Reflections on the Axiomatic Approach to Continuity

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In Hilbert's paper "Axiomatic Thinking" - the published version[1] of his 1917 Zürich talk which the present meeting commemorates - he touches on the axiomatic treatment of continuity and, as he puts it, "the dependence of the propositions of a field of knowledge on the axiom of continuity."

By the "axiom of continuity" Hilbert seems to mean a number of things. He first assimilates it to the Archimedean axiom (which he also calls the "axiom of measurement") and observes its independence of the other axioms of the theory of real numbers. Presumably he means the other axioms of the first-order theory of real numbers, since the Archimedean axiom is derivable in the second-order theory in which order-completeness is assumed.

Hilbert goes on to observe that the Archimedean axiom plays - implicitly at least - a role in physics.

It seems to me that it has principal interest in physics as well; for it leads us to the following outcome. That is, the fact that we can come up with the dimensions and ranges of celestial bodies by putting together terrestrial ranges, namely measuring celestial lengths by terrestrial measure, as well as the fact that the distances inside atoms can be expressed in terms of metric measure, is by no means a merely logical consequence of propositions on the triangular congruence and the geometric configuration, but rather an investigative result of experience. The validity of the Archimedean axiom in nature, in the sense indicated above, needs experimental confirmation just as much as does the proposition of the angle sums in triangle in the ordinary sense.

Hilbert asserts that the validity of the Archimedean axiom is "an investigative result of "experience." What he may mean here is that in comparing astronomical, terrestrial and subatomic distances, none is infinitesimal, or infinitely large, with respect to the others. Thus, in principle, the radius of an electron could be used as a unit to measure terrestrial or astronomical distances.

What has this to do with continuity? Hilbert seems to imply that, so far as measurement is concerned, the empirical validity of the Archimedean axiom means that there is a kind of continuity - a smooth

[1] Hilbert (1918).

transition - between microcosm, mesocosm and macrocosm. None of these realms is cut off from the others.

While the Arcimedean axiom is exact, the notion of "continuity" associated with it, although suggestive, is essentially qualitative (and akin to Leibniz's principle of continuity, see below). In order to formulate an exact principle of continuity Hilbert turns to physics:

In general, I should like to formulate the axiom of continuity in physics as follows: "If a certain arbitrary degree of exactitude is prescribed for the validity of a physical assertion, a small range shall then be specified, within which the presuppositions prepared for the assertion may freely vary so that the deviation from the assertion does not overstep the prescribed degree of exactitude." This axiom in the main brings only that into expression which directly lies in the essence of experiments; it has always been assumed by physicists who, however, have never specifically formulated it.

(Note the little dig at physicists with which Hilbert concludes this passage - is this a foretaste of the famous, but perhaps apocryphal remark later attributed to Hilbert that ``Physics is obviously much too difficult for the physicists.``)

Hilbert`s formulation of the principle of continuity in physics – what I shall call the *physical continuity axiom* (**PCA**) is evidently an empirical version of the familiar (ε, δ) definition of a continuous function. More precisely, the axiom asserts that any function from real numbers to real numbers *associated with a physical assertion* is (ε, δ) - continuous. This is an updated version of Leibniz``s Principle of Continuity: *Natura non facit saltus.*

Before the 19th century **PCA** would have been formulated in terms of infinitesimals, perhaps as follows:

"*If the degree of exactitude is prescribed for the validity of a physical assertion, is prescribed to be within infinitesimal limits, then also within infinitesimal limits the presuppositions prepared for the assertion may freely vary so that the deviation from the assertion does not overstep the prescribed infinitesimal limits* ."

This may be termed the *Principle of Infinitesimal Continuity* (**PIC)***:* any real function sends infinitesimally close points to infinitesimally close points.

These are all very strong "global" axioms which are to be contrasted with the "local" continuity axioms imposed on the system of real numbers such as the Archimedean principle or the order-completeness principle.

Hilbert's continuity axiom was formulated for the physical realm, but it can be extended to mathematics where it takes the form of *Brouwer's continuity principle:*

BCP *All functions from real numbers to real numbers are continuous.*

Of course, Brouwer did not regard this principle as an axiom – indeed he seems to have had a low opinion of the axiomatic method in mathematics. Rather he regarded it as a *fact* (albeit requiring demonstration) about the real numbers arising from the nature of the continuum as he conceived it.

The question of the *consistency* of this strengthened principle of continuity arises immediately. It might seem at first glance that **BCP** is inconsistent since the "blip" function $b: \mathbb{R} \to \mathbb{R}$ defined by $b(0) = 1$, $b(x) = 0$ for $x \neq 0$ is obviously discontinuous. But the condition that b is defined on the *whole* of $\mathbb{R}$ rests on the unquestioned assumption that, for any real number x, either $x = 0$ or $x \neq 0$. This in turn rests on the *Law of Excluded Middle* (**LEM**)– the logical principle, going back to Aristotle, that, for any proposition, either it or its negation must be true. While **LEM** is a core principle of classical logic, it is not affirmed in *intuitionistic* logic, the system of logic implicit in Brouwer's conception of mathematics and later made explicit by his student Heyting.

Thus, while **BCP** is inconsistent with classical mathematics, that is, mathematics based on classical logic, it can be, and in fact is, consistent with intuitionistic mathematics, that is, mathematics based on intuitionistic logic. It is easily seen that, within intuitionistic mathematics, **LEM** is *refutable* from **BCP** in the sense that

$$\mathbf{BCP} \Rightarrow \neg\forall x\in\mathbb{R}(x = 0 \vee x \neq 0).$$

Here we have an example of a mathematical axiom *actually refuting a logical axiom*. It is of interest to note here that Cantor, in introducing his transfinite numbers, had to repudiate Euclid's 5th *axiom* that the whole is always greater than the part, and Bolyai and Lobachevsky (as well as Gauss) in their formulation of non-Euclidean geometry, were compelled to repudiate Euclid's 5th *postulate*. In both of these earlier cases the question of consistency was central, and it is equally important in the case of **BCP.** In fact, just as models of non-Euclidean geometry were later constructed to establish its consistency, so models of mathematics have been constructed based on imtuitionistic logic and realizing **BCP,** so establishing the consistency of the latter.

An even stronger version of the continuity principle (implicitly adhered to in differential geometry) is:

SP *All functions from reals to reals are smooth,* i.e. arbitrarily many times differentiable. (More generally, all functions between manifolds are smooth).

Axiom **SP** can be realized by adopting what amounts to a *synthetic* approach to differential geometry.

Traditionally, there have been two methods of deriving the theorems of (classical) geometry: the *analytic* and the *synthetic* or *axiomatic.* While the analytic method is based on the introduction of numerical coordinates, and so on the theory of real numbers, the idea behind the synthetic approach is to furnish the subject of geometry with a purely geometric foundation in which the theorems are then deduced by purely logical means from an initial body of axioms.

The most familiar examples of synthetic geometry are classical Euclidean geometry and the synthetic projective geometry introduced by Desargues in the 17th century and revived and developed by Carnot, Poncelet, Steiner and others during the 19th century.

The power of analytic geometry derives very largely from the fact that it permits the methods of the calculus, and, more generally, of mathematical analysis, to be introduced into geometry, leading in particular to *differential geometry* (a term, by the way, introduced in 1894 by the Italian geometer *Luigi Bianchi*). That being the case, the idea of a "synthetic" differential geometry seems elusive: how can differential geometry be placed on a "purely geometric" or "axiomatic" foundation when the apparatus of the calculus seems inextricably involved?

To my knowledge there have been two attempts to develop a synthetic differential geometry. The first was initiated by Herbert Busemann in the 1940s, building on earlier work of Paul Finsler. Here the idea was to build a differential geometry that, in its author's words, "requires no derivatives": the basic objects in Busemann's approach are not differentiable manifolds, but metric spaces of a certain type in which the notion of a geodesic can be defined in an intrinsic manner.

The second approach, that with which I shall be concerned here, was originally proposed in the 1960s by F. W. Lawvere, who was in fact striving to fashion a decisive axiomatic framework for continuum mechanics. His ideas have led to what I shall simply call *synthetic differential geometry* (**SDG** – often referred to as *smooth infinitesimal analysis* **SIA**)[2]. **SDG** is formulated within *category theory,* the branch of mathematics created in 1945 by Eilenberg and Mac Lane which deals with mathematical form and

[2] For accounts of **SDG/SIA** see Bell (2008) and Kock (2006).

structure in its most general manifestations. As in biology, the viewpoint of category theory is that mathematical structures fall naturally into species or *categories*. But a category is specified not just by identifying the species of mathematical structure which constitute its *objects*; one must also specify the transformations or *maps* linking these objects. Thus one has, for example, the category **Set** with objects all sets and maps all functions between sets; the category **Grp** with objects all groups and maps all group homomorphisms; the category **Top** with objects all topological spaces and maps all continuous functions; and **Man**, with objects all (Hausdorff, second countable) smooth manifolds and maps all smooth functions. Since differential geometry "lives" in **Man**, it might be supposed that in formulating a "synthetic differential geometry" the category-theorist's goal would be to find an axiomatic description of **Man** itself.

But in fact the category **Man** has a couple of "deficiencies" which make it unsuitable as an object of axiomatic description:

1. It lacks exponentials: that is, the "space of all smooth maps" from one manifold to another in general fails to be a manifold. And even if it did—
2. It also lacks "infinitesimal objects"; in particular, there is no "infinitesimal" or *incredible shrinking manifold* Δ for which the tangent bundle TM of an arbitrary manifold M can be identified as the exponential "manifold" M^Δ of all "infinitesimal paths" in M. (It may be remarked parenthetically that it is this deficiency that makes the construction of the tangent bundle in **Man** something of a headache.)

Lawvere's idea was to enlarge **Man** to a category **S**—a category of so-called *smooth spaces* or a *smooth category*—which avoids these two deficiencies, admits a simple axiomatic description, and at the same time is sufficiently similar to **Set** for mathematical construction and calculation to take place in the familiar way.

The essential features of a smooth category **S** are these:

- In enlarging **Man** to **S** no "new" maps between manifolds are added, that is, all maps in **S** between objects of **Man** are smooth. (Notice that this is not the case when **Man** is enlarged to **Set.**)
- **S** is *Cartesian closed*, that is, contains products and exponentials of its objects in the appropriate sense.

- **S** satisfies the *principle of microstraightness.* Let R be the real line considered as a object of **Man,** and hence also of **S**. Then there is a nondegenerate segment Δ of R around 0 which remains *straight* and *unbroken* under any map in **S.** In other words, Δ is subject in **S** to *Euclidean motions only.*

Δ may be thought of as a *generic tangent vector.* For consider any curve C in a space M—that is, the image of a segment of R (containing Δ) under a map f into M. Then the image of Δ under f may considered as a short straight line segment lying along C around the point $p = f(0)$ of C. By considering the curve in $R \times R$ given by $f(x) = x^2$, we see that Δ may be identified with the intersection of the curve $y = x^2$ with the x-axis. That is,

$$\Delta = \{x: x \in R \wedge x^2 = 0\},$$

Thus Δ consists of *nilsquare infinitesimals,* or *micronumbers.* We use the letter ε to denote an arbitrary micronumbers.

Now classically Δ coincides with $\{0\}$, but a precise version of the principle of microstraightness—the *Principle of Microtaffineness* (or *Kock-Lawvere axiom*)—ensures that this is not the case in **S.** The principle states that

- in **S**, any map $f: \Delta \rightarrow$ R is (uniquely) *affine*, that is, for some *unique* $b \in$ R, we have, for all ε,

$$f(\varepsilon) = f(0) + b\varepsilon.$$

In essence, this asserts that that the action of any real function f on Δ is a Euclidean transformation: a translation by $f(0)$ and a rotation b.

The principle of microaffineness asserts also that the map $R^\Delta \rightarrow R \times R$ which assigns to each $f \in$ R$^\Delta$ the pair ($f(0)$, slope of f) is an isomorphism:

$$R^\Delta \cong R \times R.$$

Since $R \times R$ is the *tangent bundle* of R, so is R^Δ.

For any space M in **S**, we take the tangent bundle TM of M to be the exponential M^Δ. Elements of M^Δ are called *tangent vectors* to M. Thus a *tangent vector* to M at a point $p \in M$ is just a map $t: \Delta \to M$ with $t(0) = x$ That is, a tangent vector at p is a *micropath in M with base point p*. The *base point map* $\pi: TM \to M$ is defined by $\pi(t) = t(0)$. For $p \in M$, the fibre $\pi^{-1}(p) = T_pM$ is the *tangent space to M at p*.

Observe that, if we identify each tangent vector with its image in M, then *each tangent space to M may be regarded as lying in M.* In this sense each space in **S** is "infinitesimally flat".

We check the compatibility of this definition of TM with the usual one in the case of Euclidean spaces:

$$T(R^n) = (R^n)^\Delta \cong (R^\Delta)^n \cong (R \times R)^n \cong R^n \times R^n.$$

The assignment $M \mapsto TM = M^\Delta$ can be turned into a functor in the natural way—the *tangent bundle functor.* (For $f: M \to N$, $Tf: TM \to TN$ is defined by $(Tf)t = f \circ t$ for $t \in TM$.)

The whole point of synthetic differential geometry is to render *the tangent bundle functor representable*: TM becomes identified with the space of all maps from some fixed object—in this case Δ)—to M. (Classically, this is impossible.) This in turn simplifies a number of fundamental definitions in differential geometry.

For instance, a *vector field* on a space M is an assignment of a tangent vector to M at each point in it, that is, a map $\xi: M \to TM = M^\Delta$ such that $\xi(x)(0) = x$ for all $x \in M$. This means that $\pi \circ \xi$ is the identity on M, so that a *vector field is a section of the base point map*.

A *differential k- form* ((0, k) tensor field) on M may be considered as a map $M^{\Delta n} \to R$.

The notions of affine connection, geodesic, and the whole apparatus of Riemannian geometry can also be developed within **SDG**[3].

[3] See Kock (2009)

As an *axiomatic system,* **SIA** may be set up as a system of axioms for the (smooth) real line R involving micronumbers as already iintroduced The core axiom in **SIA** is the aforementioned *principle of microaffineness.* Writing Δ for the set of (nilsquafre) *infinitesimals* or *micronumbers,* i.e.

$$\Delta = \{x: x \in \mathbf{R} \wedge x^2 = 0\},$$

the principle can be stated:

For any $f: \Delta \rightarrow R$*, there is a* ***unique*** $b \in R$ *such that*

$$f(\varepsilon) = f(0) + d\varepsilon$$

holds for all ε. (We use ε as a variable ranging over Δ.)

This in turn gives rise to a simple definition of the derivative f' of f: given $r \in R$, $f'(r)$ is the unique $b \in R$ such that, for all ε, $f(r + \varepsilon) = f(\mathrm{r}) + b\varepsilon$ (apply microaffineness to the function $\varepsilon \mapsto r + \varepsilon$). Then we get the equation

$$f(r + \varepsilon) = f(r) + \varepsilon f'(r).$$

Similarly we obtain higher derivatives f'', f''', so that **SP** holds. This being the case, the postulates of **SIA** are *incompatible with the law of excluded middle of classical logic*.

From the principle of microaffineness we deduce the important *principle of microcancellation*, viz.

If $\varepsilon a = \varepsilon b$ *for all* ε, *then* $a = b$.

For the premise asserts that the graph of the function $g: \Delta \rightarrow R$ defined by $g(\varepsilon) = a\varepsilon$ has both slope a and slope b: the uniqueness condition in the principle of microaffineness then gives $a = b$. The principle of microcancellation supplies the exact sense in which there are "enough" infinitesimals in smooth infinitesimal analysis.

In **SIA** there is a sense in which *everything is generated by the domain of infinitesmals.* For consider the set Δ^Δ of all maps $\Delta \rightarrow \Delta$. It follows from the principle of microaffineness that R can be identified as the subset of

Δ^Δ consisting of all maps vanishing at 0. In this sense R is "generated" by Δ. Explicitly, Δ^Δ is a monoid under composition which may be regarded as acting on Δ by composition: for $f \in \Delta^\Delta$, $f \cdot \varepsilon = f(\varepsilon)$. The subset V consisting of all maps vanishing at 0 is a submonoid naturally identified as the set of *ratios of infinitesimals.* The identification of R and V made possible by the principle of microaffineness thus leads to the characterization of R itself as the set of ratios of infinitesimals. This was essentially the view of Euler, who regarded infinitesimals as formal zeros and real numbers as representing the possible values of 0/0. For this reason Lawvere[4] has suggested that R in **SIA** should be called the space of *Euler reals.*

Once one has R, Euclidean spaces of all dimensions may be obtained as powers of R, and arbitrary manifolds may be obtained by patching together subspaces of these.

From the principle of microaffineness the following are easily deduced:

- Δ is *nondegenerate,* i.e. $\Delta \neq \{0\}$.[5]
- Call $x, y \in R$ *indiscriminable* (resp., *indistinguishable*) and write $x \approxeq y$ (resp. $x \approx y$) if $x - y \in \Delta$ (resp. $\neg x \neq y$). Then $x \approxeq y$ implies $x \approx y$ (but not vice-versa).
- If J is a closed interval in R, any $f\colon J \to R$ is *indiscriminably continuous* in the sense that, for $x, y \in J$, $x \approxeq y$ implies $fx \approxeq fy$, and hence also $fx \approx fy$. (Note that it follows trivially from $x \approx y$ that $fx \approx fy$.)

It follows that **PIC (principle of infinitesimal continuity)** holds in **SIA**

A *stationary point* of a function $f\colon J \to R$ is defined to be one in whose vicinity "infinitesimal variations" fail to change the value of f, that is, a point a such that $f(a + \varepsilon) = f(a)$ for all ε Equivalently, a is a stationary point of f if f is *locally constant* around a in the sense that, for all $x \in J$, $x \approxeq a$ implies $fx = fa.$ An important axiom concerning stationary points adopted in **SIA** is the

[4] Lawvere (2011).

[5] It should be noted that, while Δ does not reduce to $\{0\}$, nevertheless 0 is the sole *element* of Δ in the (weak) sense that the assertion "there exists an element of Δ which is $\neq 0$" is refutable. Figuratively speaking, Δ is the "atom" 0 encased in an infinitesimal carapace.

Constancy Principle. *If f: $J \to R$ **is locally constant** on J in the sense that* $x \approx y$ implies $fx = fy$ for all $x, y \in J$, *then f is constant.*

It follows easily from this that (as usual) two functions with identical derivatives differ by at most a constant.

Now call a subset $D \subseteq$ **R** *discrete* if it satisfies

$$\forall x \in D \forall y \in D[x = y \vee x \neq y].$$

Notice that if D is discrete, then, for $x, y \in D$, $x \approx y$ implies $x = y$.

It follows quickly from the Constancy Principle that *any map on* **R** (*or one of its closed intervals*) *to a discrete subset of* **R** *is constant.* To see this, let f be a map of R to a discrete set D. Then from $x \approx y$ we deduce $fx \approx fy$, and hence $fx = fy$, in D. So f is locally constant, and hence constant.

In ordinary analysis R and each of its intervals is connected in the sense that they cannot be split into two nonempty subsets neither of which contains a limit point of the other. In **SIA** these have the vastly stronger property of *cohesiveness:* they cannot be split *in any way whatsoever* into two disjoint nonempty subsets[6]. This follows quickly from the Constancy Principle: if $R = U \cup V$ with $U \cap V = \varnothing$, let **2** be the discrete subset $\{0, 1\}$ of R, and define f: $R \to$ **2** by $f(x) = 1$ if $x \in U$, $f(x) = 0$ if $x \in V$. Then f is constant, that is, constantly 1 or 0. In the first case $V = \varnothing$, and in the second $U = \varnothing$.

One of the most widely discussed axioms in mathematics is the *Axiom of Choice*. Surprisingly, perhaps, this is *incompatible* with the various continuity axioms we have discussed. This is because, as discovered in the 1970s, it implies **LEM**[7] . We shall show that it is refutable in **SIA** by showing that it implies $\forall x \in R(x = 0 \vee x \neq 0)$, and hence that the discontinuous blip function is defined on the whole of R.

We take the Axiom of Choice in the particular form

AC *for any family A of nonempty subsets of R, there is a function f: $A \to R$ such that $f(X) \in X$ for every $X \in A$.*

[6] For more on cohesiveness see Bell (2009).
[7] Diaconescu (1975), Goodman and Myhill (1978).

For each $x \in R$ define

$$A_x = \{y \in R\textbf{:}\; y = 0 \vee x = 0\}$$

$$B_x = \{y \in R\textbf{:}\; y = 1 \vee x = 0\}.$$

Clearly $0 \in A_x$ and $1 \in B_x$ so these sets are both nonempty. By **AC,** we obtain a map f_x: $\{A_x, B_x\} \rightarrow R$ such that, for any $x \in R$**,** $f_x(A_x) \in A_x$ and $f_x(B_x) \in B_x$. Thus

$$[f_x(A_x) = 0 \vee x = 0] \wedge [f_x(B_x) = 1 \vee x = 0].$$

Applying the distributive law for $\vee$ over $\wedge$ (valid in intuitionistic logic), we

obtain

$$[f_x(A_x) = 0 \wedge f_x(B_x) = 1] \vee x = 0$$

whence

(*) $$f_x(A_x) \neq f_x(B_x) \quad \vee \quad x = 0.$$

Now clearly $A_0 = B_0 = R$**,** so that $f_0(A_0) = f_0(B_0)$. Thus

$$f_x(A_x) \neq f_x(B_x) \quad \rightarrow \quad x \neq 0.$$

So from (*) it follows that

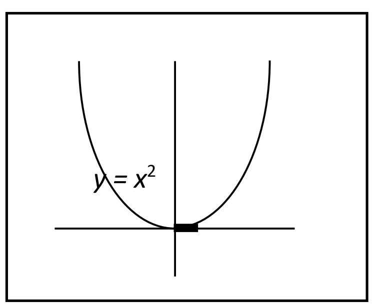
$$x \neq 0 \vee x = 0$$

whence

$$\forall x \in R(x = 0 \vee x \neq 0).$$

I conclude with some historical observations.While **SIA** was not developed until the 1960s, the idea of treating infinitesimals as nilpotent quantities was first put forward in works of 1694-6 by the Dutch physician **Bernard Nieuwentijdt** (1654–1718). Nieuwentijdt developed his account of infinitesimals - a striking example of axiomatic thinking - in conscious opposition to Leibniz's well-known theory of differentials. Nieuwentijdt postulates a domain of quantities, or numbers, subject to an ordering relation of greater or less. This domain includes the ordinary finite quantities, but it is also presumed to contain infinitesimal and infinite quantities—a quantity being infinitesimal, or infinite, when it is smaller, or, respectively, greater, than any arbitrarily given finite quantity. The whole domain is governed by a version of the Archimedean principle to the effect that zero is the only quantity incapable of being multiplied sufficiently many times to equal any given quantity. Infinitesimal quantities may be characterized as quotients b/m of a finite quantity b by an infinite quantity m. In contrast with Leibniz's differentials, Nieuwentijdt's infinitesimals have the property that the product of any pair of them vanishes[8]; in particular squares and all higher powers of infinitesimals are zero. This fact enables Nieuwentijdt to show that, for any curve given by an algebraic equation, the hypotenuse of the differential triangle generated by an infinitesimal abscissal increment e coincides with the segment of the curve between x and $x + e$. That is, a curve is locally straight, or, in 17th century parlance, an "infinilateral polygon".

In responding to Nieuwentijdt's assertion that squares and higher powers of infinitesimals vanish, Leibniz remarked that "it is rather strange to posit that a segment dx is different from zero and at the same time that the area of a square with side dx is equal to zero." Yet this oddity may be regarded as a consequence—apparently unremarked by Leibniz himself—of one of his own key principles, namely that curves may be considered as infinilateral polygons. For consider the curve $y = x^2$ below. Given that the curve is an infinilateral polygon, the infinitesimal straight portion of the curve between the abscissae 0 and dx must coincide with the tangent to the curve at the origin—in this case, the axis of abscissae—between those two points. But then the point (dx, dx^2) must lie on the axis of abscissae, which means that dx^2 = 0.



[8] Here Nieuwentijdt's theory conflicts with **SIA**, for in the latter it is not hard to refute the assertion that the product of any pair of infinitesimals vanishes. For more on this see Bell (forthcoming).

Now Leibniz could retort that this argument depends crucially on the assumption that the portion of the curve between abscissae 0 and d*x*, while undoubtedly infinitesimal, is indeed *straight*. If this be denied, then of course it does not follow that $dx^2 = 0$. But still, if one grants, as Leibniz does, that there is an infinitesimal portion of the curve between abscissae 0 and e (say) which is straight and does not reduce to a single point (so that e cannot be equated to 0), then the above argument does show that $e^2 = 0$. It follows that, *if curves are infinilateral polygons, then the "lengths" of the sides of these latter must be nilsquare infinititesimals.*[9] Accordingly, to do full justice to Leibniz's conception, *two* sorts of infinitesimals are required: first, "differentials" obeying —as laid down by Leibniz—the same algebraic laws as finite quantities; and second, the (necessarily smaller) nilsquare infinitesimals which measure the lengths of the sides of infinilateral polygons. It may be said that Leibniz recognized the need for the first, but not the second type of infinitesimal and Nieuwentijdt, *vice-versa.* It is of interest to note that Leibnizian infinitesimals (differentials) are realized in *nonstandard analysis*,[10] the other major modern account of mathematical analysis built on a theory of infinitesimals. In fact it has been shown to be possible to construct models of **SIA** which at the same time embody enough of the theory of nonstandard analysis[11] to allow for the presence of Leibnizian infinitesimals in addition to the nilsquare variety.

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[9] This is essentially the converse of Nieuwentijdt's observation above.
[10] See Robinson (1996).
[11] See Moerdijk and Reyes (1991)

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