lie algebra.

Principal so(2,1) subalgebras

The concept of principal subalgebras can be generalised to a rank r Kac-Moody algebra $\tilde{\mathbf{g}}$. We can again define a generator $J_3 = \sum_{i=1}^{r} \rho^{(i)} H_{\alpha_i}^{Cartan}$, where $H_{\alpha_i}^{Cartan}$ are the Cartan generators in the Cartan-Weyl basis, and $\rho = \sum_{i=1}^{r} \frac{2}{(\alpha_i,\alpha_i)} \lambda_i$ is the Weyl vector. Expressing the fundamental weights as

$$\lambda_i = \sum_{j=1}^{r} (\alpha_i, \alpha_i) A_{ij}^{-1} \frac{\alpha_j}{(\alpha_j, \alpha_j)}, \tag{E.8}$$

allows us to rewrite the Weyl vector as

$$\rho = \sum_{i,j=1}^{r} A_{ij}^{-1} \frac{2\alpha_j}{(\alpha_j, \alpha_j)}. \tag{E.9}$$

The generator J_3 then becomes

$$J_3 = \sum_{i,j=1}^{r} A_{ij}^{-1} H_j, \tag{E.10}$$

where H_j are the Cartan generators of $\tilde{\mathbf{g}}$ in the Chevalley-Serre basis. The generator J_3, together with

$$J^+ = \sum_{i=1}^{r} k_i E_i \quad , \quad J^- = \sum_{i=1}^{r} l_i F_i, \tag{E.11}$$

constitute a so(2,1) algebra. The hermiticity property $(J^+)^\dagger = J^-$ will be satisfied only if $k_i = l_i^*$, since $(E_i)^\dagger = F_i$. Furthermore, demanding $[J^+, J^-] = \sum_{i=1}^{r} k_i l_i H_i = -J_3$ requires

$$0 \leq |k_j|^2 = k_j l_j = -\sum_{i=1}^{r} A_{ij}^{-1} \equiv p_j. \tag{E.12}$$

The remaining Lie bracket of so(2,1), $[J_3, J^\pm] = \pm J^\pm$ are then satisfied automatically. Thus, a real principal so(2,1) subalgebra exists if and only if

$$\sum_{i=1}^{r} A_{ij}^{-1} \leq 0, \quad \forall\, j. \tag{E.13}$$

For instance, all the hyperbolic Kac-Moody algebras have a principal so(2,1) subalgebra, since all their fundamental weights lie in the forward light-cone, and thus $A_{ij}^{-1} = \frac{2}{(\alpha_i,\alpha_i)}(\lambda_i, \lambda_j) \leq 0$ for all the indices i and j [15].

The algebra $\tilde{\mathbf{g}}$ can again be decomposed into irreducible representations of the principal subalgebra. Due to the non-compactness of the group SO(2,1), all these irreducible representations will be infinite dimensional and unitary, except the adjoint representation of so(2,1), which is neither. The unitarity means that one can define a scalar product (x, y) on the representation space, being hermitian and positive definite with the properties

$$\begin{array}{rcl} ([E_i, x], y) &=& (x, [F_i, y]) \\ ([H_i, x], y) &=& (x, [H_i, y]) \end{array} , \quad \forall\, x, y \in \tilde{\mathbf{g}}. \tag{E.14}$$

The representation space is then the vector space of the algebra $\tilde{\mathbf{g}}$, with $\tilde{\mathbf{g}}$ acting on itself by the adjoint action. Assuming that the algebra possesses an invariant bilinear form $\langle\cdot|\cdot\rangle$, i.e., a generalised Killing form, we can define the scalar product on the representation space as

$$(x, y) \equiv -\langle x|\theta(y)\rangle, \tag{E.15}$$

where θ is the Cartan involution

$$\theta(E_i) = -F_i \quad , \quad \theta(F_i) = -E_i \quad , \quad \theta(H_i) = -H_i. \tag{E.16}$$

The generators J_3 and $J^{\pm}$ of the principal so(2,1) can be verified to satisfy Eq. (E.14), with J_3 being self-adjoint and $J^{\pm}$ mutually adjoint, respectively. Taking the norms

$$(J_3, J_3) = \rho^2 = -\sum_{i=1}^{r} p_i < 0 \quad \text{and} \quad (J^+, J^+) = (J^-, J^-) = \sum_{i=1}^{r} p_i > 0 \tag{E.17}$$

shows that the adjoint representation of so(2,1) is indeed not unitary. Since the representation space is decomposed into orthogonal subspaces consisting of the Cartan subalgebra and subspaces associated with each root, all the other irreducible representations in the decomposition of $\tilde{\mathbf{g}}$ will be unitary. This conclusion is made by observing that the subspace spanned by all the vectors orthogonal to the Weyl vector ρ is positive definite with respect to the scalar product (E.15).

Because of the adjoint action the spectrum of J_3 will be integral, i.e., $e^{i2\pi J_3} = 1$ and the representations arised are called single valued or unitary. Using the irreducible adjoint representation of so(2,1) we define the quadratic Casimir operator

$$\begin{aligned} \mathcal{Q} &= J_3 J_3 - J^+ J^- - J^- J^+ \\ &= J_3(J_3+1) - 2J^- J^+ = J_3(J_3 - 1) - 2J^+ J^-, \end{aligned} \tag{E.18}$$

which acts on a generator adjointly according to

$$\mathrm{ad}_{\mathcal{Q}}(x) = [J_3, [J_3, x]] - [J^+, [J^-, x]] - [J^-, [J^+, x]]. \tag{E.19}$$

The eigenvalues of this Casimir operator are then used to label the irreducible representations of so(2,1) occurring in the decomposition of $\tilde{\mathbf{g}}$. Besides the non-unitary finite dimensional representations such as the three dimensional one, so(2,1) possesses two different kinds of unitary infinite dimensional representations.

i. **The discrete series representations**
The Casimir eigenvalues for the discrete series representations are

$$\mathcal{Q} = s(s-1) > 0. \tag{E.20}$$

These representations contain a lowest (highest) weight state satisfying $J^-|s, s\rangle = 0$ (or $J^+| - s, -s\rangle = 0$), and the states of a representation are denoted by $|s, m\rangle$ (or $| - s, -m\rangle$) for $m = s, s+1, \ldots$, where $s = 2, 3, 4, \ldots$. These representations are entirely contained in the triangular subalgebras N_+ or N_- defined in Eq. (E.1). The lowest weight representations are built on states of the form

$$v^{(s)} = \sum_{j_1,\ldots,j_s} c_{j_1\ldots j_s}[E_{j_1}, \ldots, [E_{j_{s-1}}, E_{j_s}] \ldots] \quad , \quad [J^-, v^{(s)}] = 0 \tag{E.21}$$

by repeated application of J^+. Similarly, the highest weight states are obtained by acting on

$$v^{(-s)} = \sum_{j_1,\dots,j_s} c_{j_1\dots j_s}[F_{j_1},\dots,[F_{j_{s-1}},F_{j_s}]\dots] \quad , \quad [J^+, v^{(-s)}] = 0 \tag{E.22}$$

with J^-.

ii. **The continuous representations**
The continuous representations split into principal and supplementary series representations with the Casimir eigenvalues

$$\begin{array}{rr} \text{principal series}: & -\infty < \mathcal{Q} < -\frac{1}{4} \\ \text{supplementary series}: & -\frac{1}{4} < \mathcal{Q} < 0, \end{array} \tag{E.23}$$

respectively. The supplementary series can in practical never appear as unitary representations of so(2,1), since $e^{i2\pi J_3}$ can never equal unity for these. Thus, we consider only the principal series in detail.

Apart from J_3, there are $(r-1)$ linearly independent combinations of the Cartan generators of $\tilde{\mathbf{g}}$, written generally as $\sum_{i=1}^r c_i H_i$. These are determined by diagonalising the so(2,1) Casimir operator. Acting with the Casimir operator adjointly we obtain

$$\mathrm{ad}_{\mathcal{Q}}\left(\sum_i^r c_i H_i\right) = -2\sum_{i,j=1}^r c_i A_{ij} p_j H_j. \tag{E.24}$$

Note that using $\sum_{j=1}^r A_{ij}p_j = -\sum_{j,k=1}^r A_{ij}A^{-1}_{jk} = -1$ and putting $c_i = -p_i$, Eq. (E.24) becomes

$$\mathrm{ad}_{\mathcal{Q}}(J_3) = 2\sum_{i,j=1}^r p_i A_{ij} p_j H_j = 2J_3, \tag{E.25}$$

which is precisely the result expected for the adjoint representation of so(2,1). The coefficients of the remaining $(r-1)$ linearly independent combinations will be orthogonal to J_3 and thus satisfy

$$\left(\sum_{i=1}^r c_i H_i, -\sum_{j=1}^r p_j H_j\right) = -\sum_{i,j=1}^r c_i p_j A_{ij} = 0 \quad \Rightarrow \quad \sum_{i=1}^r c_i = 0. \tag{E.26}$$

The full representations are then generated by multiply commuting $\sum_{i=1}^r c_i H_i$ with J^+ and J^-. Since the eigenvalues of J_3 are bounded neither from above nor from below, the orthogonal complement of J_3 must belong to $(r-1)$ principal series representations. These representations extend simultaneously into both N_+ and N_-. It can also be verified that all the elements in the principal series have positive norm, i.e., $(x, x) > 0$.

F

The lattices $\Pi_{D-1;1}$

The lattices $\Pi_{D-1;1}$ are the unique even self-dual Lorentzian lattices, existing only in the dimensions $D = 8n+2$, $n = 0, 1, 2, \ldots$. These lattices are spanned by the vectors $x = (x^1, \ldots, x^{D-1}; x^0) \in \mathbb{R}^{D-1,1}$ satisfying

$$x \cdot r \equiv \sum_{i=1}^{D-1} x^i r^i - x^0 r^0 \in \mathbb{Z}, \tag{F.1}$$

where $r = (\frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2}) \in \mathbb{R}^{D-1,1}$, and the scalar product is defined using the flat Minkowski metric. In addition, either

$$\text{all } x^\mu \in \mathbb{Z} \quad \text{or} \quad \text{all } (x^\mu - r^\mu) \in \mathbb{Z} \tag{F.2}$$

must be obeyed [17].

Concentrating on the lattice $\Pi_{1;1}$, we find it consisting of the vectors

$$(m; 2p+m) \quad \text{and} \quad \left(n + \frac{1}{2}; 2q + n + \frac{1}{2}\right) \quad \forall\, m, n, p, q \in \mathbb{Z}. \tag{F.3}$$

It is straightforward to show that $\Pi_{1;1}$ is even

$$\left\{ \begin{array}{rcl} (m; 2p+m)^2 & = & -4p(p+m) \\ \left(n+\frac{1}{2}; 2q+n+\frac{1}{2}\right)^2 & = & -2q(2q+2n+1) \end{array} \right. \in 2\mathbb{Z}\,. \tag{F.4}$$

The self-duality can be shown by explicitly finding the dual lattice. For convenience we change the notation to $z = (z^+, z^-)$, where

$$\left\{ \begin{array}{rcl} z^+ & = & x^0 + x^1 \\ z^- & = & \frac{1}{2}(x^0 - x^1) \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{rcl} x^0 & = & \frac{1}{2}(z^+ + 2z^-) \\ x^1 & = & \frac{1}{2}(z^+ - 2z^-) \end{array} \right. . \tag{F.5}$$

The scalar product then becomes

$$\begin{aligned} x \cdot y &= \tfrac{1}{4} \{ (z^+ - 2z^-)(w^+ - 2w^-) - (z^+ + 2z^-)(w^+ + 2w^-) \} \\ &= -z^+ w^- - z^- w^+ . \end{aligned} \tag{F.6}$$

In the basis given by (z^+, z^-) the points in $\Pi_{1;1}$ can be written simply as

$$(n, m) \quad \forall\, n, m \in \mathbb{Z}, \tag{F.7}$$

e.g., the vector r will be $r = (1, 0)$. Thus, the lattice $\Pi_{1;1}$ is the integer span of the basis vectors

$$k \equiv (1, 0) \quad \text{and} \quad \bar{k} \equiv (0, -1), \tag{F.8}$$

which have be chosen to satisfy $k^2 = \bar{k}^2 = 0$ and $k \cdot \bar{k} = 1$. Also, $\pm(k + \bar{k})$ are the only two points in the whole lattice with length squared equal two.

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