Frege's Theory of Concepts and Objects and the Interpretation of Second-order Logic†

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But when she got there
The cupboard was bare

In a recent paper,[1] Jaakko Hintikka and Gabriel Sandu have sought to redress what they perceive to be the injustice that Frege is today so highly esteemed as a philosopher of mathematics:[2] They argue that to the extent that Frege lacked the notions of an arbitrary set and an arbitrary function, he also lacked the concept of an interpretation of the second-order quantifier which associates with it all properties of the individuals in the domain of the first-order quantifier. Hintikka and Sandu first present an analysis of Frege's notion of the extension of a concept in order to show that Frege cannot generate, in the usual way, *all sets*. Next they claim that this analysis, together with certain other, hitherto unrecognized features of Frege's view of real-valued functions, makes it plausible that Frege did not have the concept of *all properties*—only so many as are needed to generate a fragment of the power set of the domain of individuals—so that Frege effectively worked within a 'nonstandard' interpretation of second-order logic. According to Hintikka and Sandu, Frege never noticed that

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[1] 'The skeleton in Frege's cupboard: The standard versus nonstandard distinction', *Journal of Philosophy* **89** (1992), 290–315.

[2] Hintikka and Sandu's summary remarks give the flavor of their paper:

> ...Frege failed to relate his own work to much ...of the foundational work that really mattered to actual mathematical research and to mathematicians' own efforts to clarify the fundamentals of their own discipline. It seems to us unfortunate that philosophers habitually go to Frege for their problems and for the concepts that could help us to cope with them. Frege was far too myopic to be a fruitful source of concepts, ideas, and problems. (p. 315)

many of the major accomplishments of logicism—accomplishments like his own definition of the ancestral and Dedekind's proof of the categoricity of the Peano Postulates—require the full power set, what we today call 'the standard interpretation' of second-order logic. This is the 'skeleton in Frege's cupboard' Hintikka and Sandu claim to have uncovered.

We believe that in virtually all matters of importance, Hintikka and Sandu's discussion is incomplete or misleading. We have, however, limited ourselves to three main objectives. Our first objective is to cast doubt on Hintikka and Sandu's account of Frege's theory of concepts and extensions by showing that it misses one of Frege's most important contributions to the subject. This is taken up in Section 1 and elaborated in the Appendix. Next we show that Hintikka and Sandu have not provided us with any reason for believing Frege was committed to anything like a nonstandard interpretation of his second-order quantifiers; the remarks on the development of the concept of a real-valued function, which they offer in support of this contention, are at best misleading. We conclude with a brief discussion of Hintikka and Sandu's claim that the standard interpretation is necessary for the derivation of the Peano Postulates and the proof of their categoricity.

1. Generating new sets

In the section of their paper entitled 'Frege lacked the notion of arbitrary set', Hintikka and Sandu write:

> The inseparable Fregean linkage between sets and their defining properties prevented [Frege] from generating new sets by operations acting solely on the members of the already existing sets. (Hintikka and Sandu, p. 302.)

Hintikka and Sandu elaborate with a quote from Hao Wang [3] which they say 'emphasizes' the point just made. But Wang is far less committal than Hintikka and Sandu on the question whether Frege's theory of extensions allows for the generation of new sets by operations acting solely on the members of already existing sets. Wang says only that Frege's conception 'has little positive to say about how sets are generated', while Hintikka and Sandu claim that Frege's theory actually *prevents* him from saying anything about the generation of sets by such operations. So Wang can hardly be said to have *emphasized* Hintikka and Sandu's point, and as we shall see, even Wang's weaker claim is seriously misleading. It is in any case not entirely clear what point Hintikka and Sandu are making, since they

[3] '[Frege] identifies sets with their extensions and treats them as individuals (on the same level with individuals). This suggests immediately the idea of a type hierarchy of extensions, since extensions of predicates seem to be more closely related to predicates than to individuals. ... But such a conception has little that is positive to say about how sets are generated.' (Hao Wang, *From mathematics to philosophy* (New York: Routledge, 1974), p. 210. Quoted by Hintikka and Sandu, p. 302.)

nowhere explicitly specify what operations they have in mind. Presumably they mean operations like the formation of unordered pairs. Thus in the standard statement of the Axiom of Infinity for ZF we implicitly appeal to the free use of such an operation when we pass from a set x to its 'successor set' $x \cup \{x\}$. Evidently, the availability of this set is essential to the definition of the infinite set whose existence the axiom postulates. In the Appendix we explain in detail how one of the principal achievements of the *Grundlagen*, *viz.*, Frege's proof of the infinity of the natural number sequence, can be viewed as a contribution to set theory: Frege in effect discovered an 'axiom of infinity' which, unlike the standard axiom of ZF, does not explicitly *postulate* the existence of an infinite set, but allows for its *derivation*. This undeservedly neglected achievement is comparable to Zermelo's proof of the Well-ordering Theorem: just as Zermelo's formulation of Choice makes no explicit mention of well-ordering, so Frege's formulation of Infinity—what George Boolos[4] has called 'Hume's Principle'—makes no explicit mention of an infinite set. From a contemporary point of view, Zermelo's proof of well-ordering and Frege's proof of the existence of a Dedekind-infinite set are very closely related: as shown in the Appendix, both follow from the same fundamental lemma of Bourbaki's analysis of the Well-ordering Theorem.

Thus while Hintikka and Sandu are correct in claiming that Frege does not generate new sets by operations like the formation of unordered pairs, this makes absolutely no difference to the ability of Frege's theory to prove the existence of sets which must simply be assumed on the standard account. Moreover, Frege achieves this by a subtle appeal to a *consistent fragment* of his theory of concepts and their extensions. This appeal—*contra* Wang—depends essentially on the fact that Fregean extensions do *not* constitute a hierarchy. If the stratification of predicates was 'immediately suggestive' to others of a stratification of the corresponding extensions, this suggestion was always resisted by Frege. And for good reason: Without the assumption that extensions are objects on the same 'level' as individuals—*i.e.*, the assumption that extensions are arguments to first-level concepts—Hume's Principle would not have the remarkable properties Frege showed it to have. Thus whether or not Frege employs set-forming operations of the sort we find in the usual, axiomatic, development of the set concept is irrelevant to the ability of this fragment of his theory of concepts and their extensions to generate new sets from the members of existing sets.[5] To be sure, Frege achieves this only through the *mediation*

[4] 'The standard of equality of numbers', in *Meaning and method: essays in honor of Hilary Putnam*, George Boolos, ed. (Cambridge: Cambridge University Press, 1990).

[5] As Boolos has observed, Frege's derivation of the existence of a Dedekind-infinite set anticipates the construction of the finite von Neumann ordinals. *Cf.* 'The standard of equality of numbers', previously cited.

of concepts, and in this respect, his approach is certainly different from the iterative conception. However this should not obscure the fact that Frege achieved an infinite set by a *proof* where others had required an axiom. This achievement deserves to be recognized as a significant contribution to the subject, and the argument of the Appendix shows how 'Frege's Theorem' may be naturally incorporated into the conceptual structure of set theory.

2. Frege on the notion of a real-valued function

Hintikka and Sandu correctly point out that the development of the idea of an arbitrary set went hand in hand with the elaboration of the notion of an arbitrary real-valued function. They also correctly observe that both developments bear on our present understanding of the range of the second-order variables of quantification, since this is based on the notion of the power set of the domain over which the first-order variables range. Hintikka and Sandu recognize that it would be an anachronism to apply the terminology of standard and nonstandard interpretations directly to Frege, since he did not possess the model-theoretic framework within which we today draw this distinction. They therefore set out to transform their observations on *set*, *function*, *standard and nonstandard interpretation* into a criticism of Frege by claiming that the chief stumbling block to the development of our current conception of sets and functions came from their assimilation to 'intensional' notions like Fregean concepts and Fregean functions:

> What made possible the conception of an arbitrary set was the gradual disentanglement of the notion of set from intensional ingredients such as concepts, properties, etc., and the definition of sethood in alternative ways. (p. 305)

Now there is an obvious sense in which this remark is correct: In its full generality, the Fregean notion of the extension of a concept proved to be paradoxical; it therefore *had* to be overcome if there was to be a coherent notion of set. But this is not Hintikka and Sandu's point. Rather the objection they raise against Frege is supposed to hold independently of Russell's paradox, since they maintain that Frege's notions of concept and extension, even if coherent, were not of sufficient generality to represent the concept of an arbitrary set. They argue that if for Frege sets are represented by extensions of concepts, and concepts are, in turn, treated as a special type of function, then the adequacy of Frege's account ultimately rests on his notion of a function. Hintikka and Sandu therefore address the question of the adequacy of Frege's notion of the extension of a concept by considering whether his notion of a function was of sufficient generality to capture the concept of a real-valued function in the form in which it emerged in nineteenth century analysis. Their claim that Frege *did* restrict the notions of concept and extension rests on their analysis of Frege's notion of a real-valued function; indeed, this is what is novel about Hintikka and

Sandu's approach. It should be clear that *some* subtlety is required, since it would be obviously absurd to suggest that when, for example, Frege defined 'the concept F is equinumerous with the concept G' by the condition,

> ... there exists a relation ϕ which correlates one to one the objects falling under the concept F with the objects falling under the concept G,[6]

he intended a restriction on the range of the variable of existential quantification. To have done so would have been completely at variance with his claim to have given an analysis of statements of number which respects their generality. This is not how Hintikka and Sandu argue.

Hintikka and Sandu begin their discussion of Frege's concept of a function by granting Michael Dummett's contention that for Frege functions belong to the realm of reference, so that despite the terminology of, for example, the *Begriffsschrift*, Frege's functions are not to be identified with linguistic expressions. Their point, against Dummett, is that it does not follow from this concession that Frege did not, on some *other* ground, restrict the class of real-valued functions, and therefore, given his assimilation of concepts to functions, thereby restrict the interpretation of the second-order quantifier.[7]

The textual basis for Hintikka and Sandu's contention that Frege had a restricted notion of function, and thus was in this way committed to a nonstandard interpretation, is a passage from his 1904 paper, 'What is a function?'

> Our general way of expressing such a law of correlation is an equation, in which the letter 'y' stands on the left side whereas on the right there appears a mathematical expression consisting of numerals, mathematical signs, and the letter 'x', *e.g.*:
>
> '$y = x^2 + 3x$'.
>
> Functions have indeed been defined as being such mathematical expressions. In recent times this concept has been found too narrow. However, this difficulty could easily be avoided by introducing new signs into the symbolic language of arithmetic.[8]

It is not immediately clear what Frege means by 'new signs'. Of the various possible options, Hintikka and Sandu characteristically choose the al-

[6] *Grundlagen*, §72. The quotation is from the translation by J. L. Austin: *The foundations of arithmetic*, (Evanston: Northwestern University Press, 1980).

[7] While this observation of Hintikka and Sandu's is correct, it hardly justifies their rather ungenerous attribution that Dummett is assuming that the 'only... possible nonstandard interpretation [is] the one Henkin... considered'. (Hintikka and Sandu, p. 304.) It seems more likely that Dummett is assuming that this is the only non-standard interpretation *which there is any textual reason, however slight, to attribute to Frege*. This is a not unnatural assumption given the intellectual context of Dummett's discussion, and considering the fact that this interpretation of the *Begriffsschrift* continues to be advanced even today.

[8] *Gottlob Frege: Collected papers on mathematics, logic, and philosophy*, Brian McGuinness, ed. (Oxford: Basil Blackwell, 1984), pp. 289f. The translation is by Peter Geach. Quoted by Hintikka and Sandu, pp. 312f.

ternative that is the least favorable to Frege when they remark that this quotation

> ... boggles the mind in that it demonstrates Frege's alienation from the actual working problems in the foundations of mathematics of his time. Frege is in effect telling Dirichlet, Weierstrass, Dedekind,... that the problems they have been agonizing about could easily be solved by introducing new signs into the notation of arithmetic. (p. 313)

Evidently, by 'the introduction of new signs', Hintikka and Sandu take Frege to mean the introduction of new function names, since they write:

> ... there is no niche in [Frege's] world for... an arbitrary function in the sense in which Euler, Czuber, and others were thinking of this notion. For to think that the range of such arbitrary functions could be covered by introducing new signs into the notation of arithmetic is not to understand this notion. (p. 313)

So while Hintikka and Sandu grant that Frege is committed to the 'reality' or non-linguistic character of functions, they present him as having held that only those functions exist for which there are names in the 'symbolic language of arithmetic'—and this falls far short of the totality of *all* functions.

To claim, on this basis, that Frege so restricted the class of real-valued functions, places a rather heavy interpretive burden on a somewhat offhand remark, considerably more than the passage can reasonably be expected to bear. First of all, the context of Frege's remark—an explicit reference to the limitations that arise when the means for representing real-valued functions are restricted to finite polynomials—suggests that by the addition of new signs, Frege is referring to the admission of new algebraic expressions in the account of 'analytically representable.' This is a wholly different matter from the introduction of new function names when it includes the use of expressions for infinitary operations and the manipulation of series. But the briefest familiarity with what was occurring in mathematical analysis during the fourth quarter of the nineteenth century makes it highly implausible that Frege would have meant anything *else* by 'new signs'. So understood, Frege's remark about avoiding too narrow a specification of the class of functions would be an allusion to results of the form, 'If the notion of an algebraic expression is extended (say by allowing expressions for limiting processes, infinite sums of various types of series, or whatever), then the class of "analytically representable" functions can be expanded to include all functions of a particular class'. The names of many prominent mathematicians, including Dirichlet and Weierstrass, and later, Baire and Lebesgue, were associated with sometimes quite striking theorems of this kind.[9] Neither Frege, nor any other mathematician working in Germany at the time, would have been unaware of this research. Modulo the casual-

[9] For some examples, see Section 12 ('The analytical representation of functions') of the paper by A. P. Youschekvitch, 'The concept of function up to the middle of the 19th

ness of Frege's formulation, the mathematical tradition of which Hintikka and Sandu think him ignorant thoroughly vindicates his allusion to the *plasticity* of the concept of an 'analytically representable function', so that there is simply no justification for supposing that in this passage Frege is proposing any restriction on the class of real-valued functions.

3. The concept of a function in nineteenth century analysis

The development of the function concept in nineteenth-century analysis is of considerable importance for the history and philosophy of logic and mathematics, and Hintikka and Sandu's paper is one of only a small number of recent discussions which attempt to place this development in its proper philosophical context. It may therefore be worthwhile to remark briefly on their suggestion that the real conceptual innovation lay in achieving the definition of a real-valued function as simply a many-one correspondence between real numbers. Hintikka and Sandu point out that this definition may have been known to Euler as early as 1755. Now in fact, Euler gave *many* characterizations of the function concept, went back and forth among his various characterizations, and seems not to have been particularly committed to this one.[10] In any case, the real innovation did not consist in perceiving this simple definition. The achievement of the nineteenth century development of the function concept consisted in recognizing the utility, and more importantly, the *necessity*, of such a general definition. This was something arrived at only very gradually, and it certainly was not accomplished by the mere formulation of the idea that a function is a many-one correspondence. It is generally agreed that while Dirichlet and Lobachevski's contributions to this development were of unquestionable importance, the acceptance of the general definition of a function was due to Riemann. Thus Bottazzini[11] remarks that

> if for Dirichlet... the generality of the definition [of a real-valued function] did not go along with a consistent practice in the study of equally 'general' functions, the opposite is true for Riemann. In his ... *Habilitationsschrift* of 1854, ... he revealed to the mathematical world a universe extraordinarily rich in 'pathological' functions.

century', *Archive for the history of exact science* **16** (1975/76), 37–85. (Hintikka and Sandu's reference to this paper is incorrect.)

[10] For example, of his controversy with D'Alembert over the correct analysis of the motion of vibrating strings, Clifford Truesdell writes:

> ...it is clear from every one of the many examples and discussions given by Euler that for him a 'function' is what we now call a continuous function with piecewise continuous slope and curvature.

The passage is quoted by Ugo Bottazzini, *The higher calculus: a history of real and complex analysis from Euler to Weierstrass* (New York and Berlin: Springer-Verlag, 1986), p. 27. (Hintikka and Sandu's reference to Bottazzini is incorrect.)

[11] *The higher calculus: a history of real and complex analysis from Euler to Weierstrass*, p. 217.

Bottazzini goes on to say that when one looks at the historical development,

> the modern concept of a function of a real variable, in its full generality, began to emerge in mathematical practice only towards the end of the 1860s as Riemann's writings became more widely known.

In this connection, it should be noted, if only because the account given by Hintikka and Sandu might suggest a different impression, that proofs of the existence of pathologies often proceeded by showing how they arise from the manipulation of *non-pathological* objects, in accordance with thoroughly canonical (although, possibly infinitary) operations—in other words, by showing how some notion of *analytically representable* yields a pathological function.[12]

The historical situation in the nineteenth century seems therefore to have been roughly this: since the physical analysis of motion provided the principal source of examples of real-valued functions, it seemed implausible that anything so general as the notion of an arbitrary correspondence could exhaust what might usefully be said in a definition of the notion. The eighteenth-century controversy over the analysis of the motion of vibrating strings, though a problem of applied mathematics, was of pivotal importance for the development of pure mathematics because it brought to the forefront the problem of characterizing what was to be included in the notion of a real-valued function: A question arose concerning the admissible restrictions on the initial shapes and initial velocities of plucked strings, and this led naturally to the problem of characterizing what might count as an 'arbitrary' curve. The existence of an intension, or even a description in the language of analysis, was not in question. But the availability of techniques which would permit the effective and rigorous treatment of general solutions *was*.[13]

The pathological functions of later developments were pathological because they ran counter to expectations that were based on what was thought to be geometrically and kinematically obvious. All the paradigm examples of pathologies traded on the presentation of functions and curves with geometrically 'unimaginable' properties: continuous but nowhere differen-

12 For example, from their correct remark (on p. 297) that

> [t]he innocent looking Eulerian definition of an arbitrary function turned out to admit of surprisingly unruly denizens of the world of mathematics... [and] many of the results that had been taken more or less for granted turned out to hold only for functions with specifiable properties

a reader, unfamiliar with the history, would naturally miss the fact that the pathologies arose *within* characterizations of the concept of a real-valued function that were far more restrictive than the notion of a many-one correspondence.

13 For a thorough discussion of the methodological issues surrounding the controversy over the analysis of the vibratory motion of strings, together with applications to the theory of universals, see Mark Wilson, 'Honorable intensions,' in S. Wagner and R. Warner, eds, *Naturalism* (South Bend: Notre Dame University Press, *forthcoming*). The historical situation is well summarized by Bottazzini, Chapter 1, Section 3.

tiable, everywhere discontinuous, *etc.* The stumbling block which the nineteenth century 'extensionalists' had to overcome was not some general intensionalist doctrine, but an over-reliance on 'intuitions', largely geometrical in nature, that gave rise to expectations concerning the notion of functionality which turned out not to be fulfilled. The entire nineteenth-century trend toward 'rigor' was an attempt to free analysis of any dependence on such geometrical and kinematical constraints, and Frege's contributions to the foundational discussion fall squarely within this tradition.[14] Again, the difficulty was not with the *acceptability* of the definition of a real-valued function as an arbitrary many-one correspondence, but with the fact that this notion proved intractable to any simple theory. The idea that there should be such a simple theory motivated the search for alternative, more substantial, definitions of the concept. In some sense, the problems faced by Frege's Basic Law V form the final chapter in the search for a such a theory.

4. The necessity of the standard interpretation

By way of motivating their discussion of Frege's sensitivity to the issues raised by the standard/nonstandard distinction, Hintikka and Sandu claim that

> ... only the standard interpretation of second-order logic enables us to use it for the most important purposes it can serve in the foundations of mathematics. If we assume the standard interpretation, we can easily formulate... categorical axiomatizations for... number theory and the theory of real numbers. Likewise, the principle of mathematical induction is easily formulated in a second-order logic with standard interpretation, but cannot be formulated in an axiomatizable nonstandard higher-order logic.[15] Thus, for most foundational purposes we have to assume the standard interpretation of second-order logic. (p. 295)

The reason why it is correct to say that Frege proceeded without the standard/nonstandard distinction is *not*, as Hintikka and Sandu argue, because he restricted the range of the variables of second-order quantification, but because he made the assumption, which we today recognize to be false, that the notion of the extension of a concept, and therefore the semantical

[14] This trend toward 'rigor' was often expressed (somewhat misleadingly) in anti-Kantian terms when what was really at stake was the autonomy of arithmetic, where 'arithmetic' was always understood in a broad sense which included not only the theory of the natural numbers, but real and complex analysis, as well. While it evidently bears on Kant's views, this concern with autonomy is capable of being motivated independently of the details of Kantian epistemology. See Michael Dummett, *Frege: philosophy of mathematics*, (Cambridge: Harvard University Press, 1991), and William Demopoulos, 'Frege and the rigorization of analysis', *Journal of Philosophical Logic*, *forthcoming*, where this is argued at some length.

[15] Here we understand Hintikka and Sandu to be saying, 'cannot be formulated with its intended meaning'.

framework of his logic, can be developed independently of any particular mathematical theory. This is what Basic Law V would have permitted had it turned out to be consistent, and this is what was clearly required by his understanding of the traditional doctrine of the 'topic-neutrality' of logic. By contrast, we now see that the standard/nonstandard distinction must be understood relative to ZF or some other mathematical theory, sufficiently rich to express the underlying semantical interpretation that we associate with second-order logic. Although it can hardly be said to be uncontroversial, it can certainly be convincingly argued that, had the derivation of the Peano Postulates and the proof of their categoricity been established within a framework of the sort Frege envisaged, this would have been a result of considerable foundational significance. But if, in order to characterize the natural numbers, we have to assume, for the specification of our logical framework, the theory of *transfinite* ordinals (which is, after all, the very *least* that ZF gives us), it is unclear what, of foundational interest, has been accomplished. Certainly Hintikka and Sandu have not told us. That Frege perceived the difficulty, at least in general terms, is clear, since it was this that led him to abandon his program.

'Absolute' results, such as the categoricity of Peano Arithmetic, are established in the strong sense which Hintikka and Sandu evidently intend only if the notion of an arbitrary subset of N is taken in a commensurately 'absolute' sense. A similar remark, apparently first noted in print by Skolem,[16] applies in the case of the principle of mathematical induction. The assumption that the power set of N is 'absolute' (*i.e.*, the assumption that we perfectly well know the referent of 'PN') is strong enough to justify a healthy skepticism concerning the 'added value' of the standard-model theorems, as compared with their set-theoretic analogues. Finally we note that when assessing such results, it should be borne in mind that in the case of *all* the standard-model theorems, the mathematically fruitful observations, on which the original proofs depend, carry over intact to their first-order versions; what proved to be of mathematical importance in the original 'constructions', is therefore completely independent of whether these constructions are understood only 'internally' (from within a model of ZF), or from the 'outside'.

[16] See Wang's 'Survey of Skolem's work in logic', which appears as the introduction to Jens Erik Fenstad, ed. *Thoraf Skolem: selected works in logic* (Oslo: Universitetforlaget, 1970), p. 41. The relevant paper of Skolem ('Über einige Grundlagenfragen der Mathematik', cited by Hintikka and Sandu) is reprinted in this volume; see §7.

Appendix: Frege's Theorem

The chief purpose of this Appendix is to establish the existence of an infinite well-ordering as a (hitherto unremarked) consequence of a general version of Zermelo's Well-ordering Theorem. We will also show how both this fact and the existence of the natural number system can be derived along 'Fregean' lines within a certain system $\mathbb{F}$ of many-sorted first-order logic whose sorts correspond to Frege's domains of objects, relations, and first and second level concepts. The system of axioms we formulate within $\mathbb{F}$ constitute a consistent fragment of Frege's original (inconsistent) system sufficient for the development of arithmetic.

1. The system $\mathbb{F}$

We specify the basic constituents of $\mathbb{F}$.

Sorts (or domains)

$\mathcal{O}$ — objects
$\mathcal{B}$ — basic (first-level) concepts
$\mathcal{R}$ — relations
$\mathcal{S}_b$ — second level concepts
$\mathcal{S}_r$ — second level relational concepts

Variables and Constants

Sort	*Variable*	*Constant*
$\mathcal{O}$	$x, y, z, \ldots$	$a, b, c, \ldots$
$\mathcal{B}$	$X, Y, Z, \ldots$	$A, B, C, \ldots$
$\mathcal{R}$	$\underline{X}, \underline{Y}, \underline{Z}, \ldots$	$\underline{A}, \underline{B}, \underline{C}, \ldots$
$\mathcal{S}_b$	$\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{Z}, \ldots$	$\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \ldots$
$\mathcal{S}_r$	$\underline{\underline{X}}, \underline{\underline{Y}}, \underline{\underline{Z}}, \ldots$	$\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \ldots$

A *term* is a variable or a constant or one of the concept or relation or extension terms to be introduced shortly. A variable of sort $\mathcal{B}$ or $\mathcal{S}_b$ will be called a *concept variable* for brevity.

We assume the presence of an *identity* sign $=$ yielding atomic statements of the form $s = t$ where s and t are terms of the *same* sort. On all domains except $\mathcal{O}$, $=$ is to be thought of as *intensional* equality.

We also assume the presence of a *predication* sign η yielding atomic statements of the form $s\,\eta\,t$, $(s't')\,\eta\,u$ where s is of sort $\mathcal{O}$, $\mathcal{B}$, $\mathcal{B}$, $\mathcal{S}_b$, $\mathcal{S}_r$ respectively; and s', t' are both of sort $\mathcal{O}$ and u is of sort $\mathcal{R}$. We read '$s\,\eta\,t$' as '*s falls under t*'.

We shall assume the following comprehension scheme for concepts:

Corresponding to any formula $\Phi(x)$, $\Phi(x, y)$, $\Phi(X)$ or $\Phi(\underline{X})$ we are given a term s of sort $\mathcal{B}$, $\mathcal{R}$, $\mathcal{S}_b$, $\mathcal{S}_r$, respectively, for which we adopt as an axiom *the formula*

$$\forall \begin{Bmatrix} x \\ xy \\ X \\ \underline{X} \end{Bmatrix} \left[\begin{Bmatrix} x \\ (xy) \\ X \\ \underline{X} \end{Bmatrix} \eta\, s \leftrightarrow \Phi \begin{Bmatrix} x \\ xy \\ X \\ \underline{X} \end{Bmatrix} \right].$$

We write $\hat{x}\Phi$, $(xy)^\frown\Phi$, $\hat{X}\Phi$, $\hat{\underline{X}}\Phi$ for s, as the case may be. A term of the first, third and fourth types is called the *concept* (term) determined by Φ, and a term of the second type the *relation* (term) determined by Φ.

We define the relation $\equiv$ of *extensional equality* on the domains $\mathcal{B}$, $\mathcal{R}$, $\mathcal{S}_b$, $\mathcal{S}_r$ by

$$X \equiv Y \Longleftrightarrow_{df} \forall x(x\,\eta\,X \leftrightarrow x\,\eta\,Y),$$
$$\underline{X} \equiv \underline{Y} \Longleftrightarrow_{df} \forall x \forall y[(xy)\,\eta\,\underline{X} \leftrightarrow (xy)\,\eta\,\underline{Y}],$$
$$\underset{\sim}{X} \equiv \underset{\sim}{Y} \Longleftrightarrow_{df} \forall X[X\,\eta\,\underset{\sim}{X} \leftrightarrow X\,\eta\,\underset{\sim}{Y}],$$
$$\underline{\underline{X}} \equiv \underline{\underline{Y}} \Longleftrightarrow_{df} \forall \underline{X}[\underline{X}\,\eta\,\underline{\underline{X}} \leftrightarrow \underline{X}\,\eta\,\underline{\underline{Y}}].$$

Clearly concepts are determined uniquely by formulas up to extensional equality. We assume that $\mathbf{F}$ contains

- a term e such that $e(\mathfrak{X})$ is well-formed and of sort $\mathcal{O}$ for any concept variable $\mathfrak{X}$;
- a predicate symbol E such that $E(\mathfrak{X})$ is well-formed for any concept variable $\mathfrak{X}$.

We finally assume the axioms

1) $\forall \mathfrak{X} \forall \mathfrak{Y}[E(\mathfrak{X}) \wedge E(\mathfrak{Y}) \rightarrow [e(\mathfrak{X}) = e(\mathfrak{Y}) \leftrightarrow \mathfrak{X} \equiv \mathfrak{Y}]]$,

2) $\forall \mathfrak{X} \forall \mathfrak{Y}[E(\mathfrak{X}) \wedge \mathfrak{X} \equiv \mathfrak{Y} \rightarrow E(\mathfrak{Y})]$,

where in both 1) *and* 2) $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are concept variables of the same sort.*

If we think of $e(\mathfrak{X})$ as an object *representing* $\mathfrak{X}$, Axiom 1 above expresses the idea that *extensional equality* of any concepts satisfying E is equivalent to *identity* of their representing objects. That is, for any concept $\mathfrak{X}$ satisfying E, $e(\mathfrak{X})$ may be regarded as the *extension* of $\mathfrak{X}$. And the predicate E itself represents the property of *possessing an extension*. For these reasons Axiom 1 will be called the *Axiom of Extensions*. As for Axiom 2, it states the reasonable requirement that any concept extensionally equivalent to a concept possessing an extension *itself* possesses one (that is, $\equiv$ is a *congruence* relation with respect to E).

2. The Zermelo-Bourbaki Lemma and Frege's Theorem

A straightforward Russell type argument in $\mathbf{F}$ enables us to infer $\neg\forall\mathfrak{X}E(\mathfrak{X})$, that is,[17] *not every concept possesses an extension.* This being the case,

[17] To be explicit, define $A =_{df} \hat{x}[\forall X[e(X) = x \wedge E(X) \rightarrow \neg x\,\eta\,X]]$. Then $\neg E(A)$ is inferrable in $\mathbf{F}$.

what concepts do we need to (consistently) assume possess extensions in order to enable an infinite well-ordering to be constructed? It was Frege's remarkable *discovery* that for this it suffices to assume just that extensions be possessed by the members of a certain class of simple and natural *second-order* concepts—those that, following Boolos,[18] we shall term *numerical.*

Numerical concepts are defined as follows. First, we formulate the relation $\approx$ of equinumerosity or equipollence on $\mathcal{B}$ as usual:

$$\begin{aligned} X \approx Y \Longleftrightarrow_{df} \exists \underline{Z}[&\forall x \forall y[(xy)\,\eta\,\underline{Z} \to x\,\eta\,X \wedge y\,\eta\,Y] \\ &\wedge \forall x \forall y \forall z[(xy)\,\eta\,\underline{Z} \wedge (xz)\,\eta\,\underline{Z} \to y = z] \\ &\wedge \forall x[x\,\eta\,X \to \exists y\,(xy)\,\eta\,\underline{Z}] \\ &\wedge \forall y[y\,\eta\,Y \to \exists x\,(xy)\,\eta\,\underline{Z}]. \end{aligned}$$

With any basic concept X we associate the second level concept

$$\|X\| =_{df} \widehat{Y}\,[X \approx Y].$$

Concepts of the form $\|X\|$ are called *numerical.*

If we assume that every numerical concept possesses an extension (i.e., $\forall X E(\|X\|)$, then the extension

$$|X| =_{df} e(\|X\|)$$

is called the (cardinal) *number* of X. Objects of the form $|X|$ are called (cardinal) *numbers.* Under these assumptions it is easy to derive what Boolos calls *Hume's principle,* viz.

$$\forall X \forall Y\,[X \approx Y \leftrightarrow |X| = |Y|].$$

We shall call a concept X (Dedekind) *infinite* if $\exists Y\,[Y \subsetneqq X \wedge X \approx Y]$, where $Y \subsetneqq X$ of course stands for $\forall x(x\,\eta\,Y \to x\,\eta\,X) \wedge Y \not\equiv X$. Objects of the form $|X|$ with X infinite are called *infinite numbers.*

We are going to show how, in **F**, the existence of an infinite well-ordering (*i.e.*, an infinite well-ordered concept) may be derived as a special case of a general set-theoretic result—formulable and provable in **F**—which is normally used to derive Zermelo's Well-ordering Theorem. In its original form this result is what we shall call the

Zermelo-Bourbaki Lemma.[19] *Let E be a set, $\mathcal{F}$ a family of subsets of E and $p : \mathcal{F} \to \mathcal{E}$ a map such that $p(X) \notin X$ for all $X \in \mathcal{F}$. Then there*

[18] 'The standard of equality of numbers'.

[19] Lemma 3, §2, Ch. 3 of N. Bourbaki, *Théorie des ensembles*, 2nd Ed. (Paris: Hermann, 1963).

is a subset M of E and a well-ordering $\leq$ of M such that, writing S_x for $\{y : y < x\}$,

(i) $\forall x \in M\,[S_x \in \mathcal{F} \wedge p(S_x) = x]$

(ii) $M \notin \mathcal{F}$.

Bourbaki employs this result to construct an elegant derivation of Zermelo's Well-ordering Theorem from the Axiom of Choice. In the present context, however, it will be used to produce an equally elegant proof of what we shall call, following a suggestion of Boolos,

Frege's Theorem. *Suppose given a set E and a map $n : PE \to E$ such that*

$$\forall X \subseteq E\, \forall Y \subseteq E\,[n(X) = n(Y) \leftrightarrow X \approx Y]. \qquad (*)$$

Then E has an infinite well-ordered subset.

Proof. We apply the Zermelo-Bourbaki Lemma with $\mathcal{F}$ the family of all subsets X of E for which $n(X) \notin X$ and p the map n. We obtain $M \subseteq E$ and a well-ordering $\leq$ of M such that (*i*) $n(S_x) = x$ for all $x \in M$, (*ii*) $n(M) \in M$. Writing m for $n(M)$ we have $m \in M$ by (*ii*), whence $n(S_m) = m = n(M)$ by (*i*). Condition $(*)$ now yields $S_m \approx M$. Since $m \notin S_m$, S_m is a proper subset of M and it follows that the latter is infinite. □

Now both of these results can be translated into and proved within $\mathbb{F}$. Carrying this out for the Zermelo-Bourbaki Lemma yields the

Zermelo-Bourbaki Lemma in $\mathbb{F}$. *Let $\underset{\sim}{S}$ be any second-level concept with respect to which $\equiv$ is a congruence relation and t a term such that $t(X)$ is an object for all basic concepts X and satisfies*

$$\forall X \forall Y \left[X \equiv Y \wedge X\,\eta\,\underset{\sim}{S} \to t(X) = t(Y)\right]$$

$$\forall X \left[X\,\eta\,\underset{\sim}{S} \to \neg t(X)\,\eta\,X\right]$$

Then there is a relation R such that R is a well-ordering and, writing M for its field, and R_x for $\hat{y}\,[(yx)\,\eta\,R \wedge y \neq x]$,

(i) $\forall x \left[x\,\eta\,M \to R_x\,\eta\,\underset{\sim}{S} \wedge t(R_x) = x\right]$

(ii) $\neg M\,\eta\,\underset{\sim}{S}$.

In the case of Frege's Theorem, the same process yields

Frege's Theorem in $\mathbb{F}$. *Suppose that every numerical concept has an extension. Then there exists an infinite well-ordered concept and hence an infinite number.*

Since, as is well known, Frege's original system in the *Grundgesetze* was inconsistent, we should assure ourselves that the axioms of $\mathbb{F}$, together with the hypothesis of Frege's Theorem—that every numerical concept has an extension—are consistent. The easiest way to see this is by noting that the following set-theoretic interpretations yield a model of the axioms of $\mathbb{F}$ in which the hypothesis of Frege's Theorem holds. To wit, interpret $\mathcal{O}$ as $\omega + 1$, $\mathcal{B}$ as $P(\omega + 1)$, $\mathcal{R}$ as $P((\omega + 1) \times (\omega + 1))$, $\mathcal{S}_b$ as $PP(\omega + 1)$, $\mathcal{S}_r$ as $PP((\omega+1)\times(\omega+1))$, E as the subset of $PP(\omega+1)$ consisting of all elements of the form $\|X\| =_{df} \{Y \in P(\omega + 1) : X \approx Y\}$ with $X \in P(\omega + 1)$, and e as the map $PP((\omega + 1) \cup P(\omega + 1)) \to \omega + 1$ which sends each $\|X\|$ with $X \in P(\omega+1)$ to its cardinality $|X|$ $(\in \omega+1)$ and everything else to 0. Thus the axioms of $\mathbb{F}$—together with the hypothesis of Frege's Theorem—may be regarded as a consistent fragment of Frege's original system.

3. The Natural Number System in $\mathbb{F}$

In the *Grundlagen* Frege outlines a proof,[20] from principles similar to those laid down in $\mathbb{F}$, of the existence of the usual *natural number system*. One may accordingly ask how one would proceed to derive this fact within $\mathbb{F}$ from our version of Frege's Theorem. In fact, it is not hard to show, using the theory of ordinals, that the infinite well-ordered set $(M, \leq)$ obtained in (the set-theoretic version of) Frege's Theorem has order type $\omega + 1$. However, since the general theory of ordinals is not available in $\mathbb{F}$, the argument there is more involved and goes as follows. First, one establishes the Schröder-Bernstein Theorem in $\mathbb{F}$ and uses it to show that $|M|$ is the largest element of $(M, \leq)$. Next, one defines the *immediate successor* x^+ of x to be $|\widehat{y}\,(y \leq x)|$ and a subconcept of M to be *inductive* if it contains 0 $(=_{df}\ |\widehat{x}(x \neq x)|)$ and is closed under the operation of immediate successor. Defining N to be the common part of all inductive subconcepts of M, one finally shows that N satisfies the usual Peano axioms and that M is N together with a 'last element' $|M|$. These steps can all be carried out in $\mathbb{F}$, thus enabling the existence and essential properties of the natural number system to be established within it. (Details of this proof will be presented in a forthcoming paper.)

4. Ordinals and the Axiom of Infinity in $\mathbb{F}$

It is natural to apply the Zermelo-Bourbaki Lemma in the case where t is the *extension term* e and S the second-level 'Russellian' concept $\widehat{X}\,[E(X) \wedge \neg e(X)\, \eta\, X]$, i.e., the concept of all concepts whose extensions do not fall under them. This yields a well-ordered concept M_e such that each object x falling under it is the extension of the concept of being a predecessor of x. Thus, the objects falling under M_e are naturally construed as the *von Neumann ordinals* (and the well-ordering on M_e as the

[20] §§68–83. This proof is reconstructed in detail in Boolos, 'The standard of equality of numbers'. See also Crispin Wright, *Frege's conception of numbers as objects* (Aberdeen: Aberdeen University Press, 1983), pp. 154–69.

membership relation suitably defined.[21]) Under what conditions, it may be asked, is M_e *infinite*? (There is, after all, nothing so far to prevent M_e from being the *empty* concept!) A natural way of ensuring this is to postulate that certain concepts have extensions, specifically: the empty concept has an extension and, for any object x and any concept X possessing an extension the concept $\widehat{y}(y = x \vee y \, \eta \, X)$ also possesses one. (These correspond to the set-theoretical axioms of empty set and 'successor' sets respectively.) For under these conditions we can define the *immediate successor* x^+ of any object x such that $x \, \eta \, M_e$ to be the extension of the concept of being equal to or a predecessor of x; it is then easily shown that $x^+ \neq x$ for any $x \, \eta \, M_e$. Then the correspondence $x \mapsto x^+$ establishes a bijection of M_e with a proper subconcept, and so M_e is indeed infinite.

Now it is an easy consequence of conclusion (*ii*) of Zermelo-Bourbaki that M_e *cannot have an extension*. (This is essentially the Burali-Forti 'paradox'—the extension of M_e would have to be the largest ordinal.) So even the above assumptions are not sufficiently strong, in contrast with those underlying Frege's Theorem, to enable the existence of an infinite *object* to be established. Essentially the only way of strengthening these assumptions so as to obtain an infinite object is to *postulate* the Axiom of Infinity, which in this context would assert that there exists a subconcept of M_e which *a*) has an extension and *b*) contains 0 and is closed under immediate successor.

Finally, the relationship between 'Frege's Theorem' and Frege's actual construction of the natural number system as outlined in the *Grundlagen* can be stated in the following way. In the *Grundlagen* the cardinality function is first used to define the immediate successor relation and *then* one obtains from it the well-ordering of the natural numbers as its ancestral (transitive closure). The whole development involving 'Frege's Theorem' as presented here essentially *reverses* this procedure: the same cardinality function is used to obtain *first* the well-ordering and *then* the successor function. In both cases, however, the cardinality function is assumed to be universally defined and so leads to the same definition of immediate succession. To this extent, 'Frege's Theorem' is a faithful representation of the core of Frege's original derivation, which may therefore be regarded as an unrecognized special case of Zermelo's later procedure for constructing well-orderings.

21 E.g., by $x \in y \Longleftrightarrow_{df} \exists Y [e(Y) = y \wedge x \, \eta \, Y]$.

Note added in proof: For a further elaboration of the set and function concepts in the late nineteenth and early twentieth centuries, see the paper of John Burgess, Hintikka *et* Sandu *versus* Frege *in re* arbitrary functions', *This journal*, 3rd series **1** (1993), 50–65. On the matter of Frege's understanding of his second-order quantifiers see Richard Heck and Jason Stanley, 'Reply to Hintikka and Sandu: Frege and second-order logic', *Journal of Philosophy*, forthcoming. The present paper was written independently of these two important contributions to our understanding of these issues.

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ABSTRACT. This paper casts doubt on a recent criticism of Frege's theory of concepts and extensions by showing that it misses one of Frege's most important contributions: the derivation of the infinity of the natural numbers. We show how this result may be incorporated into the conceptual structure of Zermelo-Fraenkel Set Theory. The paper clarifies the bearing of the development of the notion of a real-valued function on Frege's theory of concepts; it concludes with a brief discussion of the claim that the standard interpretation of second-order logic is necessary for the derivation of the Peano Postulates and the proof of their categoricity.