Finite Sets and Frege Structures[1]

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Call a family $\mathscr{F}$ of subsets of a set E *inductive* if $\varnothing \in \mathscr{F}$ and $\mathscr{F}$ is *closed under unions with disjoint singletons*, that is, if $\forall X{\in}\mathscr{F}\,\forall x{\in}E{-}X(X \cup \{x\} \in \mathscr{F}]$. A *Frege structure* is a pair (E, ν) with ν a map to E whose domain $\mathrm{dom}(\nu)$ is an inductive family of subsets of E such that $\forall X\, Y \in \mathrm{dom}(\nu)(\nu(X) = \nu(Y) \Leftrightarrow X \approx Y)$.[2] In [1] it is shown in a constructive setting that each Frege structure determines a subset which is the domain of a model of Peano's axioms. In this note we establish, within the same constructive setting, three facts. First, we show that the least inductive family of subsets of a set E is precisely the family of *decidable Kuratowski finite* subsets of E. Secondly, we establish that the procedure presented in [1] can be reversed, that is, any set containing the domain of a model of Peano's axioms determines a map which turns the set into a *minimal* Frege structure: here by a minimal Frege structure is meant one in which $\mathrm{dom}(\nu)$ is the least inductive family of subsets of E. And finally, we show that the procedures leading from minimal Frege structures to models of Peano's axioms and vice-versa are *mutually inverse*. It follows that the postulation of a (minimal) Frege structure is constructively equivalent to the postulation of a model of Peano's axioms.

All arguments will be formulated within constructive (intuitionistic) set theory[3].

1. Some definitions of finiteness. Fix a set E. By "set" "family", "singleton", etc. we shall mean "subset of E", "family of subsets of E", "singleton of E", etc. For a set X define

***K**(X)* $\Leftrightarrow$ X is in every family containing $\varnothing$, all singletons, and closed under unions of pairs of its members. If ***K**(X),* we shall say that X is *Kuratowski finite*.

***L**(X)* $\Leftrightarrow$ X is in every family containing $\varnothing$ and closed under unions with singletons. If ***L**(X)*, we shall say that X is *finite*.

***M**(X)* $\Leftrightarrow$ X is in every inductive family. If ***M**(X),* we shall say that X is *strictly finite.*

***D**(X)* $\Leftrightarrow$ for every $x,y \in X$, either $x = y$ or $x \neq y$. If ***D**(X), X* is said to be *decidable.*

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[2] We write $X \approx Y$ for *there exists a bijection between X and Y*.

[3] This set theory may be taken to be intuitionistic Zermelo set theory or the local set theory presented in [2].

1.1. *Lemma*. $\forall X[\boldsymbol{M}(X) \Rightarrow \boldsymbol{L}(X)]$.

Proof. Obvious.

1.2. *Lemma.* $\forall X[\boldsymbol{K}(X) \Leftrightarrow \boldsymbol{L}(X)]$.

Proof. Clearly $\boldsymbol{L}(X) \Rightarrow \boldsymbol{K}(X)$. To prove the converse, it suffices to show that the family $\mathscr{L} = \{X: \boldsymbol{L}(X)\}$ is closed under unions of pairs. To this end let $\Phi(U)$ be the property $\forall X \in \mathscr{L}\ U \cup X \in \mathscr{L}$. It suffices to show $\forall U[\boldsymbol{L}(U) \Rightarrow \Phi(U)]$. Clearly $\Phi(\emptyset)$. Assuming $\Phi(U)$ and $X \in \mathscr{L}$ we have $U \cup X \in \mathscr{L}$ and so $U \cup X \cup \{x\} \in \mathscr{L}$ for arbitrary x, whence $\Phi(U \cup \{x\})$. Hence $\forall U[\boldsymbol{L}(U) \Rightarrow \Phi(U)]$ and the result follows.

1.3. *Lemma.* $\forall X[\boldsymbol{M}(X) \Rightarrow \boldsymbol{D}(X)]$.

Proof. Obviously $\boldsymbol{D}(\emptyset)$. If $\boldsymbol{D}(X)$ and $x \notin X$, clearly $\boldsymbol{D}(X \cup \{x\})$. The result follows.

1.4. *Lemma.* $\forall X[\boldsymbol{M}(X) \Rightarrow \forall a[\boldsymbol{D}(X \cup \{a\}) \Rightarrow (a \in X \text{ or } a \notin X)]]$.

Proof. Write $\Phi(X)$ for the condition following the first implication. Clearly $\Phi(\emptyset)$. Suppose that $\Phi(X)$ and $x \notin X$. If $\boldsymbol{D}(X \cup \{x\} \cup \{a\})$, then $\boldsymbol{D}(X \cup \{a\})$, so, since $\Phi(X)$, either $a \in X$ or $a \notin X$. Since $\boldsymbol{D}(X \cup \{x\} \cup \{a\})$, it follows that $a = x$ or $a \neq x$. Hence

$$(a \in X \;\&\; a = x) \text{ or } (a \notin X \;\&\; a = x) \text{ or } (a \in X \;\&\; a \neq x) \text{ or } (a \notin X \;\&\; a \neq x),$$

The first three disjuncts each imply $a \in X \cup \{x\}$, and the last disjunct means $a \notin X \cup \{x\}$. Accordingly $a \in X \cup \{x\}$ *or* $a \notin X \cup \{x\}$. We conclude that $\Phi(X \cup \{x\})$ and the result follows.

1.5. *Lemma.* $\forall X[\mathbf{L}(X) \;\&\; \boldsymbol{D}(X) \Rightarrow \boldsymbol{M}(X)]$.

Proof. We need to show $\forall X[\boldsymbol{L}(X) \Rightarrow \Phi(X)]$, where $\Phi(X)$ is $\boldsymbol{D}(X) \Rightarrow \boldsymbol{M}(X)$. Clearly $\Phi(\emptyset)$. Assume $\Phi(X)$ and $\boldsymbol{D}(X \cup \{a\})$. Then $\boldsymbol{D}(X)$, so, since $\Phi(X)$, it follows that $\boldsymbol{M}(X)$. Since $\boldsymbol{D}(X \cup \{a\})$, 1.4 gives $a \in X$ or $a \notin X$. In either case we deduce that $\boldsymbol{M}(X \cup \{a\})$. Hence $\Phi(X \cup \{a\})$, and the result follows.

From these lemmas we immediately infer

1.6. *Theorem.* For any set E, the families of strictly finite, decidable finite, and decidable Kuratowski finite subsets coincide. ■

2. Frege structures from models of Peano's axioms. Let $(N, s, 0)$ be a model of Peano's axioms; we use letters m, n, p as variables ranging over N. We write $<$ for the usual (constructively definable) strict order relation on N (see, e.g., Prop. 7.5 of [2]). Define $g: N \to PN$ by $g(n) = \{m: m < n\}$; then g satisfies (and in fact can be defined by) the equations

$$g(0) = \varnothing \qquad g(sn) = g(n) \cup \{n\}.$$

In order to prove the next Lemma we require the following fact, which is proved constructively as Lemma 3 of [1]:

(*) *for any sets X, Y, and any x, y such that* $x \notin X$, $y \notin Y$, *if* $X \cup \{x\} \approx Y \cup \{y\}$, *then* $X \approx Y$.

2.1. *Lemma*. For all m, n, $g(m) \approx g(n) \Leftrightarrow m = n$.

Proof. Write $\Phi(n)$ for $\forall m[g(m) \approx g(n) \Leftrightarrow m = n]$. Then clearly $\Phi(0)$. If $\Phi(n)$ and $g(m) \approx g(sn) = g(n) \cup \{n\}$, then $m \neq 0$ so that $m = sp$ for some p. Hence

$$g(p) \cup \{p\} = g(sp) = g(m) \approx g(sn) = g(n) \cup \{n\}.$$

Since $p \notin g(p)$ and $n \notin g(n)$, (*) above implies that $g(p) \approx g(n)$, so, since $\Phi(n)$, $p = n$ and $m = sp = sn$. Hence $\Phi(sp)$, and the result follows by induction.

Now suppose that E is a set such that $N \subseteq E$. Define

$$\nu = \{(X,n) \in PE \times N\colon X \approx g(n)\}.$$

2.2. *Lemma.* dom(ν) is the family of strictly finite subsets of E.

Proof. We need to show that dom(ν) is the least inductive family. First, dom(ν) is inductive. For it clearly contains $\varnothing$ and, if $X \in$ dom(ν), then $X \approx g(n)$ for some n. If $x \notin X$, then $X \cup \{x\} \approx g(n) \cup \{n\} = g(sn)$, whence $X \cup \{x\} \in$ dom(ν). And dom(ν) is the least inductive family. For suppose that $\mathscr{F}$ is any inductive family. For each n let $\mathscr{H}_n = \{X\colon X \approx g(n)\}$. We claim that $\mathscr{H}_n \subseteq \mathscr{F}$ for all n. For obviously $\mathscr{H}_0 = \{\varnothing\} \subseteq \mathscr{F}$. Now suppose that $\mathscr{H}_n \subseteq \mathscr{F}$. If $X \approx g(sn)$, then $X \approx g(n) \cup \{n\}$, so for some $x \in X$ (which may be taken to be the image of n under the bijection $g(n) \cup \{n\} \approx X$), we have $X - \{x\} \approx g(n)$. It follows that $X - \{x\} \in \mathscr{H}_n \subseteq \mathscr{F}$, and so $X = (X - \{x\}) \cup \{x\} \in \mathscr{F}$. The claim now follows by induction; accordingly dom(ν), as the union of all the $\mathscr{H}_n$, is included in $\mathscr{F}$. Therefore dom(ν) is the least inductive family and the Lemma is proved.

2.3. *Lemma*. ν is a function and $X \approx g(\nu(X))$ for all $X \in$ dom(ν).

Proof. Suppose that $(X,n) \in \nu$ and $(X,m) \in \nu$. Then $X \approx g(n)$ and $X \;\Phi\; g(m)$ whence $g(m) \approx g(n)$ and so $m = n$ by Lemma 2.1. The remaining claim is obvious.

2.4. *Lemma.* For all $X, Y \in \mathrm{dom}(\nu)$, $X \approx Y \Leftrightarrow \nu(X) = \nu(Y)$.

Proof. We have, using the previous Lemma, $\nu(X) = \nu(Y) \Leftrightarrow g(\nu(X)) \approx g(\nu(Y)) \Leftrightarrow$ $X \approx Y$.

Lemmas 2.2 - 2.4 establish

2.5. *Theorem.* (E, ν) is a minimal Frege structure. ■

(E,ν) is called the minimal Frege structure *associated* with E and $(N, s, 0)$.

Finally, we show that the processes of deriving models of Peano's axioms from minimal Frege structures and *vice-versa* are mutually inverse.

Suppose that we are given a minimal Frege structure (E,μ). As shown in [1], the associated model $(N,s,0)$ of Peano's axioms is obtained in the following way. First, the family $\mathbb{N}$ is defined as the least subfamily of $\mathrm{dom}(\mu)$ containing $\varnothing$ and such that, if $X \in \mathbb{N}$ and $\mu(X) \notin X$, then $X \cup \{\mu(X)\} \in \mathbb{N}$: it is shown that $\mu(X) \notin X$ for all $X \in \mathbb{N}$. The associated model $(N,s,0)$ of Peano's axioms is then defined by $N = \{\mu(X): X \in \mathbb{N}\}$, $s(\mu(X)) = \mu(X \cup \mu(X)\})$, and $0 = \mu(\varnothing)$.

We observe that since (E,μ) is minimal, for any $X \in \mathrm{dom}(\mu)$ there is a (unique) $X^* \in \mathbb{N}$ for which $X \approx X^*$, and so $\mu(X) = \mu(X^*)$. To prove this, it suffices to show that the set of $X \in \mathrm{dom}(\mu)$ with this property contains $\varnothing$ and is closed under unions with disjoint singletons. The first claim is obvious. If $X \in \mathrm{dom}(\mu)$, $x \notin X$, and $X \approx X^*$ with $X^* \in \mathbb{N}$, then

$$X \cup \{x\} \approx X^* \cup \{\mu(X^*)\} \in \mathbb{N},$$

since, as observed above, $\mu(X^*) \notin X^*$. This establishes the second claim, and the observation.

Now let (E,ν) be the minimal Frege structure associated with the model $(N,s,0)$ of Peano's axioms in turn associated with (E,μ). We claim that $\mu = \nu$. To prove this it suffices to show that

$$(**) \quad X \approx g(\mu(X)) \text{ for all } X \in \mathbb{N},$$

where $\mathbb{N}$ is defined as above. For then, by Lemma 2.3, we will have $g(\nu(X)) \approx X \approx g(\mu(X))$ and so $\mu(X) = \nu(X)$ by Lemma 2.1. This last equality for all $X \in \mathbb{N}$ in turn yields $\mu(Y) = \nu(Y)$ for all $Y \in \mathrm{dom}(\mu) = \mathrm{dom}(\nu)$. For then, by our observation above, $\mu(Y) = \mu(Y^*) = \nu(Y^*) = \nu(Y)$.

So it only remains to prove (**). It is clearly satisfied by $\varnothing$. If $X \approx g(\mu(X))$ with $X \in \mathbb{N}$, then, since $\mu(X) \notin X$,

$$X \cup \{\mu(X)\} \approx g(\mu(X)) \cup \{\mu(X)\} = g(s\mu(X)).$$

(**) now follows from the definition of $\mathbb{N}$. So our claim that $\mu = \nu$ is established.

Conversely, suppose we are given a set E and a model $(N, s, 0)$ of Peano's axioms with $N \subseteq E$. Let (E,ν) be the associated minimal Frege structure. We note first that, for any $n \in N$, we have $\nu(g(n)) = n$, where, as before, $g(n) = \{m: m < n\}$. For by Lemma 2.3, $g(n) \approx g(\nu(g(n))$, so that, by Lemma 2.1., $n = g(\nu(n))$. Now let $(N^*,s^*,0^*)$ be the model of Peano's axioms associated with the Frege structure (E,ν). We claim that $(N,s,0)$ and $(N^*,s^*,0^*)$ are identical.

First, $N^* = \{\nu(X): X \in \mathbb{N}^*\}$, where $\mathbb{N}^*$ is the least subfamily of dom(ν) containing $\varnothing$ and such that $X \in \mathbb{N}^*$ and $\nu(X) \notin X$ implies $X \cup \{\nu(X)\} \in \mathbb{N}^*$. Using the fact that $\nu(g(n)) = n$ for all $n \in N$, it is easily shown that $\mathbb{N}^* = \{g(n): n \in N\}$. Thus $N^* = \{\nu(X): X \in \mathbb{N}^*\} = \{\nu(g(n)): n \in N\} = \{n: n \in N\} = N$. Finally $0^* = \nu(0) = \nu(g(0)) = 0$ and

$$s^*(n) = s^*(\nu(g(n))) = \nu(g(n) \cup \{\nu(g(n))\}) = \nu(g(n) \cup \{n\}) = \nu(g(sn)) = sn,$$

so that $s^* = s$.

Thus we have established that the two processes are mutually inverse.

References

[1] Bell, John L. *Frege's Theorem in a Constructive Setting. Journal of Symbolic Logic,* vol. 64, no. 2, 1999

[2] Bell, John L. *Toposes and Local Set Theories: An Introduction.* Oxford University Press, 1988.