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Non-shell Unstable Particles in Thermal Field Theory

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Using (nonrigorous) operator-algebraic and group-theoretic techniques the particle structure of interacting real-time thermal field theory is investigated. A description in terms of elementary unstable entities without a dispersion relation is arrived at. The asymptotic fields are found to be two-parameter generalized free fields. Contact with the Licht-field description of on-shell unstable particles is thereby achieved. Poincaré-symmetry breaking and the rearrangement of spin to helicity at finite temperature are fully discussed and incorporated. A unique thermal Gell-Mann/Low formula is obtained. Feynman rules and renormalization conditions pertaining to non-shell thermal particles are given. Dissipation thereby naturally emerges. All relevant esoteric mathematics is explained.

1. INTRODUCTION

In the course of 60 years quantum field theory has evolved from an algorithm for calculating elementary particle collision cross sections into a comprehensive theoretical framework covering statistical mechanics and condensed matter physics as well. In high-energy physics many-body aspects are particularly relevant for the description of the quark-gluon plasma in a cosmological setting, or in the laboratory (heavy-ion collisions) [1], and for the analysis of phase transitions in the early universe in the context of the inflationary scenario [2]. In and near thermal equilibrium these statistical aspects are the domain of "field theory at finite temperature and density" [3], or thermal field theory, as we prefer to call the subject.

The basic observable quantities that are to be calculated in thermal field theory are the equation of state (EOS), thermodynamic response functions (e.g., the specific heat), transport coefficients (e.g., the shear viscosity), and the effective potential in the case of a nontrivial phase structure. In fact, the thermal effective

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potential (in curved spacetime) is the starting point of any inflationary scenario, whereas transport coefficients are indispensable in order to trace the space-time evolution of the quark-gluon plasma by means of hydrodynamic equations. In principle, reliable lattice calculations of these thermal observables would be preferable over (semi-) perturbative evaluations. Indeed, our understanding of the phase structure of (gauge) field theories is definitely based on lattice results [4], but with the present state of the art other thermal objects of interest have mainly been analyzed by some version of perturbation theory.

Now the fact that vacuum and thermal field theories are two branches of a single tree has apparently led to the general belief that their respective perturbation theories should be set up in uniform similitude; this means that in both cases the same free dynamics (i.e., a free Lagrangian or Hamiltonian and its associated free fields) is chosen to expand around. The presence of a nonzero temperature (T) (and/or chemical potential (μ)) is subsequently "incorporated" in a way that depends on the formalism used, namely by the introduction of T-dependent Matsubara frequencies in the imaginary-time method and by the appearance of ghost fields and thermal state conditions in the real-time approach [3]. In any case, the temperature appears in a certain way in the free propagator, and Green functions (from which all thermal observables can be derived) are computed in essentially the same way as in the vacuum theory. (Of course, this last feature was the main reason why field-theoretic methods were introduced in quantum statistical mechanics in the first place.)

The procedure may be refined by introducing T- and μ-dependent renormalization conditions defining T- and μ-dependent masses and coupling constants [5–7]; in zero-temperature many-body theory this is essentially the so-called dynamical quasi-particle approach [8, 3].

In relativistic field theory conventional vacuum perturbation theory works fine in various cases where the Haag–Ruelle collision theory [9–11] applies (cf. Section 3). Roughly, this means that the Hamiltonian must be positive and that the four-momentum operator P^2 must have an isolated discrete eigenvalue $m^2 > 0$, while in addition there must be no room for nonperturbative field configurations; that is, the Hilbert space $\mathscr{H}$ must be a Fock space spanned by the multiparticle states of mass m. There are theoretical reasons why perturbation theory works in this situation [11, 12], while it has, in a rather complicated sense, been well vindicated by experiment. Conventional perturbation theory is based on the Gell-Mann/Low formula, and the actual derivation of this formula [13] already identifies the correct free dynamics as the one pertaining to the asymptotic on-shell particles of mass m. It is important to remark that this perturbation method is consistent in that it produces an S-matrix that is unitary order by order; this follows from the cutting equations [14, 15] in combination with the fact that the free propagator $i/(p^2 - m^2 + i\varepsilon)$ contains the physical mass. Another prominent feature is the direct observable nature of the parameters occurring in the perturbation series: the expansion parameter λ is given by renormalization theory as the actual scattering amplitude at some specified energy, while m is necessarily the physical mass, as we

have seen. Since vacuum field theory in Fock space describes scattering experiments these quantities are measurable indeed.

In thermal field theory the situation is very, very different. The disparity between vacuum and thermal boundary conditions (actually, representations of the operator algebra, see below) is so large that in our opinion the standard perturbation scheme that copies the vacuum one is totally inadequate; it is the purpose of this paper to point out the differences and to analyze the situation to such an extent that an entirely new and unexpected structure emerges almost by necessity [16], be it attractive or not.

To begin with, it should be remarked that the formalism of relativistic thermal field theory has never been tested by experiment. The only area where thermal field theory has actually been confirmed is low-temperature condensed matter physics [17, 18]. This, however, is nonrelativistic second-quantized many-particle behaviour in or near the ground state rather than full-fledged thermal field physics.

A first indication that the usual perturbation method could be diseased was the discovery [19] of the infrared problem in high-temperature Yang–Mills plasmas, notably in the quark-gluon plasma [20, 1]. Namely, power counting shows that the pressure as a power series in the coupling constant g shows infrared divergences of the order g^6 and higher. This was unexpected, because the singular infrared behaviour of quantum chromodynamics (QCD) is supposed to be related to its confining nature, which lattice studies [1, 4] claim to be absent in the high T-phase (*pace* [21, 22]). Since QCD is not well understood it may be unfair to ascribe this infrared problem to the general inappropriateness of the standard thermal perturbation formalism; for the moment suffice it to say that the quite similar infrared problem in the charged sector of vacuum quantum electrodynamics, which is unrelated to confinement, too, has been shown [23] to be due to an improper identification of the photon Hilbert space.

A model-independent argument in favour of a new approach is implicit in certain results of Narnhofer *et al.* [24], who showed that, in great contrast to the state of affairs at zero temperature, there is no room for interactions between exact quasiparticles at finite temperature (cf. Section 6).

To achieve strong structural results in scattering theory or statistical mechanics (of the type in Refs. [9, 23, 24]) it is extremely convenient to assume that the system is spatially infinitely extended. This assumption may always be an idealization and in addition introduces technical mathematical complications, but it definitely forces the issue and brings certain properties of the system to a head: quantities that are negligibly small (like the interaction between scattering states) or devastatingly large (like the total energy in a thermodynamic system) will actually be zero and infinite, respectively, in an infinite system. Artificial "boxes" are eliminated in this way; self-maintained boundaries, on the other hand, can naturally be accounted for in infinite systems [18]. To illustrate how the use of infinite systems avoids certain ambiguities in thermal field theory, consider the following paradox. If we write the canonical partition function as $Z(\beta) = \mathrm{Tr} \exp(-\beta H)$, where β is the inverse temperature and H is the total Hamiltonian, and if we

assume that the Hilbert space over which the trace is taken is spanned by in- (or out-) scattering states, it follows that $Z(\beta) = \mathrm{Tr}\exp(-\beta H_0) = Z_0(\beta)$, where H_0 is the free Hamiltonian governing the time evolution of the scattering states. This follows because the trace is sensitive only to the spectrum of an operator, and in field theory existence and completeness of the asymptotic states require that H and H_0 have identical spectra; cf. Section 3. This paradox arises because one wants the best of all worlds, i.e., both the simple trace formula for $Z(\beta)$ and in–out-states spanning the Hilbert space. The paradox is resolved by noting that in an interacting system exact asymptotic states exist only if the system is infinite; in that case, however, the operator $\exp(-\beta H)$ is no longer of trace-class, and thermodynamic equilibrium should be defined abstractly by the KMS condition (cf. Section 5).

Infinite systems can be studied by abstract operator-algebraic techniques [25–27] (also cf. Sections 3 and 5); if we use the language of algebraic states and inequivalent representations the situation clears up considerably, and the essential differences between vacuum and thermal field theory are easy to formulate. Roughly speaking, the algebraic approach to quantum field theory starts from an abstractly defined algebra $\mathfrak{A}$ of (local) observables that may be coordinatized by fields [25]. A state is a positive linear functional over this algebra rather than a density matrix on some given Hilbert space, which as yet is absent. Selecting a particular state ω one is led to an explicit representation $\pi_\omega(\mathfrak{A})$ of $\mathfrak{A}$ on a certain Hilbert space $\mathscr{H}_\omega$, by applying the GNS construction [26, 27]. In infinite systems different states in general induce inequivalent representations of $\mathfrak{A}$. This is precisely what happens in vacuum vs thermal field theory: vacuum and thermal equilibrium (or KMS) representations of $\mathfrak{A}$ are not only unitarily inequivalent but also disjoint [26]. This implies that the basic structural properties of vacuum representations, such as the existence of in- and out-states and a unitary S-matrix, and the weak equivalence of the full Hamiltonian and the renormalized free Hamiltonian of the asymptotic particles [9, 18], do not carry over to thermal representations. Since, in our opinion, a good perturbation theory directly reflects such underlying properties of the Hilbert space we clearly see the need to start all over again if we wish to examine the structural properties of thermal representations in connection with the search for a consistent thermal perturbation theory.

In summary, the simple point we make here is that the objects defining perturbation theory, namely the free dynamics and the expansion parameter(s), cannot be simply read from the Lagrangian (or, more generally, the automorphism on $\mathfrak{A}$ defining the dynamics), but instead are strongly dependent on the "setting" (vacuum, thermal, etc.) that one has chosen. Zero-temperature masses and coupling constants are contextually meaningless in thermal situations; if a thermal perturbation theory is possible at all it should be formulated solely in terms of thermal observables, like the specific heat and the shear viscosity.

The preceding remarks strongly motivate the following questions: given a KMS state ω on a field-theoretic algebra $\mathfrak{A}$ and a space-time evolution α_x (indirectly defining ω, cf. Section 4), what is the explicit structure of the "thermal Hilbert space" $\mathscr{H}_\omega$, are there particle-like states, is it a Fock space in some sense, etc.? As

we shall see in detail, the known properties of the representative $\pi_\omega(\mathfrak{A})$ and its commutant $\mathfrak{M}'$, the "bottomless" nature of the effective Hamiltonian $\hat{H}$ in $\mathscr{H}_\omega$, and the group-theoretic structure (broken Lorentz symmetry) on $\mathscr{H}_\omega$ together are so restrictive that an essentially unique structure emerges. Namely, we shall find that $\mathscr{H}_\omega$ is spanned by a peculiar type of "particle" state: these "particles" have no mass-shell or dispersion relation, and even the single-particle states are unstable. We will call these objects non-shell unstable particles. (Of course, a particle without a mass-shell is no particle at all; the analogy with ordinary particle states is mathematical rather than physical.)

The situation may be compared with (axiomatic) vacuum field theory, where the corresponding problem starts from the Wightman (or Haag–Kastler) axioms [11], which also turn out to be so stringent that reasonable field theories with discrete mass-gap necessarily find an interpretation in terms of asymptotic (ordinary) particles.

Following our general programme, the next step is to derive, if possible, a perturbation method that is consistent (i.e., it satisfies the KMS condition order by order) and reflects the structure of $\mathscr{H}_\omega$. In the case in which a certain weak asymptotic condition (cf. Section 9) is satisfied, it turns out to be indeed possible to set up a diagrammatic perturbation scheme. The propagators contain a certain continuous function $Z(E, \mathbf{p})$ that effectively weighs the contribution of the non-shell particle with energy E and momentum $\mathbf{p}$. We can derive a self-consistent integral equation for Z which in principle admits a solution with two free parameters. These must subsequently be exchanged for two thermal observables plus a set of renormalization conditions; we shall demonstrate this procedure in a simple example.

Unfortunately, the completely self-consistent nature of this perturbation scheme, which in principle represents its structural power, renders it extremely difficult to use in practice, and it remains to be seen whether our method is of much use in actual computations. In any case, our criticism of the conventional method stands apart from the particular alternative we are offering here.

We now give a preview of the contents of this paper. To make our ideas accessible to nonspecialists in operator algebras we start in Section 2 with a brief, informal introduction to the subject, on the same heuristic lines as those of an earlier exposition [27]. Then (Section 3) we place the main ideas of the Haag–Ruelle scattering theory in this light in order to understand why vacuum perturbation theory is as it happens to be, and to appreciate later on the deep differences with the thermal case. In particular, the derivation of the Gell-Mann/Low formula is considered in detail. The Haag–Ruelle theory is valid for a theory with stable particles, but it can easily be accommodated to include unstable ones as long as there exists a lightest stable particle defining the Fock space. In thermal field theory all particles are effectively unstable, a fact of profound significance in our approach. To fix ideas in this direction we find it very helpful to discuss in Section 4 an approach to unstable particles at zero temperature due to Lukierski [28–30], which de-emphasizes the role of the stable particles in the theory and assigns a particular type of asymptotic field to an unstable particle, namely a so-called

(one-parameter) Licht field [31, 32]. It is precisely this feature of vacuum unstable-particle theory that we will be able to generalize to the thermal case; for group-theoretical reasons we will, however, end up with a two-parameter field.

Having discussed relevant features of vacuum field theory we pass on to thermal KMS representations; in Section 5 we discuss the operator mathematics that we need, giving "physicist's proofs" of the particularly relevant results of the Tomita–Takesaki theory [26]. As we see in Section 6, this theory leads almost instantly to the theorem of Narnhofer *et al.*, quoted before, on the noninteraction of exact thermal quasiparticles. This theorem in plain language means that the thermal self-energy always has an imaginary part on any (would-be) mass-shell, however refined the perturbation scheme. Thence it denies the possibility that the thermal Hilbert space $\mathscr{H}_\omega$ is an ordinary Fock space. The question what it should be instead is answered [16] by almost purely group-theoretical considerations in Section 7; the crucial ingredient is that $\mathscr{H}_\omega$ carries a unitary representation of the unbroken subgroup $S = SO(3)$ $(\times T_4$ of the Poincaré group. (The representations and Clebsch–Gordan series of this subgroup are derived in the Appendix.) Having identified $\mathscr{H}_\omega$ as an "energy-extended Fock space" of nonshell unstable "particle" states, we construct a complete set of free-field operators on $\mathscr{H}_\omega$ in Section 8; this step also elucidates the meaning of spin and helicity in thermal representations. The completeness of these operators here means that any operator on $\mathscr{H}_\omega$, in particular any element of $\pi_\omega(\mathfrak{A})'' \cup \pi_\omega(\mathfrak{A})'$, can be expressed in terms of them by means of the so-called generalized [27] dynamical map [33, 18]. (The one-particle spectral function $Z(E, \mathbf{p})$ enters the theory at this stage.)

Nevertheless, to construct a diagrammatic perturbation method more is needed, namely a weak LSZ-type asymptotic condition. Due to poor cluster properties in thermal representations we have been unable to prove such a condition. However, the choice of free-field operators is so flexible that it seems reasonable to assume that there exists a particular arrangement of the complete operator set such that the smeared Heisenberg field converges weakly to an element of this set. On this assumption we can derive a generalized Gell-Mann/Low formula for certain spectrally decomposed Green functions (Section 9), and following a two-step strategy we eventually arrive at explicit propagators and other Feynman rules (Section 10).

The perturbation scheme thus obtained is incomplete without an explicit expression for the function $Z(E, \mathbf{p})$ which enters in the propagator. In a canonical theory, Z is in principle determined by the equal-time commutation relations, which in the case where Z is a function give rise to a self-consistent integral equation for this function. It is not clear how to attack this most complicated and implicit equation. Therefore, in order to develop some intuition as to what its solution may look like, we perform a certain rearrangement of the conventional perturbation series to derive a new expansion that also follows from our original method for a particular choice of $Z(E, \mathbf{p})$ (Section 11). Contact with the so-called nonequilibrium thermo field dynamics [34–37] is thereby established. Instead of attempting to solve the integral equation we then simply adopt this choice for $Z(E, \mathbf{p})$ in order to discuss the final ingredient of our perturbation method. This is

the process of exchanging the parameters in the diagrams for physical thermal observables and renormalization conditions. In doing so in Section 12 we derive simple diagrammatic expressions for the viscosity and the specific heat, which are also useful outside our particular framework.

We close the paper with conclusions and some remarks on possible extensions of our work. Also the feedback to our original motivation, the infrared problem in thermal QCD, is discussed.

To avoid misinterpretation we feel obliged to make a qualifying statement concerning the level of mathematical analysis in this paper (also cf. [27]). We employ various techniques from functional analysis and operator algebra theory that are usually associated with rigorous mathematical physics [26]. In contrast we use these methods rather heuristically, that is, more or less analogously to the way most physicists utilize path integrals. However, the very nature of our operatorial analysis in principle should allow an analyst to amend this paper so as to render its derivations and results rigorous.

2. Operator Algebras

The motivation for the use of abstract operator algebras, in particular C^*-algebras, in statistical mechanics and field theory is given in Refs. [25, 26] (also cf. [39, 27]). Basically they allow a precise and conceptually transparent discussion of phase transitions, symmetry breaking, etc., in infinite systems; in addition their use is profitable in emphasizing structural properties in field theory. Our discussion below understates technical aspects, and we mostly omit precise mathematical definitions; for these cf. [26].

A convenient starting point is the abstract algebra $\mathfrak{A}$ of bounded local observables [25] defined without direct reference to a particular Hilbert space. This algebra may in some sense be regarded as a noncommutative manifold that can be coordinatized by quantum fields; to avoid problems with the abstract treatment of algebras of unbounded operators, however, it is preferable to postpone this step until a concrete representation of $\mathfrak{A}$ has been constructed.

States and Representations

A state ω is abstractly defined as a normalized positive linear functional on $\mathfrak{A}$; the number $\omega(A)$ that ω assigns to $A \in \mathfrak{A}$ is interpreted as the expectation value of the observable A in the state ω. States can be used to construct representations of the abstract algebra by operators on a concrete Hilbert space by means of the GNS construction [26] (for a simple presentation cf. [3, 27]). All we will need here is the final result: every state ω induces a representation of $\mathfrak{A}$, called $\pi_\omega(\mathfrak{A})$, by operators on a Hilbert space $\mathscr{H}_\omega$; this space contains a cyclic vector $|0\rangle_\omega$, which satisfies

$$\omega(A) = {}_\omega\langle 0|\pi_\omega(A)|0\rangle_\omega; \tag{2.1}$$

$$\mathscr{H}_\omega = \overline{\pi_\omega(\mathfrak{A})|0\rangle_\omega}. \tag{2.2}$$

In words, the state ω is "purified" in $\mathscr{H}_\omega$, whether or not it is a pure state originally. In addition, any vector in $\mathscr{H}_\omega$ can be arbitrarily well approximated by acting with representatives of $\mathfrak{A}$ on $|0\rangle_\omega$. In vacuum representations (see below) the vector $|0\rangle_\omega$ is just the ordinary vacuum state $|0\rangle$; in thermal (KMS) representations it is what is known in thermo field dynamics as the "thermal vacuum" $|0(\beta)\rangle$ [18, 38, 3].

It is technically convenient to extend the set of operators $\pi_\omega(\mathfrak{A})$ a little bit by closing it in the weak operator topology [26] or, equivalently, by taking its bicommutant $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$. (The commutant of an algebra of bounded operators is the set of bounded operators that commute with all elements of the algebra; this commutant itself has a commutant which is the bicommutant of the original algebra.) The commutant of $\mathfrak{M}$ is called $\mathfrak{M}'$; it will play a decisive role in thermal representations. The elements of $\mathfrak{M}$ are bounded observables, but for practical purposes we prefer to work with the usual quantum operator fields. We therefore assume that it is possible to "coordinatize" $\mathfrak{M}$ by the usual zoo of quantum fields, which are supposed to be affiliated with $\mathfrak{M}$. (Apart from domain problems [26], this means that they commute with $\mathfrak{M}'$, and that their unitary elements and bounded spectral projections actually lie in $\mathfrak{M}$.) We make this assumption in order to be able to utilize both nice C^*-algebra theory and the practical apparatus of field theory.

Symmetries

In the algebraic approach symmetries are defined as *-automorphisms of the operator algebra [25, 26, 39]; this means that the symmetry group G is realized by operators α_g on $\mathfrak{A}$ satisfying

$$\alpha_g \alpha_{g'} = \alpha_{gg'}; \tag{2.3}$$

$$\alpha_g[A]\,\alpha_g[B] = \alpha_g[AB]; \tag{2.4}$$

$$\alpha_g[A^*] = \alpha_g[A]^* \tag{2.5}$$

for all $A, B \in \mathfrak{A}$. Here * is the adjoint operation. A symmetry group G prolongates into the representation π_ω if ω is G-invariant in the sense that

$$\omega(\alpha_g[A]) = \omega(A) \tag{2.6}$$

for all $g \in G$. If (2.6) is satisfied G can be unitarily implemented in $\mathscr{H}_\omega$ by constructing the unitary representative $U(g)$ of $g \in G$ as

$$U(g)\,\pi_\omega(A)|0\rangle_\omega = \pi_\omega(\alpha_g[A])|0\rangle_\omega \tag{2.7}$$

for all $A \in \mathfrak{A}$. This indeed defines $U(g)$ as (2.2) holds. If follows from (2.3)–(2.7) that the $U(g)$ are indeed unitary operators representing G and leaving $|0\rangle_\omega$ invaiant. We also have

$$U(g)\,\pi_\omega(A)\,U^*(g) = \pi_\omega(\alpha_g[A]). \tag{2.8}$$

In infinite systems G cannot be unitarily implemented if (2.6) is violated and, assuming that the α_g commute with space translations $\alpha_{\mathbf{x}}$, if in addition, ω is not related by localized operations [25] to a state for which (2.6) does hold.

Whenever G is unitarily implementable, the direct integral decomposition of the representation $U(g)$ on $\mathscr{H}_\omega$ is an extremely useful tool in the structural analysis of the Hilbert space. We will quote a particularly relevant mathematical result for the special case that G is a so-called type I group [40]; this suffices for our applications. As such the following is a combination of theorems of von Neumann, Mautner, and Naimark-Guichardet [40]. We must first define the dual object $\hat{G}$ of a group G; this is the space of equivalence classes of inequivalent irreducible unitary representations of G. Elements of $\hat{G}$ are denoted $\hat{g}$, and the carrier space (up to unitary equivalence) of the representation $\hat{g}$ is called $\mathscr{H}(\hat{g})$. The theorem states that there exists a unitary operator F (the generalized Fourier transform) which maps $\mathscr{H}_\omega$ into an equivalent Hilbert space $\hat{\mathscr{H}}_\omega$ admitting a direct integral decomposition in the following way:

$$F\mathscr{H}_\omega = \hat{\mathscr{H}}_\omega = \bigoplus_{k=1}^{\infty} k \int_{\hat{G}} d\mu_k(\hat{g})\, \mathscr{H}(\hat{g}). \tag{2.9}$$

Here the μ_k are measures on $\hat{G}$, and k is the multiplicity of the Hilbert space $\mathscr{H}_k := \int d\mu_k(\hat{g})\, \mathscr{H}(\hat{g})$, i.e., the number of times it occurs in $\hat{\mathscr{H}}_\omega$. Of course, if $\mu_j \equiv 0$ then the multiplicity j does not occur at all in the decomposition of $\mathscr{H}_\omega$. Elements of the direct integral Hilbert space $\mathscr{H}_k$ are by definition [40] (μ_k-measurable) functions u on $\hat{G}$ such that $u(\hat{g}) \in \mathscr{H}(\hat{g})$, and the inner product

$$\langle u | v \rangle_{\mathscr{H}_k} = \int d\mu_k(\hat{g}) \langle u(\hat{g}) | v(\hat{g}) \rangle_{\mathscr{H}(\hat{g})} \tag{2.10}$$

is finite. Hence the structure of $\mathscr{H}_\omega$, and thereby its physical interpretation, is largely determined by the measures $\mu_k(\hat{g})$; these are, in turn, governed by both the dynamics of the theory and the state ω.

A further element in the analysis of $\mathscr{H}_\omega$ and $\pi_\omega(\mathfrak{A})$, which is crucial for the formulation of perturbation theory, is the identification, if possible, of a manageable complete set of operators. By complete we here mean that it must be possible to express all elements as well as the coordinatizing fields of $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$ in terms of this set ("dynamical map" [18, 36]); in particular, (2.2) implies that $|0\rangle_\omega$ must be cyclic for the complete set. By manageable we mean that their commutation relations and transformation behaviour under G must be explicitly known. A good example of such a complete set is given by the creation and annihilation operators if $\mathscr{H}_\omega$ is a (standard) Fock space. As we will see especially in the thermal case, the identification of such a complete manageable set is intimately related to the group-theoretic analysis in the preceding paragraph; mathematically this is associated with the infinite-dimensional generalization of the Burnside theorem [40].

To close this possibly dry section we announce that all of the above will actually be used in our analysis of thermal field theory in Sections 5–9.

3. Haag-Ruelle Theory and Gell-Mann/Low Formula

In relativistic quantum field theory the symmetry group G in any case contains the Poincaré-group P; in the sequel we will ignore possible internal or conformal symmetries and we will just assume $G = P$, and label its elements by p. In particular, translations are labeled $x \in \mathbb{R}^4$, and we write $\alpha_x[A] = A(x)$; clearly α_t specifies the dynamics.

To arrive at a concrete representation of the operator algebra we must select a state ω on $\mathfrak{A}$. This essentially means that we make up our mind about a particular type of problem we are interested in: accelerator physics needs (approximate) vacuum states, while thermodynamics needs thermal (equilibrium) states, etc. Given a state and a "theory," i.e., an algebra $\mathfrak{A}$ and its symmetries (particularly its space-time evolution α_x), it is necessary to determine the physical interpretation of the theory *in the state* ω. For example, one may ask whether there is a particle structure, or whether the original symmetries are realized , etc. Answering such questions amounts to analyzing the representation space $\mathscr{H}_\omega$ and the representative $\pi_\omega(\mathfrak{A})$. For instance, $\mathscr{H}_\omega$ is a conventional Fock space if the theory admits a complete particle interpretation. (If such is not the case, as in realistic gauge theories, it is very relevant to ask what $\mathscr{H}_\omega$ looks like instead [41].) In some cases it may happen that the above analysis identifies a particularly natural or even unique consistent perturbation theory, i.e., a choice of the "free" dynamics.

Input

A good example of essentially such an analysis is the Haag–Ruelle scattering theory [9–11]. Here ω is taken to be a vacuum state (i.e., a state satisfying (2.6) for all $g = p \in P$), and (vacuum expectation values of) the fields coordinatizing $\pi_\omega(\mathfrak{A})$ on $\mathscr{H}_\omega$ are supposed to satisfy the Wightman axioms [9, 11]. In addition there is a spectral assumption which in fact puts in part of the final result by hand. (As we shall see, it will be this assumption that in general will not be satisfied in interacting thermal field theories.) In the language of the previous section this assumption states that the measures $\mu_k(\hat{p})$ in (2.9) contain one or several point-contributions concentrated at positive masses, i.e., $\mu_k(m_i^2, s) = 1$ for some multiplicity k, spin s and $m_i^2 > 0$. Furthermore, the absolutely continuous contribution to the μ_k must start at some value $m^2 > \max_i\{m_i^2\}$. (Note that here $\hat{G} = \hat{P}$, where $\hat{p} \in \hat{P}$ are labeled by $m^2 > 0, s$; $m^2 = 0, \ldots$, etc. The contribution from the vacuum representation $|0\rangle := |0\rangle_\omega$ is considered understood.) It follows from the spectral representation [18] that these point-contributions correspond to poles in certain two-point functions. In the sequel we will for simplicity assume that the point-measures occur for $k = 1$ and $s = 0$. Each different mass m_i corresponds to a proper subspace $\mathscr{H}_i^{(1)} \subset \mathscr{H}_\omega$ such that $(P^2 = P_0^2 - \mathbf{P}^2)$

$$P^2 \mathscr{H}_i^{(1)} = m_i^2 \mathscr{H}_i^{(1)}. \tag{3.1}$$

Here the P^μ are defined by $U(x) = \exp iP \cdot x$, where $U(x)$ is given by (2.7) with

$P \ni g = x$. Again for simplicity we assume that $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$ can be coordinatized by a single (Heisenberg picture) neutral scalar field $A(x)$, which is chosen in such a way that it couples the vacuum to all subspaces $\mathscr{H}_i^{(1)}$.

Output

We can now discuss certain elements of the Haag–Ruelle (-Hepp [9]) theory that are particularly relevant for our purpose of comparing vacuum and thermal representations and perturbation methods. The principal result is that $\mathscr{H}_\omega$ necessarily contains a Fock space over the one-particle subspace $\mathscr{H}^{(1)} = \oplus_i \mathscr{H}_i^{(1)}$:

$$\mathscr{H}_\omega \supset \mathrm{Fock}(\mathscr{H}^{(1)}) := \mathbb{C}\,|0\rangle \oplus \Big(\bigoplus_{p=1}^{\infty} \bigotimes^{p} \mathscr{H}^{(1)} \Big). \tag{3.2}$$

(To avoid confusion in terminology, we mention that in the context of a canonical formalism this result does not imply that the representation π_ω of the Weyl CCR-algebra [26] defined by the time-zero field is a Fock representation, i.e., admits a vacuum state.) The decomposition in (3.2) is not multiplicity-free, and therefore nonunique for $p > 1$, and one usually chooses in- or out-multiparticle states (see below).

Since it is sufficient for our purpose (cf. Section 7), we will restrict ourselves to the case of asymptotic completeness, in which the inclusion sign in (3.2) is replaced by an equality sign. Then the free fields $\phi_i(x) := \phi_{m_i}(x)$ with mass m_i form a complete set of operators on $\mathscr{H}_\omega$ in the sense of Section 2. The actual choice of this set is tied to the choice of the decomposition in (3.2); i.e., the fields can be chosen as in- or out-fields, etc.; see below. The following property [9] cannot be overemphasized: the unitary representatives $U(p)$ of the Poincaré-group, which are defined in terms of the automorphism α_p via (2.7), have the same action on $\mathscr{H}_\omega$ as that in a free-field theory. Expressing the $U(p)$ in terms of the field A and omitting the label p, we write symbolically

$$U[A] =_\omega U_0[\phi], \tag{3.3}$$

where $U_0[\phi]$ denotes the standard expression of the U's in terms of the free fields ϕ_i. The symbol $=_\omega$ signifies equality in the representation space $\mathscr{H}_\omega$ (note that the $U[A]$ are *not* representatives of elements of $\mathfrak{A}$; they are in the bicommutant $\mathfrak{M}$). It is remarkable that (3.3) holds true even if the dynamics α_t in $\mathfrak{A}$ is nontrivial; physically it expresses the fact that there is no interaction between asymptotic particles. A similar relation is valid for the generators of the $U(p)$; in particular the Hamiltonian in $\mathscr{H}_\omega$ reads

$$H[A]\,|\psi\rangle = H_0[\phi]\,|\psi\rangle = \tfrac{1}{2} \sum_i \int d^3x : ((\partial_0 \phi_i(x))^2 + \phi_i(x)\, \varepsilon_i(\nabla)^2\, \phi_i(x)) : |\psi\rangle \tag{3.4}$$

on any $|\psi\rangle$ in its dense domain of definition ($\varepsilon_i(\nabla) = \sqrt{-\Delta + m_i^2}$).

The remarkable properties (3.2)–(3.4) are established by means of a strong

asymptotic condition on the one-particle excitation operators $B_i(x)$ defined by their Fourier transforms

$$\hat{B}_i(p) = Z_i^{-1/2} h_i(p^2)\, \hat{A}(p). \tag{3.5}$$

Here the $h_i(p^2)$ are test functions with support in $S_i = (m_i^2 - \varepsilon, m_i^2 + \varepsilon)$, where ε is chosen such that $S_i \cap S_j = 0$ for $i \neq j$; moreover, $h_i(m_i^2) = 1$. By construction one has

$$B_i(f)|0\rangle := \int d^4x f(x)\, B_i(x)|0\rangle \in \mathscr{H}_i^{(1)} \tag{3.6}$$

for a test function f; the constants Z_i in (3.5) are chosen such that these one-particle states are properly normalized. For later comparison with the thermal situaton it is important to note that although (3.1) implies

$$B_i(f, t)|0\rangle := (e^{iHt} B_i(f)\, e^{-iHt})|0\rangle = B_i(e^{i\varepsilon_i(\nabla)t} f)|0\rangle, \tag{3.7}$$

the same equation without $|0\rangle$, i.e., the corresponding operator equation, is valid only in free-field theories. As we see in Section 6, in thermal representations Eq. (3.7) *does* imply the corresponding operator equation.

Now define

$$(g_i \overleftrightarrow{\partial}_0 B)(t) = \int_{x_0 = t} d^3x (g_i(x)\, \partial_0 B_i(x) - \partial_0 g_i(x)\, B_i(x)), \tag{3.8}$$

where $g_i(x)$ is understood to be a regular, positive-frequency solution of the Klein–Gordon equation with mass m_i (wave packet). The strong asymptotic condition that essentially implies the structure of $\mathscr{H}_\omega$ sketched above is

$$\begin{aligned} s - \lim_{t \to \mp\infty} \prod_{l=1}^{n} i(g_{i_l}^{\alpha_l} \overleftrightarrow{\partial}_0 B_{i_l})(t)|0\rangle \\ = \prod_{l=1}^{n} i(g_{i_l}^{\alpha_l} \overleftrightarrow{\partial}_0 \phi_{i_l}^{\text{out}})|0\rangle = \prod_{l=1}^{n} a_{i_l}^{*\text{in}}{}_{\text{out}}(g_{i_l}^{\alpha_l})|0\rangle, \end{aligned} \tag{3.9}$$

where any i_l is one of the i's considered above, while the α_l label possibly different wave packets. Of course, s-lim means that the limit is considered in the strong operator topology. We now see explicitly that (3.7) without $|0\rangle$ would lead to $a^{*\text{in}} = a^{*\text{out}}$, i.e., absence of scattering, hence of interaction.

Gell-Mann/Low Formula

The Haag–Ruelle strong asymptotic condition (3.9) is crucial for the analysis of $\pi_\omega(\mathfrak{A})$ and $\mathscr{H}_\omega$, but it cannot be used for the derivation of a perturbation expansion for Green functions (i.e., the Gell-Mann/Low formula), because these are expressed in terms of the full Heisenberg fields A rather than the restricted one-particle excitation operators B. Fortunately, it can be shown [9] that (3.9) is even valid

when all B_{i_l} are replaced by $Z_i^{-1/2}A$ if the limit is taken in the weak operator topology (i.e., in matrix elements). (This seemingly technical difference in fact modifies the physical meaning of the asymptotic condition considerably.) Starting from this observation, the Gell-Mann/Low formula has been derived with varying degrees of rigor and correctness: the standard textbook derivation based on the interaction picture is clearly wrong both mathematically and physically (it violates Haag's theorem [39] and fails to uniquely identify the choice of H_0 leading to an order-by-order consistent perturbation theory, see below), whereas the most superior treatment [13] in the operator formalism known to this author is so complete that it solves the existence problem of the models for which it applies simultaneously, and therefore takes up an entire book. Our derivation below interpolates between these two extremes, in that it does not address the existence problem and is formal in the sense that it leads to the usual difficulties involving infinite renormalization constants. Nevertheless we shall give it here, because it illustrates the interplay between Hilbert-space structure and perturbation theory in the well-known vacuum context, while its proof will easily extend to the thermal case (cf. Section 9).

Thus the starting point is the (LSZ) asymptotic condition

$$w-\lim_{t\to\mp\infty}\prod_{l=1}^{n}(g_{i_l}^{\alpha_l}\overleftrightarrow{\partial}_0 A)(t)|0\rangle=\prod_{l=1}^{n}(Z_{i_l}^{1/2}\,g_{i_l}^{\alpha_l}\overleftrightarrow{\partial}_0\phi_{i_l}^{\text{in/out}})|0\rangle. \tag{3.10}$$

It is convenient to use a two-Hilbert-space formalism [10]. We define a fiducial reference Hilbert space $\mathscr{H}_M$ ($M=\{m_i\}$) which is isometrically isomorphic to $\mathscr{H}_\omega$. This $\mathscr{H}_M$ is just the space Fock ($\mathscr{H}^{(1)}$) (cf. (3.2)) with a vacuum state called $|\Omega\rangle$. In contrast to the physical vacuum $|0\rangle$ in $\mathscr{H}_\omega$, however, the state $|\Omega\rangle$ is defined as the vacuum associated with the free fields $\Phi_i:=\Phi_{m_i}$ at $t=0$. Similar to the ϕ_i^{in} on $\mathscr{H}_\omega$, the Φ_i form a complete set of operators on $\mathscr{H}_M$. We now define a map $K\colon \mathscr{H}_M\to\mathscr{H}_\omega$ that, roughly, maps a free multiparticle state into an interacting one. To do so, it is necessary to decompose the field $A(x)$ as

$$A(x)=\sum_i Z_i^{1/2}A_i(x). \tag{3.11}$$

Here the A_i must be chosen in such a way that they have the LSZ limit (cf. (3.10)) $A_i\to\phi_i^{\text{in}}$ for $t\to-\infty$, which is always possible in view of (3.10). This demand is satisfied if A_i couples the vacuum to $\mathscr{H}_i^{(1)}$ defined in (3.1). Beyond this demand, the decomposition (3.11) is quite arbitrary. We then define K according to

$$K\left(\prod_{l=1}^{n}g_{i_l}^{\alpha_l}\overleftrightarrow{\partial}_0\Phi_{i_l}\right)|\Omega\rangle=\prod_{l=1}^{n}(g_{i_l}^{\alpha_l}\overleftrightarrow{\partial}_0 A_{i_l})(t=0)|0\rangle. \tag{3.12}$$

This is well-defined, because any vector in $\mathscr{H}_M$ can be written uniquely in the form occurring on the l.h.s. for suitably chosen g's and i_l's. In the case of asymptotic com-

pleteness (that we exclusively deal with) the inverse K^{-1} is then also well-defined. Choosing $n=0$ in (3.12) we have

$$K|\Omega\rangle = |0\rangle. \tag{3.13}$$

Also note the intertwining relations

$$K\Phi_i = A_i K; \tag{3.14}$$

$$\Phi_i K^{-1} = K^{-1} A_i. \tag{3.15}$$

Furthermore, we introduce the evolution operators $U(t_1, t_2): \mathscr{H}_M \to \mathscr{H}_M$ according to

$$U(t_1, t_2) = e^{iH_0 t_1} K^{-1} e^{-iH(t_1 - t_2)} K e^{-iH_0 t_2}, \tag{3.16}$$

where $H_0 = H_0[\Phi]$ is the free Hamiltonian on $\mathscr{H}_M$, and $H = H[A]$ is the full Hamiltonian on $\mathscr{H}_\omega$. Finally, we define the quasi-Møller wave operators $\Omega^{\pm}: \mathscr{H}_M \to \mathscr{H}_\omega$ by

$$\Omega^{\pm} = w - \lim_{t \to \mp\infty} KU(0, t). \tag{3.17}$$

The existence of these limits follows from (3.10) (compare this with Ref. [10], where similar wave operators are constructed in the single-mass case, and with A in (3.12) replaced by B. The limit (3.17) then exists strongly). Note that the relations

$$H|0\rangle = H_0|\Omega\rangle = 0 \tag{3.18}$$

imply that

$$\Omega^{\pm}|\Omega\rangle = |0\rangle. \tag{3.19}$$

We are now in a position to prove the Gell-Mann/Low formula on the assumption that the field theory and the dynamics derived from H actually exist. After all preparatory work the demonstration reduces to the standard textbook manipulations, but now placed in their proper context, and with due care taken of the multi-mass spectrum and Eq. (3.11). We choose t_i smaller and t_f larger than other relevant times. Then

$$\begin{aligned} iG(x_1, \ldots, x_n) &:= \langle 0|\, T[A(x_1)\cdots A(x_n)]\,|0\rangle \\ &= \sum_{i_1 \cdots i_n} Z_{i_1}^{1/2} \cdots Z_{i_n}^{1/2} \langle 0|\, T[A_{i_1}(x_1)\cdots A_{i_n}(x_n)]\,|0\rangle \\ &= \sum_{i_1 \cdots i_n} Z_{i_1}^{1/2} \cdots Z_{i_n}^{1/2} \langle 0|\, U(0, t_f)\, T[\Phi_{i_1}(x_1)\cdots\Phi_{i_n}(x_n) \\ &\quad \times U(t_f, t_i)]\, U(t_i, 0)|0\rangle. \end{aligned} \tag{3.20}$$

We then use (3.19), take the limits $t_i \to -\infty$ and $t_f \to \infty$, and use (3.17) and the isometric nature of Ω^- to find

$$iG(x_1 \cdots x_n) = \sum_{i_1 \cdots i_n} Z_{i_1}^{1/2} \cdots Z_{i_n}^{1/2} \frac{\langle \Omega | \, T[\Phi_{i_1}(x_1) \cdots \Phi_{i_n}(x_n) \, U(\infty, -\infty)] \, | \Omega \rangle}{\langle \Omega | \, U(\infty, -\infty) \, | \Omega \rangle}. \quad (3.21)$$

The denominator is actually equal to unity:

$$\langle \Omega | \, (\Omega^-)^{-1} \, \Omega^+ \, | \Omega \rangle = \langle 0 | 0 \rangle = 1,$$

but not so in naive perturbation theory, where it is the sum of all closed diagrams; hence we have explicitly exhibited it (vacuum renormalization). Note that in real-time thermal field theory this factor *does* equal unity identically [3]. To identify (3.21) with the standard Gell-Mann/Low formula, we remark that (3.15) implies that (3.16) is actually equal to

$$\begin{aligned} U(t_1, t_2) &= e^{iH_0[\Phi]t_1} e^{-iH[\Phi](t_1 - t_2)} e^{-iH_0[\Phi]t_2} \\ &= T \exp\left(-i \int_{t_2}^{t_1} dt \, H_I(t) \right), \end{aligned} \quad (3.22)$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture defined by H_0, as usual. It is the limiting procedure $t_i \to -\infty$, $t_f \to +\infty$ that calls for a relatively refined procedure based on the weak asymptotic condition (3.10).

Remarks

We can summarize the preceding as follows: the strong asymptotic condition (3.9) is used to identify the structure of $\mathscr{H}_\omega$, whereas the weak asymptotic condition (3.10) is used to derive the Gell-Mann/Low formula (3.21), hence diagrammatic perturbation theory. In the full Haag–Ruelle theory, the choice of the free dynamics implicit in (3.21) is uniquely governed by (3.4), which in turn is closely related to the Hilbert space property (3.1).

For later comparison with the thermal case it is important to remark that the correct choice of the free Hamiltonian in (3.21) is already implicit in the demand that this formula can actually be derived. As we have seen, this amounts to demonstrating that the limit in (3.17) exists and is nonvanishing, which in turn is equivalent to proving the LSZ limit (3.10). The latter limit, however, brings a consistency requirement with it, namely [18]

$$H[A] = w - \lim_{V \to \infty} \int_V d^3x \mathscr{H}_0[\phi_i^{\text{in}}(x)], \quad (3.23)$$

where $\mathscr{H}_0$ is the free Hamiltonian density expressed in terms of the scalar fields ϕ_i^{in}.

This consistency condition can be derived from the Haag–GLZ expansion, which is a weak expansion of Heisenberg operators in terms of in-fields, and which follows directly from (3.10) [33]. Of course, Eq. (3.4) of the Haag–Ruelle theory implies (3.23), but the point is that the latter equation must necessarily hold whenever the weak LSZ condition (3.10) is valid (and the operator set $\{\phi_i^{\text{in}}\}$ is complete), whether or not there exists an underlying Haag–Ruelle-like theory based on a strong asymptotic condition. As we see in Sections 7–9, in thermal representations we have no strong Haag–Ruelle theory at our disposal; the structure of $\mathscr{H}_\omega$ will be determined from quite different considerations, and we must *postulate* a relation similar to (3.10). The choice of H_0 in the Gell-Man/Low formula will then be dictated by the consistency condition (3.23).

Feynman diagrams follow by substituting (3.22) in (3.21) and expanding the exponentials. The (momentum-space) free propagators are $i/(p^2 - m_i^2 + i\varepsilon)$. The masses m_i^2 are the eigenvalues of P^2 in the one-particle subspaces $\mathscr{H}_i^{(1)}$ and can be identified with the physical masses of the asymptotic particles. The propagators with these masses are the unique ones that guarantee that the S-matrix obtained from the Green functions (3.20) is unitary order by order in perturbation theory, i.e., without resummations. This fact, which follows from the cutting equations [14, 15], illustrates the overall consistency and tightness of perturbation theory based on operator methods. Namely, there is a one–one correspondence between the free Hamiltonian and the free propagator; the correct choice (asymptotic condition and Gell-Mann/Low formula) of the former ascertains the correct choice (unitarity) of the latter, and vice versa.

The next step is renormalization, i.e., the translation of bare parameters, say m_0^2 and λ_0, in $H[A]$ into *contextually relevant* physical parameters. This point is discussed in connection with thermal field theory in Section 12.

To close this section it should be mentioned that the spectral assumptions underlying the Haag–Ruelle theory are not met in any realistic model [41]. Nevertheless, the example it sets is very useful in situations where its postulates are approximated in some sense. For us it is the chain of ideas leading to the Gell-Mann/Low formula that is particularly relevant for understanding the state of affairs in thermal field theory.

4. Unstable Particles and Licht Fields

In this section we assume that the spectral assumption (3.1) is satisfied only for a single mass m. An interesting situation arises when a local coordinatization of the bicommutant M is only possible using more than one field, say two, called $A(x)$ and $C(x)$. Without loss of generality we may assume that the pole at $p^2 = m^2$ occurs in the two-point function of the A-field. Then $A \to Z^{1/2}\,\phi_{\text{out}}^{\text{in}}$, and $C \to 0$ for $t \to \mp\infty$ in the LSZ sense (3.10). The Fock space $\mathscr{H}_\omega$ is completely spanned by multiparticle states "belonging" to the A-field.

Model

For concreteness' sake we consider the Veltman model [14] described by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu A \partial^\mu A + \frac{1}{2}\partial_\mu C \partial^\mu C - \frac{1}{2}m^2 A^2 - \frac{1}{2}M^2 C^2 - \frac{\lambda}{2}A^2 C + \text{c.t.} \tag{4.1}$$

Here c.t. are the renormalization counterterms, and M is the renormalized mass of the C-field, which satisfies $M > 2m$. (The precise physical meaning of the quantity M is, of course, less clear than that of m; it is fixed by the renormalization conditions. For example, the real part of the C-self-energy can be chosen to vanish for $p^2 = M^2$.) It is then not completely wrong to state that the would-be particle associated to the C-field is unstable and can decay into two stable A-particles. This decay, however, is not exactly expressed by the LSZ condition $C \to 0$; instead, this seems to suggest that the C-particle dissipates into nothingness. A related problem concerns the choice of the C-propagator $D_c(p)$ in perturbation theory, as the naive attempt $D_c(p) = (p^2 - M^2 + i\varepsilon)^{-1}$ violates unitarity in every finite order of perturbation theory [14]. The reason is, of course, that this propagator in the cutting equations gives rise to intermediate on-shell states of a particle with mass M which in reality is absent.

The first step towards a solution of these difficulties consists in the realization that a state containing two on-shell A-particles in s-wave pairing with invariant mass $P^2 = s$, $4m^2 \leqslant s < \infty$, may just as well be regarded as a state containing a single C-particle whose mass can assume continuous values s [28]. The continuity of this mass spectrum may then be seen to be the cause of the unstableness (as soon as there is an interaction opening allowed decay channels). From a spectral and group-theoretic point of view there is indeed no distinction between these two pictures.

Licht Field

To formalize this idea one introduces a so-called one-parameter Licht field [31] $\phi_s(x)$ which satisfies the Klein–Gordon equation with mass-squared s, s ranging from $4m^2$ to infinity. The field ϕ_s is constructed in the usual way from annihilation and creation operators $c^{(*)}(\mathbf{p}, s)$, which now contain the extra parameter s and satisfy the commutation relations

$$[c(\mathbf{p}, s), c^*(\mathbf{p}', s')] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\, \delta(s - s'). \tag{4.2}$$

We may regard $c^*(\mathbf{p}, s)$ as an operator that creates an *on-shell* ($p^2 = s$!) *unstable particle* [28, 30] from the vacuum. Because of the equivalence of the two pictures alluded to above, we can ultimately express $c^*(\mathbf{p}, s)$ in terms of the stable-particle creation operators $a^*(\mathbf{p})$ ($p^2 = m^2$) belonging to the A-field by employing the Clebsch–Gordan series for the Poincaré-group [30]. This, however, is not advantageous if we wish to study the C-field and its associated unstable particle.

It is now physically obvious that the free field $\phi_s(x)$ should be related to the asymptotic limit of the field $C(x)$. This is indeed the case, and the crucial step is to decompose $C(x)$ in a manner similar to (3.11), but now in a continuous way:

$$C(x) = \int_{4m^2}^{\infty} ds\, Z^{1/2}(s)\, C_s(x). \tag{4.3}$$

As in (3.11), it is required that $C_s(x)$ connect the (this time unstable) particle state with $P^2 = s$ to the vacuum; otherwise the decomposition is quite arbitrary. The functon $Z(s)$ is a continuous generalization of the wavefunction renormalization constants Z_i of the previous section; there, i is a discrete index referring to the discrete mass m_i^2 in the spectrum of P^2, while here the relevant part of this spectrum is continuous, so that the emergence of such a continuous function $Z(s)$ could have been expected. $Z(s)$ effectively weighs the contribution to $C(x)$ of the Licht field creating a single-unstable particle state with $P^2 = s$. As with (3.11), the whole point of the decomposition (4.3) is that the $C_s(x)$ supposedly have LSZ-limits equal to $\phi_s(x)$ [28, 30, 31]. Formally the appropriate asymptotic condition should read

$$-\phi_s(x) = w - \lim_{y_0 \to -\infty} \int d^3y\, \Delta(x-y; s)\, \overleftrightarrow{\partial}_{y_0} C_s(y), \tag{4.4}$$

if $\phi_s(x)$ refers to incoming two-(A-)particle states; of course, the same construction can be applied to outgoing states and limits (here $\Delta(x-y; s)$ is the Pauli–Jordan commutator function). Actually, this condition can be satisfied only in the range $4m^2 \leqslant s < 16m^2$, since the higher-order decay of the C-particle into four or more stable particles is ignored.

Propagator

It is important to remark that neither $C(x)$ nor the $C_s(x)$ have strong (Haag–Ruelle) asymptotic limits; as we have seen, the existence of a strong limit is directly tied to the existence of a discrete point in the spectrum of P^2, so that in the model (4.1) only the A-field (more precisely, its associated one-particle excitation operator) has such a limit. We repeat that it is this strong limit that leads to the Hilbert-space structure of the vacuum sector of the theory (4.1), namely that of a Fock space of A-particle states. Nevertheless, as we have seen in the previous section, weak limits can be used to derive the Gell-Mann/Low formula, so that the result (4.4) implies that we can indeed set up a perturbation theory. We actually do this in Section 9 for the thermal case, which, as we schall see, is strikingly similar to the situation above, albeit *without* the stable A-particle and its field. The reasoning of Section 9 would here lead to the C-propagator (cf. [29, 30])

$$iD_c(p) = \int_{4m^2}^{\infty} ds\, Z(s) \frac{i}{p^2 - s + i\varepsilon}. \tag{4.5}$$

This choice does not give rise to problems with unitarity, because its cuts now describe on-shell two-(A-)particle states that *do* exist in the Hilbert space under consideration.

The "free" propagator (4.5) formally looks like the spectral representation of the full two-point function. The latter, however, contains the spectral function $\rho_c(s)$ of the C-field, of which $Z(s)$ is only a part containing information about the asymptotic behaviour of the fields $C_s(x)$. Thus $Z(s)$ is the analogue of the delta function $\delta(s-m^2)$ which is the first term of the spectral function of a stable-particle field. This actually renders the ordinary stable-particle propagator a special case of the unstable one. Unfortunately, the actual determination of even the "free" spectral function $Z(s)$ for an unstable field is a very difficult task. Like its discrete counterpart Z_i in the stable case it is in principle determined by the dynamics of the full theory in a fairly complicated manner [29] (also cf. Section 11). In practice, however, it may be sufficient just to parametrize $Z(s)$ in a more or less opportunistic manner, the actual values of the parameters being taken either from experiment or from some lowest-order perturbative approximation to the decay parameters. For example, a two-parameter Breit–Wigner form

$$Z(s)=\frac{\Gamma/\pi}{(s-M^2)^2+\Gamma^2} \tag{4.6}$$

will often do for practical pruposes. In the case where $\Gamma \ll M^2$ (so that the integration in (4.5) may be extended from $4m^2$ to $-\infty$) this leads to an approximate second-sheet pole [29, 42] of $D_c(p)$ for $p^2=M^2-i\Gamma$. The occurrence of this approximate pole in the *free* propagator of the C-field signifies that its instability is already taken into account at the free-field level, as it should be.

As we shall see, thermal representations of interacting field theories give rise to similar structures. The underlying stable-particle Hilbert space will be absent, however, whereas the remaining unstable states will not have a mass-shell. Also for group-theoretical reasons we will end up with two-parameter Licht fields. In spite of these differences, unstable particle theory in vacuum representations is an important background for understanding the thermal situation.

5. Thermal (KMS) Representations

KMS *Condition*

We now turn to our main subject, viz. the investigation of the properties of representations of $\mathfrak{A}$ induced by thermal equilibrium states, also called KMS states [26]. Given a time evolution α_t on $\mathfrak{A}$, a KMS state ω at inverse temperature β is by definition a state satisfying the KMS condition

$$\omega(A\alpha_t[B])=\omega(\alpha_{t-i\beta}[B]\,A). \tag{5.1}$$

(Here and in the following we refrain from giving the mathematically most precise versions [26] of definitions and theorems in order to preserve clarity.) The use of the KMS condition as a characterization of thermal equilibrium states can be motivated by stability arguments [26, 43] or, in a more low-brow way, by the observation that this condition is satisfied by finite-volume (grand-) canonical states $\omega(A) = Z^{-1} \operatorname{Tr} A \exp(-\beta(H_V - \mu N_V))$ (note that the chemical potential μ drops out of the KMS condition because we assumed that A, $B \in \mathfrak{A}$ are (local) observables; for its incorporation in connection with the field algebra cf. Ref. [26]).

A number of important structural properties of KMS representations which follow from (5.1) have been known for some time now. We will state and prove these properties below, because they all play a crucial role in our reasoning in the remainder of this paper. From the general point of view of Sections 2 and 3, and Refs. [25, 27], it is very instructive to see how these properties actually can be derived from the single condition (5.1); as such this analysis partly complements the Haag–Ruelle theory for vacuum representations. The results below may be found in the mathematical literature [26] which, however, appears to be relatively inaccessible to nonspecialists. For this reason we will give "physicist's" or "tree-level" proofs of the relevant theorems; the "mathematical" or "loop" corrections to these proofs are not small, however. They mostly involve taking into account the unboundedness and domain problems of certain operators.

We will use the following notation: the cyclic vector $|0\rangle_\omega$ of Eqs. (2.1) and (2.2) will be called $|0(\beta)\rangle$ as ω is a KMS state at $T^{-1} = \beta$; as before we define the bicommutant $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$, and its elements are generically called A. In fact, we will not make any notational distinction between $A \in \mathfrak{A}$ and $A := \pi_\omega(A) \in \mathfrak{M}$ whenever no confusion can arise.

Thermal Hamiltonian

To begin, with $A = \mathbb{1}$ in (5.1) it immediately follows from analyticity arguments that ω is time-translation invariant, so that (2.6) is satisfied with $g = t$. Hence (Section 2) time translations can be unitarily implemented in $\mathscr{H}_\omega$. This implementation is done indirectly by (2.7); we write

$$U(t) = \exp i\hat{H}t, \tag{5.2}$$

which defines the *effective* "thermal" Hamiltonian $\hat{H}$, being a representation-dependent object. It follows from (2.7) with $A = \mathbb{1}$ that $\hat{H}$ must annihilate the thermal vacuum:

$$\hat{H}|0(\beta)\rangle = 0. \tag{5.3}$$

Equation (2.8) implies

$$\pi_\omega(\alpha_t[A]) =: A(t) = e^{i\hat{H}t} A e^{-i\hat{H}t}. \tag{5.4}$$

Commutant

The principal feature of KMS states which distinguishes them clearly from vacuum states is that $|0(\beta)\rangle$ is *separating* for $\mathfrak{M}$. This means that for any $A \in \mathfrak{M}$ the property $A|0(\beta)\rangle = 0$ implies $A = 0$ (so that $\omega(A^*A) = 0$ only if $A = 0$ in clear contrast with the vacuum case). The proof proceeds in two steps; the first is to show that $A|0(\beta)\rangle = 0$ implies $A^*|0(\beta)\rangle = 0$. To wit, consider the function of a complex variable z,

$$F(z) = \omega(A^*\alpha_z[A]) = \langle 0(\beta)|A^*A(z)|0(\beta)\rangle. \tag{5.5}$$

It follows from the KMS condition that F is analytic in the strip $0 < \operatorname{Im} z < \beta$ and continuous on its boundary. Hence $A|0(\beta)\rangle = 0$ implies by (5.4) and (5.3) that F identically vanishes in this strip, and, by continuity, $F(i\beta) = 0$. However, by (5.1)

$$F(i\beta) = \omega(AA^*) = \langle 0(\beta)|AA^*|0(\beta)\rangle = 0, \tag{5.6}$$

which indeed implies $A^*|0(\beta)\rangle = 0$. This completes the first step. It then easily follows that $|0(\beta)\rangle$ is indeed separating for $\mathfrak{M}$: $A|0(\beta)\rangle = 0$ implies $B^*A|0(\beta)\rangle = 0$ for all B; if $B \in \mathfrak{M}$ then the result above gives $A^*B|0(\beta)\rangle = 0$ for all $B \in \mathfrak{M}$; since $|0(\beta)\rangle$ is cyclic for $\mathfrak{M}$ (cf. Section 2), this implies $A^* = A = 0$.

This result is important because it implies that $|0(\beta)\rangle$ is cyclic not only for $\mathfrak{M}$ (as in all GNS representations) but also for its commutant $\mathfrak{M}' = \pi_\omega(\mathfrak{A})'$:

$$\mathscr{H}_\omega = \overline{\pi_\omega(\mathfrak{A})'|0(\beta)\rangle}. \tag{5.7}$$

In words, any vector $|x\rangle \in \mathscr{H}_\omega$ can be arbitrarily well approximated by vectors of the form $\tilde{B}_x|0(\beta)\rangle$, with $\tilde{B}_x \in \mathfrak{M}'$. (This shows that $\mathfrak{M}'$ is nontrivial, and we adopt the convention that a tilde tops its elements.) To prove (5.7) we first define a projection operator P that projects on the space $\overline{\mathfrak{M}'|0(\beta)\rangle} =: \mathscr{H}_c$. Clearly (5.7) is equivalent to $P = \mathbb{1}$. We have $\mathscr{H}_\omega = \mathscr{H}_c \oplus \mathscr{H}_c^\perp$, and from this it follows that P commutes with all elements in $\mathfrak{M}'$: $P\tilde{A}|x\rangle = \tilde{A}|x\rangle = \tilde{A}P|x\rangle$ for all $|x\rangle = \tilde{B}_x|0(\beta)\rangle \in \mathscr{H}_c$, while $P\tilde{A}|y\rangle = 0 = \tilde{A}P|y\rangle$ for all $|y\rangle \in \mathscr{H}_c^\perp$. The first member of the last pair of equalities follows by taking the inner product with an arbitrary vector, and then using the first pair. Hence $P \in \mathfrak{M}$, but then also $P - \mathbb{1} \in \mathfrak{M}$, and since $\mathbb{1} \in \mathfrak{M}'$ we have $P|0(\beta)\rangle = |0(\beta)\rangle$ and therefore $P = \mathbb{1}$ by the previous theorem. This proves (5.7). As we see in Section 6 it is basically this result that prohibits a particle picture in thermal representations.

Tomita–Takesaki Theory

$\mathfrak{M}'$ must obviously be fairly large in order for (5.7) to be satisfied (in contrast to the vacuum case, where $\mathfrak{M}' = \mathbb{C}\mathbb{1}$), and in fact a more detailed analysis will show that $\mathfrak{M}'$ is as large as $\mathfrak{M}$: there is a one–one correspondence $\mathfrak{M} \ni A \leftrightarrow \tilde{A} \in \mathfrak{M}'$. This analysis forms the contents of the so-called Tomita–Takesaki theory [26]. It starts from the observation that the operator S given by

$$SA|0(\beta)\rangle = A^*|0(\beta)\rangle \tag{5.8}$$

is well-defined, because the fact that $|0(\beta)\rangle$ is separating for $\mathfrak{M}$ implies that a vector $|x\rangle$ can be written in at most one way as $|x\rangle = A_x|0(\beta)\rangle$ for some $A_x \in \mathfrak{M}$. We define the positive self-adjoint operator $\Delta := S^*S$ and the *anti* unitary operator $J = S\Delta^{-1/2}$, so that we have the polar decomposition

$$S = J\Delta^{1/2}. \tag{5.9}$$

Note that S is *anti*linear, so that its adjoint is defined by

$$\langle x|\, S^*\,|\, y\rangle = \langle y|\, S\,|x\rangle, \tag{5.10}$$

i.e., without complex conjugation of the r.h.s. A similar relation holds for J. It should be remarked that S, J, and Δ are not in $\mathfrak{M}$.

The aim of the following manipulations is to explicitly construct the isomorphism between $\mathfrak{M}$ and $\mathfrak{M}'$ and to relate Δ to the Hamiltonian defined in (5.2). If follows from (5.8) that $S^2 = \mathbb{1}$, and hence $J\Delta^{1/2} = \Delta^{-1/2}J^*$, from which we see that the operator $J^2\Delta^{1/2}$ equals $(J\Delta^{-1/4})(J\Delta^{-1/4})^*$, so that it is positive. But J^2 is unitary because J is antiunitary, so the polar decomposition of $J^2\Delta^{1/2}$ is on the one hand unitary $\times$ positive operator $= J^2 \times \Delta^{1/2}$ and on the other hand $\mathbb{1} \times J^2\Delta^{1/2}$. Uniqueness of the polar decomposition therefore implies

$$J^2 = \mathbb{1}; \qquad J^* = J, \tag{5.11}$$

where the second equation follows from the antiunitarity of J combined with the first one. From the reasoning above it also follows that $J\Delta^{1/2}J = \Delta^{-1/2}$, and calling

$$\Delta = e^{-\hat{K}}, \tag{5.12}$$

this implies

$$J\hat{K}J = -\hat{K}. \tag{5.13}$$

This property will ultimately lead to the instability of all thermal (would-be) particles. Choosing $A = \mathbb{1}$ in (5.8) yields $S|0(\beta)\rangle = |0(\beta)\rangle$, and this result combined with (5.10) shows that the same equation is satisfied by S^*. Hence by (5.9), (5.11), and (5.12) we find from $S^*S = \Delta$ that

$$\hat{K}|0(\beta)\rangle = 0. \tag{5.14}$$

The resemblance with (5.3) is not accidental; see (5.23). Use of (5.8) with $A = \mathbb{1}$, (5.9), (5.12), and (5.14) also shows that

$$J|0(\beta)\rangle = |0(\beta)\rangle. \tag{5.15}$$

The definition (5.8) leads to $[SAS, B] = 0$ for all $A, B \in \mathfrak{M}$ by operating with this

commutator on an arbitrary vector $C|0(\beta)\rangle$. Hence $S\mathfrak{M}S \subset \mathfrak{M}'$. It follows from (5.8) and (5.10) that

$$S^*\tilde{A}\,|0(\beta)\rangle = \tilde{A}^*\,|0(\beta)\rangle \tag{5.16}$$

for $\tilde{A} \in \mathfrak{M}'$, and then the reasoning above leads to $S^*\mathfrak{M}'S^* \subset \mathfrak{M}$. Therefore, $\Delta\mathfrak{M}\Delta^{-1} = S^*S\mathfrak{M}S^{-1}S^{*-1} \subset \mathfrak{M}$. Iteration gives $M^n := \Delta^n\mathfrak{M}\Delta^{-n} \subset \mathfrak{M}$ for all $n \in \mathbb{N}$. The sequence $\mathfrak{M}^n$ has a limit point at infinity, so [26] we in fact have $\mathfrak{M}^z \subset \mathfrak{M}$ for any $z \in \mathbb{C}$. Taking $z = -1$, it then follows that $\mathfrak{M} = \Delta\Delta^{-1}\mathfrak{M}\Delta\Delta^{-1} \subset \Delta\mathfrak{M}\Delta^{-1}$, and hence $\Delta\mathfrak{M}\Delta^{-1} = \mathfrak{M}$. Iterating this equation and using the reasoning above once again give, on use of (5.12),

$$e^{i\hat{K}t}\mathfrak{M}e^{-i\hat{K}t} = \mathfrak{M} \tag{5.17}$$

for all t. For real t the so-called modular automorphism

$$\sigma_t[A] := e^{i\hat{K}t}Ae^{-i\hat{K}t} \in \mathfrak{M} \tag{5.18}$$

is an *-automorphism (i.e., it satisfies (2.4), (2.5)) which maps $\mathfrak{M}$ into itself. We are now in a position to construct the isomorphism $\mathfrak{M} \leftrightarrow \mathfrak{M}'$: we have $J\mathfrak{M}J = J\Delta^{1/2}\mathfrak{M}\Delta^{-1/2}J = S\mathfrak{M}S \subset \mathfrak{M}'$, and similarly $J\mathfrak{M}'J \subset \mathfrak{M}$, which on use of (5.11) together yield

$$J\mathfrak{M}J = \mathfrak{M}'; \qquad J\mathfrak{M}'J = \mathfrak{M}. \tag{5.19}$$

Hence the isomorphism $\mathfrak{M} \ni A \leftrightarrow \tilde{A} \in \mathfrak{M}'$ is given by

$$\tilde{A} = JAJ. \tag{5.20}$$

Thermal Field Theory Regained

We can extend the automorphism σ_t to act on $\mathfrak{M}'$, and use of (5.13), (5.17)–(5.20) shows that

$$J\sigma_t[A]\,J = \sigma_t[\tilde{A}] = e^{i\hat{K}t}\tilde{A}e^{-\hat{K}t} \in \mathfrak{M}'. \tag{5.21}$$

The whole point is now that σ_t satisfies the KMS condition at $\beta = 1$:

$$\langle 0(\beta)|\, A\sigma_t[B]\,|0(\beta)\rangle = \langle 0(\beta)|\, \sigma_{t-i}[B]\, A\,|0(\beta)\rangle, \tag{5.22}$$

as follows from a simple direct computation employing (5.8), (5.9), (5.12), (5.18) and the commutant properties above. Therefore, $\sigma_{t/\beta}$ satisfies the KMS condition at β, and because an automorphism that satisfies the KMS condition is unique [26], we may identify $\sigma_{t/\beta}$ with the actual time evolution (5.4). Comparing with (5.18) yields

$$\hat{K} = \beta\hat{H}, \tag{5.23}$$

so that (5.13) implies

$$J\hat{H}J = -\hat{H}, \tag{5.24}$$

and (5.8) translates to the so-called thermal state condition [38]

$$e^{-(1/2)\beta\hat{H}}A\,|0(\beta)\rangle = \tilde{A}^*\,|0(\beta)\rangle, \tag{5.25}$$

where we have substituted (5.9) with (5.12) and (5.23) and used (5.20), (5.15). Note that (5.11) implies that the tilde operation commutes with taking the adjoint. Finally, (5.23) and (5.14) reproduce (5.3).

Equation (5.24) shows that the effective thermal Hamiltonian $\hat{H}$ must have a spectrum that is symmetric around zero: its (possibly improper) eigenvectors $|E\rangle$ and $J|E\rangle$ have opposite eigenvalues, so that

$$J|E\rangle = |-E\rangle. \tag{5.26}$$

In an infinite system $\hat{H}$ is always an unbounded operator, and we see that it must be *unbounded from below* (bottomless). This property will lead to the instability alluded to before. In fact, it can be shown [44] that the spectrum of $\hat{H}$ as a set is equal to the real axis. In local field theory we can introduce a Hamiltonian density $\hat{\mathscr{H}}(x)$ such that

$$\hat{H} = \int d^3x \hat{\mathscr{H}}(\mathbf{x}). \tag{5.27}$$

The properties (5.18), (5.20), (5.21), (5.24), and (5.3) then imply

$$\hat{\mathscr{H}}(x) = \mathscr{H}(x) - \tilde{\mathscr{H}}(x), \tag{5.28}$$

where $\mathscr{H}(x)$ is the ordinary Hamiltonian density composed of the fields that coordinatize $\mathfrak{M}$, so we presume that $\mathscr{H}(x)$ is affiliated with $\mathfrak{M}$ and $\tilde{\mathscr{H}}$ with $\mathfrak{M}'$. Although the operator $H = \int d^3x \mathscr{H}(x)$ makes sense in vacuum representations and also in finite-volume KMS representations, it is a *meaningless object* in general thermal (KMS) representations (for example, it sends the thermal vacuum $|0(\beta)\rangle$ away to infinite norm). The formal decomposition [18, 34–38]

$$\hat{H} = H - \tilde{H} \tag{5.29}$$

is only meaningful if it is read as a (strong) limit

$$\hat{H} = \lim_{V \to \infty} \int_V d^3x (\mathscr{H}(x) - \tilde{\mathscr{H}}(x)). \tag{5.30}$$

We emphasize this point, because for our purpose it is extremely important to realize that the spectrum of $\hat{H}$ is altogether unrelated to that of H, as the latter is defined in vacuum representations and, like $\hat{H}$, is a representation-dependent object.

As we have seen in Section 3, correct perturbation theory in infinite field-theoretic systems is strongly related to spectral properties of the relevant symmetry generators, and the remarks above indicate that the situation in thermal representations will be radically different from the vacuum case.

A parenthetical remark is that there exist more intrinsic, representation-independent notions of the spectrum of an automorphism group (like α_t), namely the Arveson spectrum and the Connes spectrum [45]. The practical relevance of these partial characterizations of the dynamics for the type of analysis performed in this paper ought to be closely examined.

To close, we mention that the commutant structure $A \leftrightarrow \tilde{A}$ and the properties (2.1), (5.3), (5.25), and (5.29) are the basis of (equilibrium) thermo field dynamics [18, 38, 3]; as we have seen, the whole structure follows from the single KMS condition (5.1). The connection between thermo field dynamics and the operator-algebraic formalism was recognized by Ojima [46].

6. Thermal Quasiparticles?

Free Fields

The theory defined by a free neutral relativistic scalar field $A(x)$ with mass m can, of course, be exactly solved in thermal representations. The results are the following [18, 3]. We decompose

$$A(x) = \int \frac{d^3p}{(2\pi)^3\,(2\varepsilon_p)^{1/2}}\, e^{-i\varepsilon_p t + i\mathbf{p}\cdot\mathbf{x}} a(\mathbf{p}) + \text{h.c.} \tag{6.1}$$

with $\varepsilon_p = \sqrt{\mathbf{p}^2 + m^2}$, and $[a(\mathbf{p}), a^*(\mathbf{p}')] = (2\pi)^3\,\delta(\mathbf{p} - \mathbf{p}')$, as in the vacuum case. The $a^*(\mathbf{p})$ create (unnormalized) one-particle states with energy (i.e., eigenvalue of $\hat{H}$) ε_p from the thermal vacuum

$$a^*(\mathbf{p})|0(\beta)\rangle = |\mathbf{p}_j; \varepsilon_{\mathbf{p}}\rangle, \tag{6.2}$$

and according to (5.26) we also have

$$\tilde{a}^*(\mathbf{p})|0(\beta)\rangle = |-\mathbf{p}_j; -\varepsilon_{\mathbf{p}}\rangle \tag{6.3}$$

(we have $\hat{\mathbf{P}} = \mathbf{P} - \tilde{\mathbf{P}}$ in the sense of (5.28)). However, the $a(\mathbf{p})$ and $\tilde{a}(\mathbf{p})$ do *not* annihilate the thermal vacuum, for this would contradict (5.25) and (6.3). In fact, $a(\mathbf{p})$ *creates* a negative energy particle à la (6.3) (cf. (6.4), (6.5)). Instead, (5.25) allows the construction of "thermal" annihilation operators by the Bogoliubov transformation

$$\beta(\mathbf{p}) = \sqrt{1 + N(\varepsilon_{\mathbf{p}})}\, a(\mathbf{p}) - \sqrt{N(\varepsilon_{\mathbf{p}})}\, \tilde{a}^*(\mathbf{p}), \tag{6.4}$$

where N is the Bose distribution function [3]. By (5.25) and (6.2), (6.3) we indeed have

$$\beta(\mathbf{p})\,|0(\beta)\rangle = \tilde{\beta}(\mathbf{p})\,|0(\beta)\rangle = 0. \tag{6.5}$$

Note that β and $\tilde{\beta}$ are not affiliated with $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, but with $\mathfrak{M} \cup \mathfrak{M}'$, although $\tilde{\beta}$ is still defined by (5.20). Up to normalization, Eqs. (6.2), (6.3) are also satisfied with a, a^* replaced by β, β^*; to indicate that $\beta^*(\mathbf{p})$ creates a state with energy $\varepsilon_{\mathbf{p}}$ we may call it $\beta^*(\mathbf{p}, \varepsilon_{\mathbf{p}})$. We thus have

$$\tilde{\beta}^*(-\mathbf{p}, \varepsilon_{\mathbf{p}}) = \beta^*(\mathbf{p}, -\varepsilon_{\mathbf{p}}). \tag{6.6}$$

Explicit construction [18, 26] based on the results above shows that for a free-field theory $\mathscr{H}_\omega$ is a "doubled" Fock space; we may symbolically write

$$\mathscr{H}_\omega = \mathfrak{F} \otimes \tilde{\mathfrak{F}} \tag{6.7}$$

where $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ are both isomorphic to the Hilbert space Fock$(L^2(\mathbb{R}^3))$. The difference lies in the fact that states in the subspace $\mathfrak{F}$ have positive energy while those in $\tilde{\mathfrak{F}}$ have negative energy. Since $\mathfrak{F}$ is a Fock space, the operators $a^{(*)}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ form a complete set on $\mathfrak{F}$, and so do the $\tilde{a}^{(*)}(\mathbf{p}, -\varepsilon_{\mathbf{p}})$ on $\tilde{\mathfrak{F}}$. Hence the $a^{(*)}$ and $\tilde{a}^{(*)}$ together form a complete set on $\mathscr{H}_\omega$. (The $a^{(*)}$ alone are sufficient to "span" $\mathscr{H}_\omega$ in the sense of (2.2), but they are not complete because the tilde operators cannot be expressed in terms of them.) Equivalently, the $\beta^{(*)}$, $\tilde{\beta}^{(*)}$ form a complete set on $\mathscr{H}_\omega$. This set is more convenient because well-defined operators on $\mathscr{H}_\omega$ must be normal-ordered with respect to the $^{(\tilde{\beta})(*)}$ and *not* with respect to the $^{(\tilde{a})(*)}$. For example, the free Hamiltonian reads [18]

$$\hat{H} = \hat{H}_0 = \int \frac{d^3p}{(2\pi)^3}\, \varepsilon_{\mathbf{p}} (\beta^*(\mathbf{p}, \varepsilon_{\mathbf{p}})\, \beta(\mathbf{p}, \varepsilon_{\mathbf{p}}) - \tilde{\beta}^*(\mathbf{p}, \varepsilon_{\mathbf{p}})\, \tilde{\beta}(\mathbf{p}, \varepsilon_{\mathbf{p}})). \tag{6.8}$$

It is very instructive to recast the structure of $\mathscr{H}_\omega$ in group-theoretic language, in particular to establish contact with (2.9). Because KMS states explicitly break boost symmetry, the implementable symmetry group is not the full Poincaré-group P but its boostless subgroup $S = SO(3)$ $(\times T_4$. As explained in the Appendix, the irreducible unitary representations of S are labeled by three numbers: $E \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$, and $n \in \mathbb{Z}$ (so that $\hat{S} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{Z}$). In the scalar case we can ignore n $(=0)$ for free fields because of the Fock structure (6.7) combined with the properties of the Clebsch–Gordan series (A.19) for $n = 0$, so the decomposition (2.9) in the present case reads

$$\mathscr{H}_\omega = \bigoplus_{k=1}^{\infty} k \int_{-\infty}^{\infty} dE \int_0^{\infty} d\sigma^2\, \mu_k'(E, \sigma^2)\, \mathscr{H}(E, \sigma^2, 0). \tag{6.9}$$

Here we have (sloppily) written $\mu_k'(E, \sigma^2) = d^2\mu_k(E, \sigma^2)/dE d\sigma^2$, which we allow to be a product of two delta functions in the case where μ_k is a point measure, and a

product of a single delta function and a continuous function in the case where μ_k is a singularly continuous measure. In the case where μ_k is absolutely continuous, $\mu_k'(E, \sigma^2)$ is just a continuous function of E and σ^2, which may be taken to be a constant. The existence of one-particle states (6.2), (6.3) then implies that apart from the vacuum contribution $\delta(E)\,\delta(\sigma^2)$ the measure contains two delta functions,

$$\mu_1'(E, \sigma^2) = \delta(E - \varepsilon(\sigma^2)) + \delta(E + \varepsilon(\sigma^2)) + \cdots, \tag{6.10}$$

plus an absolutely continuous part, whose explicit form may be calculated from the Clebsch–Gordan series (A.19). As in the vacuum case, the measures μ_k' in (6.9) contain direct spectral information.

Interacting Theories

We now ask what remains of this structure in the interacting case. It is a highly distinctive feature of the Haag–Ruelle theory for vacuum representations that the spectral structure of $\mathscr{H}_\omega$ *does not change at all* in passing from the free to the interacting theory. In particular, $\mathscr{H}_\omega$ remains a Fock space in the sense of (3.2). What does change is the representation of the canonical commutation relations; the physical vacuum $|0\rangle$ is no longer annihilated by the annihilation operators associated with the time-zero fields but by those associated with the in- or out-field. In perturbation theory the conditions for the Haag–Ruelle theory may be checked by examining whether or not the self-energy vanishes on-shell.

What happens in thermal representations? The standard perturbation theory that is universally used employs the propagators of the free-field theory [3, 18, 20], and this can be justified only if we have the Haag–Ruelle situation even at nonzero temperature; i.e., if the measure in (6.9) is not affected by the presence of interactions. This perturbation theory, however, predicts its own collapse: the perturbatively calculated self-energy does not vanish on-shell. The next- simplest assumption is that the spectral structure (6.10) remains intact, but with an energy $\varepsilon(\sigma^2)$ that differs from its vacuum value and may become T-dependent. In other words, one has a dynamical quasi-particle structure [7]. This assumption is based on experience with zero-temperature finite density representations [18], where it is often satisfied, e.g., in the case of normal Fermi liquids. Bound states c.q. collective excitations, as in superfluid systems, can be incorporated by adding more delta-function contributions of the type (6.10) to the measure in (6.9). The energy $\varepsilon(\sigma)$ and therewith the free propagator are then determined by self-consistent computations [18].

We have seen, however, that thermal representations differ radically from ground-state (including vacuum) ones. In the present context it is the typically thermal property (5.7), showing that $|0(\beta)\rangle$ is cyclic for the commutant $\mathfrak{M}'$, that essentially precludes such a dynamical thermal quasi-particle picture. The crucial point is that contrary to vacuum and ground-state (quasi-) particles it is impossible to add interactions between thermal quasiparticles. Hence they do not scatter, and their contribution to thermodynamic quantities would be trivial. This does not actually

prohibit their presence in thermal representations, but is does show that they completely decouple from the rest of the theory if they exist, and do not influence any dynamical features. This would be in very sharp contrast with the ground-state c.q. vacuum case, where the (quasi-) particle sector makes up the whole theory in the case of asymptotic completeness!

Narnhofer–Requardt–Thirring Theorem

The basic argument leading to this result is due to Narnhofer, Requardt, and Thirring [24]. It is not absolutely rigorous in a mathematical sense, because it presumes that the one-particle excitation operators creating the quasi-particle states from the thermal vacuum (cf. Section 3) are affiliated with $\mathfrak{M}$, which has not yet been proven. Nevertheless, this shortcoming is what we earlier have been calling a "loop correction" to the argument, and we shall proceed on the assumption that it does not affect the end result.

If one-particle states

$$|f\rangle = \int d^3p \hat{f}(\mathbf{p}) |\mathbf{p}, \varepsilon_{\mathbf{p}}\rangle \tag{6.11}$$

exist, there must be one-particle excitation operators $B(f)$ creating them from the thermal vacuum, as in the Haag–Ruelle theory (cf. Section 2; the actual construction of $B(f)$ from the field $A(x)$ is more complicated in the thermal case, because the discrete point $\varepsilon(\mathbf{p})$ is embedded in a continuum [24]). By construction, they must satisfy the analogue of (3.7):

$$B(f, t)|0(\beta)\rangle := e^{i\hat{H}t} B(f) e^{-i\hat{H}t} |0(\beta)\rangle = B(e^{i\varepsilon(\nabla)t} f)|0(\beta)\rangle. \tag{6.12}$$

Now, however, this property implies that the $B(f, t)$ are actually free-field operators! As mentioned above, we assume that $B(f, t)$ commutes with all elements in $\mathfrak{M}'$ (which would definitely be true if B were bounded, cf. (5.18)). But then (6.12) is true with $|0(\beta)\rangle$ replaced by an arbitrary vector $|x\rangle = \tilde{B}_x |0(\beta)\rangle$, $\tilde{B}_x \in \mathfrak{M}'$, so that

$$B(f, t) = B(e^{i\varepsilon(\nabla)t} f) \tag{6.13}$$

and from the construction of in- and out-operators in Section 2 it then follows that $a^{\text{in}} = a^{\text{out}}$; i.e., the S-matrix is unity and there is no interaction between the quasiparticles. To be quite certain of this result, the authors of [24] also evaluated the n-point correlation functions of the one-particle excitation operators and showed that they factorize in a free-field way.

To this result we add the physical argument that a Hamiltonian which is unbounded from below, like $\hat{H}$ in thermal representations, cannot admit stable particle states in the presence of interaction (and allowed decay channels). On top of this, even improved perturbation theory with self-consistently determined temperature-dependent energies in the propagator leads to a nonvanishing value of the

imaginary part of the self-energy on the (improved) energy-shell [18]. Combining these arguments we feel forced to conclude that thermal representations of interacting relativistic field theories do not have a particle structure. But what do they look like instead?

7. Hilbert-Space Rearrangement

Thermal Measure

To at least partly answer the question of the preceding paragraph we should determine the measures $\mu_k(\hat{g})$ in (2.9) for $\hat{g}=\hat{s}=(E, \sigma^2, n)$. For the moment, we shall restrict ourselves to a situation in which a single real scalar field $A(x)$ suffices to coordinatize M. It may be worthwhile to remark that we cannot exclude the possibility that representation spaces $\mathscr{H}$ $(E, \sigma^2, n \neq 0)$ carrying nonzero helicity n (cf. (A.8)) occur in the decomposition (2.9), even though these are absent in the free-field case (6.9). For simplicity we just assume that such subrepresentations do not materialize in interacting theories. One should keep in mind, however, that by this assumption we may miss a natural incorporation of certain (linearized) hydrodynamic modes as state vectors in $\mathscr{H}_\omega$. Foregoing this opportunity, we are back to the problem of determining the functions $\mu_k'(E, \sigma^2)$ in (6.9).

As we have seen, the Ansatz (6.10) seems to be excluded in the interacting case. A few basic properties are helpful in specifying an alternative. In theories where the field $A(x)$ satisfies canonical equal-time commutation relations one needs support from the entire σ^2-axis. Furthermore, the spectrum of the effective Hamiltonian $\hat{H}$ is the real axis $\mathbb{R}$ [44]. Finally, the thermal vacuum $|0(\beta)\rangle$ is assumed to be invariant under the entire group $S=SO(3)$ $(\times T_4$, so that it is annihilated by its generators (cf. (2.7), (5.3)). All these features are incorporated by the postulate [16] that up to unitary equivalence the function μ_k' in (6.9) is the sum of a delta function picking up the thermal vacuum, and an absolutely continuous part, which without loss of generality we may take to be $\sigma/4\pi$. Hence

$$\hat{\mathscr{H}}_\omega = \mathscr{H}(0, 0, 0) \oplus \infty \cdot \hat{\mathscr{H}}^{(1)}, \tag{7.1}$$

with

$$\hat{\mathscr{H}}^{(1)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \int_0^\infty d\sigma^2 \, \sigma \mathscr{H}(E, \sigma^2, 0) \tag{7.2}$$

for *interacting* thermal scalar field theories. Any multiplicity other than infinity in (7.1) would be most puzzling indeed.

Function Space

Let us now investigate the structure of the Hilbert space (7.1) more closely. First, it follows from the Clebsch–Gordan series (A.19) for S that $\hat{\mathscr{H}}_\omega$ in (7.1) is in fact nothing but a Fock space over $\hat{\mathscr{H}}^{(1)}$,

$$\hat{\mathscr{H}}_\omega = \mathrm{Fock}(\hat{\mathscr{H}}^{(1)}), \tag{7.3}$$

in the sense of (3.2); cf. the text below (A.20). This feature is a consequence of the fact that, in contrast to the corresponding situation for the Poincaré-group, the tensor product of scalar (i.e., $n=0$) representations of S contains only scalar subrepresentations. Therefore, the study of $\hat{\mathscr{H}}_\omega$ is greatly simplified, and for most intents reduces to that of $\hat{\mathscr{H}}^{(1)}$.

According to the definition of a direct integral of Hilbert spaces (cf. (2.10) and surrounding text), elements of $\hat{\mathscr{H}}^{(1)}$ (7.2) are L^2-functions $\Psi(\cdot)$ with argument (E, σ^2) and taking values in $\mathscr{H}(E, \sigma^2, 0)$. Since elements of the latter are L^2-functions $\Phi_{E,\sigma^2}(\cdot)$ on the two-sphere, with inner product $\langle \cdot | \cdot \rangle_{\mathscr{H}(E,\sigma^2)}$ given by (A.4), it follows from (2.10) that elements of $\hat{\mathscr{H}}^{(1)}$ are in fact ordinary complex-valued functions $\Phi(\cdot)$ on $\mathbb{R}^4$ with argument $(E, \sigma^2, \theta, \phi)$, for which the inner product in $\hat{\mathscr{H}}^{(1)}$,

$$\langle \Psi(\cdot) | \Phi(\cdot) \rangle_{\hat{\mathscr{H}}^{(1)}} = \int \frac{dE}{2\pi} d\sigma \sigma^2 \langle \Psi_{E,\sigma^2}(\cdot) | \Phi_{E,\sigma^2}(\cdot) \rangle_{\mathscr{H}(E,\sigma^2,0)}$$

$$= (2\pi)^{-4} \int dE\, d\sigma\, d\Omega\, \sigma^2 \bar{\Psi}(E, \sigma^2, \theta, \phi)\, \Phi(E, \sigma^2, \theta, \phi), \quad (7.4)$$

is finite. We may assemble the variables $E, \sigma^2, \theta, \phi$ into the four-vector $p=(E, \mathbf{p})$ with $\mathbf{p}^2=\sigma^2$, and write (7.4) in the appealing form

$$\langle \Psi(\cdot) | \Phi(\cdot) \rangle = \int \frac{d^4p}{(2\pi)^4} \bar{\Psi}(p)\, \Phi(p). \quad (7.5)$$

Consequently, $\hat{\mathscr{H}}^{(1)}$ is isomophic to $L^2(\mathbb{R}^4)$. The different components of p are not on equal footing; however, $p_0=E$ and $\mathbf{p}^2=\sigma^2$ together label different orbits in $\hat{T}_4$ (cf. Appendix), while θ and ϕ identify points on a given orbit. Note that the mass-shell factor $\theta(p_0)\,\delta(p^2-m^2)$ characterizing the corresponding inner product for scalar representations of the Poincaré-group is absent from (7.5).

Generalized Fourier Transform

The theory of induced representations has provided us with a concrete realization of $\hat{\mathscr{H}}^{(1)}$ as the function space $L^2(\mathbb{R}^4)$ with elements $\Phi(\cdot)$, and of $\hat{\mathscr{H}}_\omega$ as a Fock space over $L^2(\mathbb{R}^4)$. We now return to the original representation space $\mathscr{H}_\omega$, which is related to $\hat{\mathscr{H}}_\omega$ by (2.9). According to (7.3) it is sufficient to restrict ourselves to

$$\mathscr{H}^{(1)} := F^{-1} \hat{\mathscr{H}}^{(1)}, \quad (7.6)$$

where F is the generalized Fourier transform appearing in (2.9), because (7.3) clearly implies that $\mathscr{H}_\omega$ is a Fock space over $\mathscr{H}^{(1)}$. We denote elements of $\mathscr{H}^{(1)}$ by

$$|\Phi\rangle := F^{-1}\, \Phi(\cdot). \quad (7.7)$$

By unitarity of F

$$\langle \Psi | \Phi \rangle = \langle \Psi(\cdot) | \Phi(\cdot) \rangle, \quad (7.8)$$

where the right-hand side is given by (7.5). We can bring (7.7) into a more explicit form by invoking the nuclear spectral theorem [40]. This ascertains the existence of a linear functional $\langle p|$, defined on a dense nuclear subspace $F^{-1}N \subset \mathscr{H}^{(1)}$, satisfying

$$\langle p | \Phi \rangle = \Phi(p). \tag{7.9}$$

(One may, for example, choose N as the Schwartz space $\mathscr{D}(\mathbb{R}^4)$.) It then follows from (7.5), (7.8), and (7.9) that

$$|\Phi\rangle = \int \frac{d^4p}{(2\pi)^4} \Phi(p) | p \rangle, \tag{7.10}$$

where $|p\rangle$ is the antilinear functional on $F^{-1}N$ defined by $\langle \Phi | p \rangle = \overline{\langle p | \Phi \rangle}$. Equation (7.10) establishes the explicit correspondence between $|\Phi\rangle \in F^{-1}N \subset \mathscr{H}^{(1)}$ and $\Phi(\cdot) \in N \subset \hat{\mathscr{H}}^{(1)} = L^2(\mathbb{R}^4)$.

Non-shell States

The improper kets $|p\rangle = |E, \sigma^2, \theta, \phi\rangle$ are at the basis of our physical interpretation of the thermal representation space $\mathscr{H}_\omega$. By the Fock structure (7.3) the vectors $|0(\beta)\rangle$, $|p\rangle$, ..., $|p_1 \cdots p_N\rangle, \cdots$ form an improper basis in $\mathscr{H}_\omega$. This resembles a basis of ordinary multiparticle states that spans $\mathscr{H}_\omega$ in the vacuum case, cf. Section 3. The crucial difference is that now p_0 and $\mathbf{p}$ are altogether unrelated. We express this absence of a mass-shell restriction $p_0 = \varepsilon(\mathbf{p})$ by calling $|p\rangle$ a non-shell particle state. As the notation already suggests (and as we show in the next section), $|p\rangle$ is a generalized eigenvector of the generator $\hat{P}^\mu$ of the unitary representation $U(x)$ of $x \in T^4 \subset S$ in the representation space $\mathscr{H}_\omega$ with generalized eigenvalue p^μ. To obtain proper elements of $\mathscr{H}_\omega$ we must smear in *both* p_0 and $\mathbf{p}$ separately (cf. (7.10)). This resembles the description of multi(-stable-)particle states in vacuum representations, where one must smear in the momentum $\mathbf{p}$ and the mass $s = p^2$. As we have seen in Section 4, one may regard an s-wave two-particle state as a single-particle state of an unstable particle with continuous mass s, and thereby partly forget the underlying description in terms of stable particles. In the thermal case such an underlying level is absent, and only the description in terms of unstable particles remains. We summarize these ideas by calling proper vectors in $\mathscr{H}_\omega$ *non-shell unstable-particle states.* Paraphrasing Lukierski [28] we may call the generalized state $|p\rangle$ an elementary non-shell unstable object.

We cannot overemphasize the peculiarity that in contrast to the vacuum case (without symmetry breaking) the introduction of interactions in thermal representations completely changes the representation Hilbert space. This process is somewhat similar to what occurs in zero-temperature (vacuum or finite density) representations in which an internal symmetry is broken spontaneously or dynamically. The spectrum of the theory, and therewith the complete set of operators tied to the representation π_ω, $\mathscr{H}_\omega$, then completely changes [18]. One may call this phenomenon an interaction-induced *Hilbert-space rearrangement.*

At zero temperature such a rearrangement is always accompanied by the emergence of Goldstone bosons (or fermions, if a supersymmetry is broken). Since in our description so much emphasis is put on the breakdown of the Poincaré-group P to its boostless subgroup S in thermal representations, one may wonder where the accompanying Goldstone bosons are. It turns out that these are absent indeed, because the breakdown of P to S in thermal representations is of a very special type. To wit, the zero-energy (intermediate) states that are necessary for the fulfillment of the Ward–Takahashi identities associated with the symmetry breaking are automatically present, because the spectrum of the effective thermal Hamiltonian $\hat{H}$ is the entire real axis. Adding special gapless particles is unnecessary: the zero modes can be formed by combining positive and negative energy states [47, 48], and the Ward–Takahashi relations are identically satisfied [49].

8. Covariant Operator Fields

In order to further analyze the thermal representation $\pi_\omega(\mathfrak{A})$, $\mathscr{H}_\omega$, keeping in mind the possible derivation of a Gell-Mann/Low formula, we proceed with the construction of a complete set of operators on $\mathscr{H}_\omega$ in the sense explained at the end of Section 2. The implementable symmetry group is now $S = SO(3)$ $(\times T_4$. We must, therefore, construct a set of covariant operator fields $\phi(x)$ whose transformation behaviour under S, in particular its subgroup T_4 describing space-time evolution, is explicitly known. In this context "covariant" means that the $\phi(x)$ are tensor operators transforming under a so-called covariant representation of S; cf. (A.10).

S-Covariance

The covariance of the fields $\phi(x)$ ultimately derives from the automorphism (2.3)–(2.5) for G chosen as the Poincaré-group P. At the level of coordinatizing fields $A_a(x)$ this automorphism is always generated by a covariant representation T^f of P according to

$$\alpha_p[A_a(x)] = (T^f(p^{-1})\, A)_a\,(x). \tag{8.1}$$

The representation T^f is, in turn, given by a representation $\hat{l}$ of the Lorentz group L acting on the index a of A, viz.

$$(T^f(p)\, A)_a\,(x) = D^f(\Lambda)_{ab}\, A_b(\Lambda^{-1}(x-a)), \tag{8.2}$$

where $p = (\Lambda, a)$; cf. (A.10). Then (2.3) is identically satisfied, while (2.4) and (2.5) are supposed to hold on account of the particular algebraic structure imposed on the algebra $\mathfrak{A}$ [25, 27]. As we have seen, the automorphism α_s can be unitarily

implemented in the thermal representation space $\mathscr{H}_\omega$ for $s \in S \subset P$. From (2.8), (8.1), and (8.2) we then have for $s = (R, a)$

$$\alpha_s[A_a(x)] = U(s)\, A_a(x)\, U^*(s) = D^{\hat{I}}(R^{-1})_{ab}\, A_b(Rx + a), \tag{8.3}$$

where $Rx = (x^0, R\mathbf{x})$. (Here and in (8.1) we have made no notational distinction between the automorphism α acting on $\mathfrak{A}$ and that on the fields coordinatizing $\pi_\omega(\mathfrak{A})''$.) Now since the complete set of fields $\{\phi(x)\}$ is chosen in such a way that the fields $A_a(x)$ can be polynomially expressed in terms of them (dynamical map [18]) it follows from the unitarity of the $U(s)$ that the $\phi(x)$ must satisfy (8.3), i.e., transform covariantly under S, too. Note that the representation $\hat{I}$ subduced from the Lorentz-group L to its subgroup $SO(3)$ is reducible in nontrivial cases.

Non-shell Particle Content

Apart from transforming covariantly, the fields $\phi(x)$ should also carry all information about the non-shell particle content of the thermal space $\mathscr{H}_\omega$. This information is contained in the direct integral decomposition (2.9) (for $G = S$), a special case of which is (7.1)–(7.3) for the scalar case, together with the generalized Fourier transform F. In order to explicitly demonstrate that our formalism does not hinge upon the restriction to scalar representations we shall consider the Hilbert space (7.3) with $\hat{\mathscr{H}}^{(1)}$ given by (A.8) rather than the special case (7.2). The Hilbert space (A.8) is the minimal choice for a non-shell one-particle space with nonzero "helicity" n, since the occurrence of the subrepresentation (E, σ^2, n) automatically implies the presence of $(-E, \sigma^2, -n)$ as well: the two of these are connected by the modular conjugation J defined in Section 5 (cf. (5.24) and (8.15)). Representations with a more general non-shell particle content may then be constructed by forming tensor products or direct sums, as in the vacuum case. Note, however, that even with this generalized choice of $\hat{\mathscr{H}}^{(1)}$ we do not cover the most comprehensive situation thinkable, because we *derived* the (generalized) Fock space structure (7.3) for the scalar case (7.2) only. For practical applications, however, the formalism below seems to be sufficiently general.

Consider, then, the representation (A.9) of S acting on the function space $\hat{\mathscr{H}}_n^{(1)} = L^2(\mathbb{R}^4)$. The actual non-shell one-particle subspace $\mathscr{H}_n^{(1)} \subset \mathscr{H}_\omega$, over which $\mathscr{H}_\omega$ is a Fock space, is given by the generalized Fourier transform (7.6), which has been made concrete by (7.9). The action of the unitary operator $U(s)$, defined by (2.7) (for $G = S$) or by the first member of (8.3), on $\mathscr{H}_n^{(1)}$ is given by the generalized Fourier transform of (A.9). Using (7.7), (7.9), and (A.5) this yields

$$\langle p; n |\, U(s)\, | \Phi \rangle = (U_n^{(1)}(s)\, \Phi)(p) = \exp i(pa + n\alpha(R, p)) \langle R^{-1}p; n \,|\, \Phi \rangle, \tag{8.4}$$

where $s = (R, a)$, and the Wigner rotation α is defined below (A.5). We have written $\langle p; n|$ rather than $\langle p|$ to explicitly indicate that this is a functional on (a dense

subspace of) $\mathscr{H}_n^{(1)}$. Taking the conjugate of (8.4), omitting the arbitrary state $|\Phi\rangle \in F^{-1}N \subset \mathscr{H}^{(1)}$ (cf. Section 7), and using the unitarity of $U(s)$ we find

$$U(s)|\, p; n\rangle = \exp i(p_0 a_0 - R\mathbf{p}\cdot\mathbf{a} + n\alpha(R,p))|(p_0, R\mathbf{p}); n\rangle. \tag{8.5}$$

In particular, we may choose $S=(\mathbb{1}, (t, \mathbf{0}))$ and use (5.2) to find

$$\hat{H}\,|\, p; n\rangle = p_0\,|\, p; n\rangle, \tag{8.6}$$

as anticipated in the previous section.

Thermal Construction Operators

We subsequently introduce a thermal creation operator $\beta_n^*(p)$, which is the non-shell analogue of $\beta^*(\mathbf{p})$ in Section 6, so that

$$|\, p; n\rangle = \beta_n^*(p)|\, 0(\beta)\rangle. \tag{8.7}$$

Its transformation behaviour follows from (8.5), (8.7) as

$$U(s)\,\beta_n^*(p)\,U^*(s) = e^{iRpa + in\alpha}\beta_n^*(Rp), \tag{8.8}$$

with $Rp=(p_0, R\mathbf{p})$, and $\alpha=\alpha(R,p)$. Hence

$$U(s)\,\beta_n(p)\,U^*(s) = e^{-iRpa - in\alpha}\beta_n(Rp) \tag{8.9}$$

for the annihilation operator β_n satisfying

$$\beta_n(p)|\, 0(\beta)\rangle = 0, \tag{8.10}$$

cf. (6.5). We emphasize that β_n and β_n^* are affiliated with $\mathfrak{M}\cup\mathfrak{M}'$ and not with $\mathfrak{M}$; cf. Section 6. Their commutator follows from (8.7), (8.10) and (7.5), (7.8) as

$$[\beta_n(p), \beta_{n'}^*(p')] = (2\pi)^4\,\delta_{nn'}\,\delta^4(p-p'). \tag{8.11}$$

These results may be compared with the corresponding rules for massless particles in vacuum field theory [50]. One finds that the integer $n\in\mathbb{Z}$ plays the same role for non-shell particles as the helicity for massless on-shell particles.

It follows from the results of the Appendix that irreducible non-shell one-particle states may be obtained by restricting p to the orbit $\mathscr{O}_{E,\sigma^2}=\{p_0=E, \mathbf{p}^2=\sigma^2\}$. The corresponding irreducible creation and annihilation operators are denoted $\beta_{\hat{s}}^*(\cdot)$ and $\beta_{\hat{s}}(\cdot)$, respectively; their argument can be chosen as p, where it is understood that $p\in\mathscr{O}_{E,\sigma^2}$, or as θ, ϕ; cf. the text after (A.3). The element $\hat{s}=(E,\sigma^2,n)$ of the dual $\hat{S}$ is defined in the Appendix; below it is understood that $\sigma^2>0$, and we omit the suffix $>$ in $\hat{s}_>$. We also use the abbreviations $\hat{s}_- := (E,\sigma^2,-n)$, and $s_- :=$

$(R, -a) \in S$. We then see from (A.5), (8.3), (8.8), and (8.9) that the following transformation rules hold,

$$\alpha_s[\beta_{\hat{s}}(p)] = (U_{\hat{s}}(s^{-1})\,\beta_{\hat{s}})(p); \tag{8.12}$$

$$\alpha_s[\beta^*_{\hat{s}}(p)] = (U_{\hat{s}_-}(s_-^{-1})\,\beta^*_{\hat{s}})(p); \tag{8.13}$$

cf. (8.1). Hence $\beta_{\hat{s}}$ transforms as a positive-frequency operator with helicity n, while its adjoint is a negative-frequency operator with helicity $-n$, in entire analogy with the massless on-shell case [50].

Covariant Field

These rules allow a simple construction of irreducible S-covariant free operator fields: we must only read (A.13) as an equation involving operators rather than c-number functions and replace the irreducible representation $\mathscr{D}^j$ of $SO(3)$) by the reducible one $D^{\hat{l}}$ occurring in (8.3). To be able to do so, the "helicities" $\pm n$ must both occur in the reduction of $D^{\hat{l}}(SO(3))$ to a direct sum of representations of $SO(2)$; cf. the text after (A.11). As we explain in the Appendix, *irreducible* covariant representations carry the same label $\hat{s}$ as the canonical ones. Combining these remarks with (8.12), (8.13) we arrive at the expression

$$\phi^{\hat{s}}_a(x) = \sigma^{-2} \int \frac{d^4p}{(2\pi)^3}\, e^{-ipx} \delta(|\mathbf{p}| - \sigma)\{\delta(p_0 - E)\, D^{\hat{l}}(\rho(p))_{ab}\, u_{b\mu_+}\, \beta_{\hat{s}}(p)$$

$$+ \delta(-p_0 - E)\, D^{\hat{l}}(\rho(-p))_{ab}\, u_{b\mu_-}\, \gamma^*_{\hat{s}_-}(-p)\} \tag{8.14}$$

for a covariant irreducible operator field transforming under S according to (8.3). Here $\gamma^*_{\hat{s}_-}$ is the *thermal* creation operator (i.e., its adjoint annihilates the thermal vacuum) of the possible non-shell antiparticle of the particle created by $\beta^*_{\hat{s}}$, that is, carrying the opposite quantum numbers and "helicity" n. If no conserved internal charges are present then $\gamma = \beta$. The case with $\gamma \neq \beta$ corresponds to the situation where the non-shell one-particle subspace $\hat{\mathscr{H}}^{(1)}$ is the tensor product of $\hat{\mathscr{H}}^{(1)}_n$ and $\hat{\mathscr{H}}^{(1)}_{-n}$ defind in (A.8). The possible existence of non-shell antiparticles is unrelated to the modes $|-E, -n\rangle$ that are necessarily present, whether or not γ equals β, because of the existence of the modular conjugation J in KMS representations; cf. Section 5. The rotation $\rho(p)$ is described after (A.5), while the connecting quantities $u_{b\mu}$ are specified after (A.11). In this case the indices $\mu_\pm$ refer to the subspaces $\mathscr{H}_{\mu_\pm}$ (of the space $\mathscr{H}(\hat{l})$ carrying the representation $\hat{l}$ of the Lorentz group) carrying the helicity $n_\pm$-representations of $SO(2)$ occurring in the decomposition of $\mathscr{H}(\hat{l})$, which is analogous to that of the $SO(3)$-irreducible space $\mathscr{H}^j$ described after (A.11). The normalization factor in (8.14) is related to the normalization chosen in (A.8) and (7.5).

In fact, (8.14) is not the most general expression, because the non-shell particle and antiparticle contributions may enter with a different (p-independent) weight factor. As it stands, Eq. (8.14) is correct if particle–antiparticle symmetry is main-

tained, i.e., in the special KMS representation in which the chemical potential μ corresponding to the particular charge carried by the (anti-) particle vanishes. In general one may expect μ-dependent weight factors in (8.14), which ultimately cause the μ-dependence of the real-time thermal propagator. In the sequel we will assume $\mu = 0$.

Two-parameter Licht Fields

Since $\hat{s} = (E, \sigma^2, n)$ while according to Section 7 all values of E and $\sigma^2 \geqslant 0$ actually participate in the representation, we find that the covariant field (8.14) depends on two continuous parameters; in analogy with the one-parameter Licht field (with spin [30]) emerging in Section 4 we may call $\phi^{\hat{s}}(x)$ a two-parameter Licht field with helicity. The fact that in thermal representations one needs a two- rather than a one-parameter field to describe unstable particles has a group-theoretical origin: canonical representations of the Poincaré-group happen to be characterized by one continuous parameter m^2 (plus a discrete one) while those of the "thermal symmetry group" $S = SO(3)$ ($\times T_4$ are characterized by two continuous numbers E and σ^2 (plus a discrete one); cf. Appendix. As we have seen in Section 2, the choice of Licht fields as a complete set (in a given sector) was a matter of choice in the vacuum case, for ultimately they could be expressed in terms of the discrete-mass free fields relating to the stable particle. In the interacting thermal case, on the other hand, there is no such underlying description in terms of stable particles, and we have no choice but the set of two-parameter Licht fields describing non-shell unstable particles.

As a consequence of the alleged Fock structure (7.3) of $\hat{\mathscr{H}}_\omega$, hence of $\mathscr{H}_\omega$, the operator set $\{\phi_a^{\hat{s}}(x)\}$ is indeed complete on $\mathscr{H}_\omega$ in the sense of Section 2, so that all operators on $\mathscr{H}_\omega$ may be expressed in terms of them. However, the Heisenberg fields $A(x)$ and $\tilde{A}(x)$ coordinatizing $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$ and $\mathfrak{M}'$, respectively, have a structure different from that of the fields $\phi^{\hat{s}}$ in that they are (up to domain problems) affiliated with $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, while $\hat{\phi}^s$ is affiliated with $\mathfrak{M} \cup \mathfrak{M}'$; cf. the remark after (8.10).

Bogoliubov Transformation

The same situation occurs in free-field theory (cf.. Section 6) and is easily remedied by a Bogoliubov transformation similar to (6.4). To do so, we note that according to (8.6), (8.7), and (5.24) we have, with $\hat{s} = (E, \sigma^2, n)$,

$$\beta^*_{(-E, \sigma^2, -n)}(p) = \tilde{\beta}^*_{(E, \sigma^2, n)}(p_0, -\mathbf{p}), \tag{8.15}$$

where the tilde operation is defined by (5.20); cf. (6.6). We accordingly may restrict the domain of the energy variable to $E > 0$ and replace the set of operators $\beta^*_{(E, \sigma^2, n)}$ for $E \in \mathbb{R}$, $n = \pm |n|$ by the two sets $\beta^*_{(E, \sigma^2, n)}$ and $\tilde{\beta}^*_{(E, \sigma^2, n)}$ both restricted to $E > 0$ and $n = |n|$; cf. (A.7), (A.8). Then, by (8.15), $\beta^*_{\hat{s}}$ and $\tilde{\beta}^*_{\hat{s}}$ create positive and negative energy modes, respectively, from the thermal vacuum. The same holds for the

possible antiparticle operator $\gamma_{\hat{s}}^*$. We subsequently perform an (inverse) Bogoliubov transformation (cf. (6.4)) defining the operator

$$a_{\hat{s}}(p) = \sqrt{1+N(E)}\,\beta_{\hat{s}}(p) + \sqrt{N(E)}\,\tilde{\beta}_{\hat{s}}^*(p), \tag{8.16}$$

as well as the hermitian and tilde conjugate transformation, where $N(E)$ is the Bose distribution function $(\exp(\beta E)-1)^{-1}$, $E>0$. Quite similar to the free-field case it follows from (5.25) that $a_{\hat{s}}^{(*)}$ and $\tilde{a}_{\hat{s}}^{(*)}$ are affiliated with $\mathfrak{M}$ and $\mathfrak{M}'$, respectively (ignoring domain problems). Antiparticle operators $c^{(*)}$ can be constructed in a similar way from $\gamma^{(*)}$ and $\tilde{\gamma}^{(*)}$. In order to construct irreducible covariant operator fields $F_a^{\hat{s}}(x)$ and $\tilde{F}_a^{\hat{s}}(x)$, which are affiliated with $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, we must only replace β by ${}^{(}\tilde{a}{}^{)}$ and γ^* by ${}^{(}\tilde{c}{}^{)*}$ in (8.14). We repeat that $E>0$ in $\hat{s}=(E,\sigma^2,n)$ so that $F_a^{\hat{s}}$ corresponds to the canonical representation $(E>0,\sigma^2,n)$ while $\tilde{F}_a^{\hat{s}}$ corresponds to $(E<0,\sigma^2,-n)$. In the scalar case $n=0$, to which we shall restrict ourselves in the sequel, we thus have

$$F_{E,\sigma}(x) = \sigma^{-2}\int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\delta(|\mathbf{p}|-\sigma)\{e^{-iEt}a(E,\mathbf{p}) + e^{iEt}a^*(E,-\mathbf{p})\}, \tag{8.17}$$

where we simply write $F_{E,\sigma}$ rather than $F^{(E,\sigma^2,0)}$. The normalization follows from (8.11) and (8.16) as

$$[a(E,\mathbf{p}), a^*(E',\mathbf{p}')] = (2\pi)^4\,\delta(E-E')\,\delta^3(\mathbf{p}-\mathbf{p}'); \tag{8.18}$$

cf. (4.2). The corresponding formulae for $\tilde{F}_{E,\sigma}$ are obtained by placing a tilde on top of the operators.

Remarks

Before we continue with the scalar field (8.17) in the next section it may be appropriate to comment on the physical significance of the group-theoretic results achieved above. The most telling feature is the decoupling of different components of a field with spin into "helicity" states, which are no longer connected by rotations. This means that they are not necessarily degenerate, as already remarked by Borchers and Sen [51] in the context of ground-state representations. There is not even a direct connection between the number of independent spin states in vacuum representations and the number of helicity states in thermal representations (except for free fields). For example, in superconductivity [18] the electromagnetic gauge field excites modes with $n=0$ and $n=\pm 1$ from the ground state, neither of which is massless.

Indeed, the connection between gauge invariance and mass is lost in thermal representations. In vacuum representations massless particle states with helicity ± 1 can be described covariantly by a gauge-equivalence class of vector fields [50, 52], whereas at nonzero temperature the use of ordinary fields suffices to describe any representation of S covariantly. This explains why a gauge field may approximately

generate a "thermal mass" without breaking gauge invariance [20]. Note, however, that the considerations on exact mass generation in Section 6 also apply to gauge fields!

9. Thermal Gell-Mann/Low Formula

We now discuss to what extent one can derive a self-consistent diagrammatic perturbation scheme based on a Gell-Mann/Low formula. To keep matters relatively simple we will do this for the case in which a single neutral scalar field $A(x)$ coordinatizes the bicommutant $\mathfrak{M} = \pi_\omega(\mathfrak{A})''$ of the representation of the observable algebra (so that $\tilde{A}(x)$ coordinatizes $\mathfrak{M}'$, cf. Section 5). Then, according to Section 7, the thermal representation space $\mathscr{H}_\omega$ may be taken to be a Fock space over the non-shell one-particle space $\mathscr{H}^{(1)}$ defined by (7.6) and (7.2) (with the reservations expressed in the beginning of that section). The set of S-covariant free fields (8.17), with E and σ both ranging from zero to infinity, together with its tilde conjugate is complete in $\mathscr{H}_\omega$. Hence $A(x)$ can be (weakly) expressed in terms of the $F_{E,\sigma}$ (dynamical map):

$$A(x) = A[F_{E,\sigma}, x]. \tag{9.1}$$

Desiderata

As we explained at length in Section 3 for vacuum representations, the validity of the Gell-Mann/Low formula (3.21) hinges upon the following requirements:

(i) the completeness of the fields ϕ_i^{in} in $\mathscr{H}_\omega$, hence of the fields Φ_i in the reference Hilbert space $\mathscr{H}_M$;

(ii) the weak asymptotic condition (3.10); and

(iii) the weak equality of the full Hamiltonian $H[A]$ and the free one $H_0[\phi_i^{\text{in}}]$ on $\mathscr{H}_\omega$ (Eq. (3.23)).

As we have seen, in the asymptotically complete case these requirements are consequences of the Haag–Ruelle theory [9], based on the strong asymptotic condition (3.9). In thermal representations the situation is rather different: we have no strong asymptotic condition at our disposal, because there exist no on-shell one-particle states. Nevertheless, we do have, *mutatis mutandis*, (i) and (iii) (see below) with ϕ_i^{in} replaced by $F_{E,\sigma}$ (cf. (8.17)) and its tilde counterpart, and $\hat{H}$ instead of H. In fact, the Fock nature of $\mathscr{H}_\omega$ (7.3), combined with (7.2), (8.6), (8.7), and (8.15), implies the weak equality

$$\hat{H} = \int_0^\infty dp_0 \int \frac{d^3p}{(2\pi)^4} p^0 \{\beta^*(p)\,\beta(p) - \tilde{\beta}^*(p)\,\tilde{\beta}(p)\}. \tag{9.2}$$

Since the expression in curly brackets is invariant under the Bogoliubov transformation (8.16), this may be written as

$$\hat{H} = \int_0^\infty dE \int \frac{d^3p}{(2\pi)^4} E\{a^*a - \tilde{a}^*\tilde{a}\}, \tag{9.3}$$

with $a = a(E, \mathbf{p})$. Finally, using (8.17) we find [16]

$$\begin{aligned}\hat{H} &= \hat{H}_0[F, \tilde{F}] \\ &= \int d^3x \int_0^\infty \frac{dE}{8\pi E} \int_0^\infty d\sigma_1 d\sigma_2 \sigma_1^2 \sigma_2^2 \{(\dot{F}_{E,\sigma_1}\dot{F}_{E,\sigma_2} + E^2 F_{E,\sigma_1}F_{E,\sigma_2}) - \text{t.c.}\},\end{aligned} \tag{9.4}$$

where $F = F(x)$, $\dot{F} = \partial_0 F$ is the canonical momentum conjugate to F, and t.c. means tilde conjugate, i.e., the same expression with a tilde on top of the operators. Note that (9.4) is automatically normal-ordered with respect to the thermal creation and annihilation operators (the "vacuum" contributions from the tilde and the nontilde parts cancel out). According to (8.17), (8.18), and (9.4) the time evolution of the fields $F_{E,\sigma}(x)$ is generated by $\hat{H}_0$, as it should.

Asymptotic Condition

What is still lacking is the precise relation between the Heisenberg field $A(x)$ and the complete set $\{F_{E,\sigma}\}$. As in the vacuum unstable-particle case (Section 4) a strong (Haag–Ruelle) asymptotic limit for $A(x)$ does not exist, whereas a naive weak LSZ asymptotic limit vanishes on account of the lack of a discrete point in the joint spectrum of $\hat{H}$, $\hat{\mathbf{P}}$. Nevertheless, we have seen in Section 4 that the field $C(x)$ describing an unstable particle admits the decomposition (4.3) such that the components $C_s(x)$ do have a nonvanishing LSZ-limit (4.4) between s-wave multiparticle states; indeed, asymptotic conditions provide a most natural setting for Licht fields to emerge [31, 32]. Since in the thermal case we have the two-parameter Licht fields $F_{E,\sigma}(x)$ at our disposal we may attempt a similar construction here. We therefore decompose the Heisenberg field $A(x)$ according to

$$A(x) = \int_0^\infty dE \int_0^\infty d\sigma\, \sigma^2 (4\pi E)^{-1/2} Z^{1/2}(E, \sigma^2)\, A_{E,\sigma}(x). \tag{9.5}$$

The prefactor of $Z^{1/2}$ has been extracted for later convenience. In order, then, to be able to derive a Gell-Mann/Low formula it is required that $A_{E,\sigma} \to F_{E,\sigma}$ for $t \to -\infty$ in the LSZ sense, i.e., weakly and smeared with a solution of the free-field equation satisfied by $F_{E,\sigma}$; cf. (3.10) and (4.4). More precisely, we should have

$$w - \lim_{t' \to -\infty} \Delta_E(t - t')\, \overleftrightarrow{\partial}_0 A_{E,\sigma}(t', \mathbf{x}) = F_{E,\sigma}(x), \tag{9.6}$$

with

$$\Delta_E(t) = i(e^{iEt} - e^{-iEt})/2E. \tag{9.7}$$

Since $A_{E,\sigma}$ and $F_{E,\sigma}$ are distributions in E and σ^2, Eq. (9.6) should in fact also be smeared with test functions in these variables.

The idea behind the asymptotic condition (9.6) is similar to that underlying the ordinary LSZ-condition: for large (negative) t', matrix elements of $A_{E,\sigma}(t', \mathbf{x})$ are expected to approach a free-field time evolution with prefactors $e^{-iEt'}$ and $e^{iEt'}$ for the particle and antiparticle contributions in $A_{E,\sigma}$, respectively; these factors are canceled by the opposite ones in the function (9.7) so that the limit will exist. This also shows how the decomposition (9.5) should be chosen: $A_{E,\sigma}(x)$ represents the contribution to $A(x)$ that exhibits the above asymptotic behaviour. This still renders (9.5) highly nonunique, like the decompositions (3.11) and (4.3), but this lack of uniqueness is immaterial for the Gell-Mann/Low formula, which uses only the asymptotic behaviour (cf. Section 3). This argument is, however, heuristic only, and we stress that (9.6) is a postulate and not a theorem. Indeed, the poor cluster properties in KMS representations [24, 26] indicate that an actual proof of a condition like (9.6), if it holds at all, would be excessively hard (the situation is much more favourable in vacuum representations, where the LSZ condition (3.10) can be proved from the Wightman axioms [9]).

Apart from the mere existence of the limit on the left-hand side of (9.6) it should be ascertained that this limit actually equals the right-hand side. If the left-hand side has the correct free-field (energy-extended) commutation relations, however, we simply define $F_{E,\sigma}$ to be this limit. We are free to do so, because up to this point we did not have to specify which non-shell one-particle space $\hat{\mathscr{H}}^{(1)}$ out of the infinitely many occurring in (7.1) was selected to construct the Fock space (7.3). We choose the particular $\hat{\mathscr{H}}^{(1)}$ as that created from the thermal vacuum by the (smeared) $F_{E,\sigma}$ defined by (9.6). It then follows that $A_{E,\sigma}(x)$ in (9.5) must have a nonvanishing matrix element between the state $|E, \mathbf{p}\rangle \in (F^{-1}N)^*$ (cf. Section 7), with $\mathbf{p}^2 = \sigma^2$, and the thermal vacuum. This situation should be compared with the decomposition (3.11). The difference between the two cases is that in contrast to $\mathscr{H}^{(1)}$ above, the subspaces $\mathscr{H}_i^{(1)}$ (cf. text below (3.11)) occur multiplicity-free in the decomposition (2.9), and thus are uniquely identified from the start. Finally, the correct normalization of the left-hand side of (9.6) is guaranteed by the presence of the function $Z^{1/2}$ in (9.5), which in a canonical theory is determined in principle (cf. Section 11).

There It is

Suppose, then, that the LSZ-like condition (9.6) holds, so that the fields $F_{E,\sigma}(x)$ are in-fields. The three conditions (i)–(iii) below (9.1) are thereby satisfied, and a thermal Gell-Mann/Low formula may accordingly be derived. All the work has already been done in Section 3: we must just replace the discrete index i in (3.11) by the continuous pair E, σ in (9.5). Following the same route as that in Section 3, we introduce a reference Hilbert space $\mathscr{H}_E$ (analogous to $\mathscr{H}_M$ in Section 3) with thermal vacuum $|0, \beta\rangle$ (analogous to $|\Omega\rangle$). The Hamiltonian on this space is $\hat{H}_0[\Phi, \tilde{\Phi}]$ (cf. (9.4)) which is expressed in terms of the covariant Licht fields ${}^{(}\tilde{\Phi})_{E,\sigma}(x)$

(analogous to the Φ_i in Section 3, and not to be confused with the c-number functions $\Phi_{E,\sigma^2}(p)$ in the Appendix). The thermal state condition (5.25) on $\mathscr{H}_E$ reads

$$\exp\left(-\frac{\beta}{2}\hat{H}_0[\Phi, \tilde{\Phi}]\right) A\,|0, \beta\rangle = \tilde{A}^*\,|0, \beta\rangle, \tag{9.8}$$

which implies

$$\hat{H}_0[\Phi, \tilde{\Phi}]\,|0, \beta\rangle = 0. \tag{9.9}$$

Following, *mutatis mutandis*, the same steps as those leading to (3.21) we then arrive at the thermal Gell-Mann/Low formula

$$\begin{aligned} iG^{(r)}(x_1 \cdots x_n) &:= \langle 0(\beta)|\, T[A^{r_1}(x_1) \cdots A^{r_n}(x_n)]\,|0(\beta)\rangle \\ &= \int \prod_{i=1}^{n} d\mu(E_i, \sigma_i) \\ &\quad \times \langle 0, \beta|\, T[\Phi^{r_1}_{E_1,\sigma_1}(x_1) \cdots \Phi^{r_n}_{E_n,\sigma_n}(x_n)\, \hat{U}(\infty, -\infty)]\,|0, \beta\rangle \end{aligned} \tag{9.10}$$

with the abbreviation

$$\int d\mu(E, \sigma) := \int_0^\infty dE \int_0^\infty d\sigma\, \sigma^2 (4\pi E)^{-1/2} Z^{1/2}(E, \sigma^2); \tag{9.11}$$

cf. (9.5). Analogous to (3.22), we have

$$\hat{U}(\infty, -\infty) = T \exp\left(-i \int_{-\infty}^{\infty} dt\, \hat{H}_I(t)\right) \tag{9.12}$$

with the interaction Hamiltonian

$$\hat{H}_I(t) = \hat{H}[\Phi, \tilde{\Phi}] - \hat{H}_0[\Phi, \tilde{\Phi}], \tag{9.13}$$

in which the time evolution of the $\overset{(\sim)}{\Phi} = \overset{(\sim)}{\Phi}(x)$ in (9.13) is governed by the free Hamiltonian $\hat{H}_0$, which is given by (9.4), with F replaced by Φ. As in Section 3, $\hat{H}[\Phi, \tilde{\Phi}]$ is obtained by replacing $\overset{(\sim)}{A}$ by $\overset{(\sim)}{\Phi}$ in $\hat{H}[A, \tilde{A}]$; in this case this means that first (9.5) should be substituted in $\hat{H}[A, \tilde{A}]$, whereupon $A_{E,\sigma}$ is to be replaced by $\Phi_{E,\sigma}$. We omitted the denominator (vacuum renormalization) in (9.10) (cf. (3.21) and text thereafter) because this factor is identically equal to one even in perturbation theory [3]. Finally, we employed the thermal doublet notation [18, 38, 3] $A^r = (A, \tilde{A}^*)$. The Feynman rules implied by (9.10) are presented in the next section.

10. Non-shell Feynman Rules

Interaction Hamiltonian

For simplicity we consider a theory with a quartic interaction. According to (5.27)–(5.30) the Hamiltonian on $\mathscr{H}_\omega$ reads

$$\hat{H}[A, \tilde{A}] = \frac{1}{2}\int d^3x \left\{(\dot{A}^2 + (\nabla A)^2 + m_0^2 A^2 + \frac{\lambda_0}{12} A^4\right) - \text{t.c.}\Big\}, \tag{10.1}$$

with $A = A(x)$ and canonical momentum $\dot{A}$; m_0 and λ_0 are the bare mass and coupling constant, respectively. The free Hamiltonian that is to be used in the derivation of the Feynman rules is $\hat{H}_0[\Phi, \tilde{\Phi}]$, i.e., (9.4) with $F_{E,\sigma}$ replaced by $\Phi_{E,\sigma}$ which is the covariant irreducible operator field in the interaction picture. As explained in the previous section, the interaction Hamiltonian in the interaction picture determined by the free Hamiltonian $\hat{H}_0$ follows from (9.4), (9.5), (9.13), and (10.1) as

$$\begin{aligned} H_I(t) = \int d^3x \Big\{ & \frac{\lambda_0}{4!}\int \prod_{i=1}^{4} (d\mu(E_i, \sigma_i)\, \Phi_{E_i,\sigma_i}) \\ & + \frac{1}{2}\int \prod_{j=1}^{2} (d\mu(E_j, \sigma_j))[\nabla\Phi_{E_1,\sigma_1}\cdot\nabla\Phi_{E_2,\sigma_2} \\ & + (1 - \delta(E_1 - E_2)\, Z^{-1/2}(E_1, \sigma_1^2)\, Z^{-1/2}(E_2, \sigma_2^2))\, \dot{\Phi}_{E_1,\sigma_1}\dot{\Phi}_{E_2,\sigma_2} \\ & + (m_0^2 - E_1^2\delta(E_1 - E_2)\, Z^{-1/2}(E_1, \sigma_1^2)\, Z^{-1/2}(E_2, \sigma_2^2))\, \Phi_{E_1,\sigma_1}\Phi_{E_2,\sigma_2}] \\ & - \text{t.c.}\Big\}, \end{aligned} \tag{10.2}$$

where $d\mu$ is given in (9.11), t.c. means tilde conjugate, and the **x**-integration is understood in the sense of (5.30). All terms but the first in (10.2) are counterterms.

We are going to derive the Feynman rules in two steps: first for Green functions of the operator $A_{E,\sigma}(x)$ (cf. (9.5)) and second for those of the full operator $A(x)$, which is the quantity of physical interest.

First Step

We first remark that the Wick theorem holds for the right-hand side of (9.10) as in conventional real-time perturbation theory [38, 3]. The computation of the free propagator employs (9.8), (8.17), and (9.4) and is analogous to the conventional case [38, 3]. In terms of the doublet $\Phi^r = (\Phi, \tilde{\Phi}^*)$ one finds

$$\begin{aligned} iG_0^{(r_1 r_2)}(E_1, \sigma_1, x_1; E_2, \sigma_2, x_2) &:= \langle 0, \beta |\, T[\Phi^{r_1}_{E_1,\sigma_1}(x_1)\, \Phi^{r_2}_{E_2,\sigma_2}(x_2)]\, |0, \beta\rangle \\ &= 4\pi i E_1 \sigma_1^{-4}\delta(E_1 - E_2)\, \delta(\sigma_1 - \sigma_2) \\ &\quad \times \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_1 - x_2)}\delta(|\mathbf{p}| - \sigma)\, \check{D}^{(r_1 r_2)}(p_0, E), \end{aligned} \tag{10.3}$$

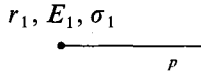
with

$$\check{D}^{(r_1 r_2)}(p_0, E) = M(|p_0|)\begin{pmatrix} \dfrac{1}{p_0^2 - E_1^2 + i\varepsilon} & 0 \\ 0 & \dfrac{-1}{p_0^2 - E_1^2 - i\varepsilon} \end{pmatrix} M(|p_0|) \quad . \tag{10.4}$$

Here $M(|p_0|)$ is the well-known 2×2-matrix [3] with entries

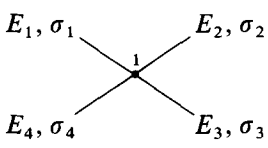
$$\begin{aligned} M^{11} &= M^{22} = \sqrt{1 + N(|p_0|)}; \\ M^{12} &= M^{21} = \sqrt{N(|p_0|)}, \end{aligned} \tag{10.5}$$

where N is the Bose distribution function. We denote the momentum-space propagator by a line and have our first Feynman rule:



$$= 4\pi i E_1 \sigma_1^{-4} \delta(E_1 - E_2)\, \delta(\sigma_1 - \sigma_2) \times \delta(|\mathbf{p}| - \sigma_1)\, \check{D}^{(r_1 r_2)}(p_0, E). \tag{10.6}$$

Since the interaction (10.2) does not mix tilde and nontilde operators there are two types of vertices: one type containing type 1 (nontilde) legs only, and one containing type 2 (tilde) legs only. The Feynman rules of these two types differ by a sign only. Below we will, therefore, give the type 1 vertices only. (This situation is the same as that in conventional real-time perturbation theory [3].) First there is a four-point vertex

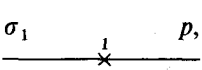


$$= -i\lambda_0 f_1 f_2 f_3 f_4 \tag{10.7}$$

with

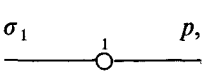
$$f_i = f(E_i, \sigma_i^2) = \sigma_i^2 (4\pi E_i)^{-1/2}\, Z^{1/2}(E_i, \sigma_i^2). \tag{10.8}$$

There are two qualitatively different types of two-point counterterms: one coming from $\hat{H}$ and one from $\hat{H}_0$. They follow from (10.2) as



$$= i f_1 f_2 (p^2 - m_0^2), \tag{10.9}$$

and

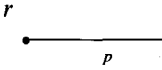


$$= -i\sigma_1^2 \sigma_2^2 (4\pi E_1)^{-1}\, \delta(E_1 - E_2)(p_0^2 - E_1^2). \tag{10.10}$$

A diagrammatic expansion for an n-point function of $A^{\Gamma}_{E,\sigma}$ with external momenta p_i and external "non-shell parameters" E_i, σ_i, that is, the Fourier transform of the integrand of (9.10), is now obtained by drawing Feynman diagrams with rules (10.6)–(10.10), and the corresponding ones for type 2 vertices, in the usual way [3]. Apart from the loop momenta one should now also integrate over all *internal* variables E, σ from zero to infinity. This completes the first stage in the derivation of the non-shell thermal Feynman rules.

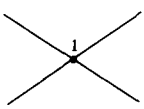
Second Step

We now pass to the derivation of the Feynman rules for the full Green functions of the operator $A(x)$, which according to (9.10) are obtained from the Green functions constructed above by also integrating over the external variables E_i, σ_i with weight function f_i given by (10.8). The delta functions in (10.6) imply that all σ-integrations and one-half of the E-integrations can be trivially performed. Two different situations then arise. External points and vertices of types (10.7) and (10.9), collectively called regular vertices, contribute a factor $Z^{1/2}$. A propagator connecting two regular vertices therefore will contain a factor Z. Each such propagator carries its own E-variable which is integrated over with weight function $Z(E, \mathbf{p}^2)$. This yields the "regular" propagator (cf. (4.5))

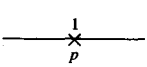


$$=: iG_0^{(rs)}(p) = i\int_0^{\infty} dEZ(E, \mathbf{p}^2)\,\check{D}^{(rs)}(p_0, E). \qquad (10.11)$$

Regular here means that the external points are regular, i.e., not of type (10.10). In contrast, the vertex (10.10) contributes a delta function rather than a factor Z. Hence two propagators connected by the vertex (10.10) carry the same "energy" E. Eventually a regular vertex will be encountered on both sides, so that a factor Z will be picked up in any case. However, all propagators connecting these outer regular vertices carry the same "energy" E, and there is only one factor Z and one E-integration for such a chain. Since all awkward factors cancel between propagators and vertices, the type 1 vertices in the E, σ-integrated diagrams are just

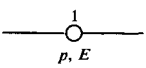


$$= -i\lambda_0, \qquad (10.12)$$



$$= i(p^2 - m_0^2), \qquad (10.13)$$

and



$$= -i(p_0^2 - E^2). \qquad (10.14)$$

Equations (10.11)–(10.14) are the final non-shell thermal Feynman rules. One

should draw all diagrams as usual, but with the propagator (10.11) and the extra vertex (10.14), keeping in mind the prescription given above for the special treatment of the latter. The whole situation is illustrated by the following diagram (see Fig. 1) with all vertices of type 1 for simplicity:

$$A = \lambda_0^2 \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \prod_{i=1}^{6} (dE_i Z(E_i, \mathbf{p}_i^2)\, \check{D}^{(11)}(p_i^0, E_i))(q^2 - m_0^2)$$

$$\cdot \int_0^\infty dE_7 Z(E_7, \mathbf{k}^2)(\check{D}^{(11)}(k_0, E_7))^2\, (k_0^2 - E_7^2), \tag{10.15}$$

with $p_5 = p_6 = q = k - p_1 - p_2$. The point to remember is that the two propagators connected by the vertex (10.10) together acquire only one E-integration and one factor Z, while those connected by the vertex (10.9) have a weight Z and an E-integration each.

Consistency

It is to be remarked that a diagrammatic expansion for the full two-point function employing the propagator (10.11) identically satisfies the KMS condition [3] for any choice of the function Z. This is because (10.11) formally has the same structure as the spectral representation of the full 2×2-real-time propagator [3], with the full spectral function $\rho(\omega, \mathbf{p})$ replaced by $Z(E, \mathbf{p})$. This feature, which forms the basic consistency condition for real-time thermal perturbation theory, has nowhere been put in by hand; it results naturally as a consequence of the unique choice of the free Hamiltonian (9.4) for which the asymptotic limits in the Gell-Mann/Low formula exist. This is analogous to the situation in the Haag–Ruelle theory: as we have seen in Section 3, the unique choice of the free vacuum propagator that leads to a unitary S-matrix automatically results from the derivation of the vacuum Gell-Mann/Low formula (cf. the text after (3.21)).

In spite of the formal similarity between (10.11) and the full spectral representation, the function Z should not be confused with the spectral function ρ. In vacuum perturbation theory Z would equal $\delta(E - \varepsilon(\sigma^2))$ for the on-shell energy $\varepsilon(\sigma^2)$; i.e., in that case it represents only the one-particle contribution to the full spectral function. In the thermal case at hand, Z is prohibited from being a delta

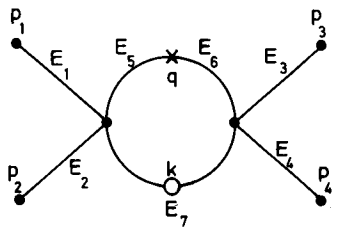


FIG. 1. Example of a non-shell diagram.

function by the results of Sections 6 and 7; it now represents the contribution to the spectral function coming from the non-shell one-particle Hilbert space $\mathscr{H}^{(1)}$. Its actual determination is the subject of Section 11 and 12.

Ansatz

For the moment we find it most illuminating to yet further elaborate the Feynman rules above for a model choice of Z. Guided by conventional wisdom in vacuum unstable-particle theory (cf. (4.6)) we adopt the normalized Breit–Wigner form

$$Z(E, \sigma^2) = \kappa/\pi[(E - \varepsilon(\sigma^2))^2 + \kappa^2]^{-1}. \tag{10.16}$$

Here κ is a parameter, and $\varepsilon(\sigma^2)$ is a certain dispersion relation. In the standard picture Eq. (10.16) represents the contribution to the spectral function coming from an unstable quasiparticle with lifetime κ^{-1}. In our operator formalism it weighs the relative contribution of a state $|E, \mathbf{p}\rangle \in (F^{-1}N)^*$ with $\mathbf{p}^2 = \sigma^2$. In any case, (10.16) is useful only if $\kappa \ll \varepsilon(\sigma^2)$. Then the E-integration in (10.11) may be extended to minus infinity (introducing errors of order κ/ε), and using (10.4) the propagator (10.11) becomes

$$\check{G}_0^{(rs)}(p) = M(|p_0|)\begin{pmatrix} \dfrac{1}{p_0^2 - (\varepsilon_{\mathbf{p}} - i\kappa)^2} & 0 \\ 0 & \dfrac{-1}{p_0^2 - (\varepsilon_{\mathbf{p}} + i\kappa)^2} \end{pmatrix} M(|p_0|), \tag{10.17}$$

with $M(|p_0|)$ given by (10.5). (It is actually possible, although not very illuminating, to do this computation exactly using techniques developed in Ref. [53].)

We can also explicitly evaluate the effect of the counterterms (10.13) and (10.14). A computation of the relevant two-point diagrams (i.e., the two internal lines in Fig. 1), taking care of the special rule pertaining to (10.14) and summing over type 1 and type 2 insertions, shows that in the case of (10.16) both of these counterterms may be replaced by a single, complex one that mixes type 1 (nontilde) and type 2 (tilde) fields, namely

$$r \xrightarrow[p]{\otimes} s = i\{\varepsilon(\mathbf{p})^2 - \varepsilon_0(\mathbf{p})^2 - \kappa^2)\,\tau_3 - 2i\varepsilon(\mathbf{p})\,\kappa\tau_3 M(|p_0|)^2\,\tau_3\}^{rs}. \tag{10.18}$$

Here $\varepsilon_0(\mathbf{p})^2 = \mathbf{p}^2 + m_0^2$, τ_3 is the third Pauli matrix, and r and s are thermal indices taking values 1, 2. To arrive at (10.18) we used the identity

$$M(|p_0|)\,\tau_3 M(|p_0|) = \tau_3. \tag{10.19}$$

Since the special nature of (10.14) has been explicitly taken into account in deriving (10.18) one may use (10.18), (10.17), and (10.12) without any further prescriptions. The propagator (10.17) and the counterterm (10.18) have a very interesting structure, which was originally discovered, in a rather different setting, in the context of

nonequilibrium thermo field dynamics [34, 35]. We defer a further discussion to the next section. For the moment we just remark that in the limit $\kappa \to 0$ the above degenerates into the standard set of real-time thermal Feynman rules [3]. In this limit (10.16) reduces to a delta function, which entails the form (6.10) of the thermal measure. In view of the results of Section 6 this is not allowed for interacting theories, and we conclude that if the Breit–Wigner form (10.16) applies it must be for strictly finite κ.

11. Choice of the Function $Z(E, \sigma^2)$

Canonicity

We now turn to the question of determining the function $Z(E, \sigma^2)$, which enters in the "free" propagator (10.11), in general. For this purpose we return to its defining equation (9.5). Since the normalization of $A_{E,\sigma}$ appearing there is fixed by the asymptotic condition (9.6), with $F_{E,\sigma}$ given by (8.17), with the normalization (8.18), it follows that Z completely determines the normalization of the Heisenberg field $A(x)$. On the other hand, in a canonical theory the latter is fixed by the equal-time commutation relations. The fact that both the overall normalization and the functional form of $Z(E, \sigma^2)$ are governed by the canonical relations may be illustrated by the example of an interacting theory possessing a multi-mass spectrum at zero temperature: there the renormalization constant for each asymptotic field (cf. (3.11), (3.12)) is separately dictated by imposing the equal-time commutators.

Actually, at zero temperature the canonical relations are not explicitly needed in order to calculate Z (which in the single-mass case reduces to a constant). The reason is that the renormalized full two-point functon has a pole in $p^2 = m^2$ with residue i. Perturbation theory can be set up in such a way that the propagator is just the renormalized free two-point function, so that Z appears only in the two-point counterterm $i(Z-1)p^2$. Then Z is determined order by order by the consistency condition $Z - 1 = (\partial/\partial p^2 \Sigma)(p^2 = m^2)$, where the contribution of given order to the self-energy by definition excludes the counterterm above of the same order. Consistency here does not lead to self-consistency, because the self-energy itself contains only lower-order contributions from Z, which had been determined in a previous step. All this is true basically because the free propagator itself is independent of Z altogether.

This favourable situation does not extend to thermal representations: first the two-point Green function has no stable-particle pole, and second the "free" propagator (10.11) itself contains Z. At the moment we do not see any possibility other than calculating the function $Z(E, \sigma^2)$ the hard way. This means that the dynamical map (9.1) should be evaluated explicitly (the first term being (9.5) with A replaced by F on the right-hand side) after which the canonical commutator may be derived. There are two ways of accomplishing this end.

Basic Techniques

The first method consists of making an Ansatz for the expansion of $A(x)$ in terms of the asymptotic fields ($F_{E,\sigma}$ in our case) and then calculating the mapping coefficients by the Heisenberg equation (perturbation theory in the Heisenberg picture, cf. Ref. [18] for the zero-temperature case). This scheme always works and in principle allows one to check the condition (9.6).

In the second method one calculates all retarded Green functions in perturbation theory, in our case employing the consistent Feynman rules (10.11)–(10.14), whereupon the dynamical map follows in the guise of the so-called Haag–GLZ expansion [33]. This method works if the asymptotic condition (9.6) holds.

It should be remarked that, in contrast to the state of affairs at zero temperature, in real-time thermal field theory retarded functions can be expressed immediately in terms of time-ordered ones [54] so that one indeed has a direct perturbation method for the former.

In either method, imposing the canonical relation leads to a nonlinear inhomogeneous integral equation for the function Z, which enters both directly as in (9.5) and indirectly because it is contained in the mapping coefficients. In the first method above this is caused by the Z-dependence of the interaction Hamiltonian (10.2), while in the second it is due to the appearance of $Z(E, \mathbf{p}^2)$ in the "free" propagator (10.11). Actually, at this stage it is rather unclear what is to be understood by a perturbation expansion (for the retarded functions) since we have not yet identified what the physical expansion parameters are (let alone ascertain their smallness), so that the counterterms (10.13), (10.14) are undetermined, as is in fact the "coupling constant" in (10.12). We further elaborate on this point in the next section.

The preceding prescriptions to evaluate $Z(E, \sigma^2)$ are excessively complicated and in practice turn out to be untractable to the extent that we seem to be compelled to significantly modify our strategy. Rather than attempting to derive Z from first principles, we are going to motivate the Feynman rules (10.17), (10.18) directly. This, then, will lead us back to the Breit–Wigner form (10.16) as a reasonable first approximation to the function Z.

Practical Approach: Quasi-naive GML *Formula*

The starting point is the following *formal* Gell-Mann/Low formula [55],

$$\begin{aligned} iG^{(r)}(x_1 \cdots x_n) = \langle 0, \beta |\, U\left(\infty - i\frac{\beta}{2}, \infty\right)[T\Phi^{r_1}(x_1) \cdots \Phi^{r_n}(x_n)\, \hat{U}(\infty, -\infty)] \\ \times U\left(-\infty, -\infty + i\frac{\beta}{2}\right)|0, \beta\rangle / \langle 0, \beta |\, U\left(\infty - i\frac{\beta}{2}, \infty\right) \\ \times \hat{U}(\infty, -\infty)\, U\left(-\infty, -\infty + i\frac{\beta}{2}\right)|0, \beta\rangle. \end{aligned} \tag{11.1}$$

Here $iG^{(r)}(x_1 \cdots x_n)$ is the full n-point Green function defined in (9.10). The notation is similar to that in Section 9, with the essential difference that now $\Phi(x)$ is a field in the conventional interaction picture determined by the free Hamiltonian $\hat{H}_0 = H_0 - \tilde{H}_0$ (in the sense of (5.30)), *with* H_0 *the free Hamiltonian of the vacuum theory.* The evolution operator U is defined by (3.22); in contrast to $\hat{U}$ (cf. (9.12)) it does not contain contributions from the tilde fields. Unlike our non-shell thermal Gell-Mann/Low formula (9.10), which is meaningful as it stands, the expression (11.1) is formal in the sense that neither the infinite-volume limit nor the asymptotic time limits have been treated properly. Namely, the operator U diverges with the volume, while the limits $t \to \pm\infty$ in U and $\hat{U}$ do not exist in the strong operator topology and vanish in the weak topology for the conventional (vacuum) choice of H_0 (cf. Sections 6 and 9). Hence (11.1) should be properly regularized by a space and time cutoff, whereupon the divergences in taking the infinite space-time limit in the numerator and the denominator supposedly cancel out. The regularized expression corresponding to (11.1), however, is valid for any choice of H_0, so that the Gell-Mann/Low formula loses its function of identifying the unique consistent choice of the free propagator in perturbation theory (cf. Sections 1, 3, 9, and 10).

A related difficulty is that (11.1) does not give rise to the usual Feynman diagram method because of the presence of the factors $U(\infty - \frac{1}{2}i\beta, \infty)$ and $U(-\infty, -\infty + \frac{1}{2}i\beta)$. It has been shown [56] that these factors are essential in order for the KMS condition to be satisfied, except in the case where the fields in the interaction picture carry the same masses as the asymptotic fields in the usual sense of vacuum field theory. In that case the contributions of the operators U to the denominator and the numerator in (11.1) cancel out. However, we have seen that this condition cannot be satisfied in interacting thermal systems.

As we have argued, the way out of all these difficulties is to modify the choice of the free Hamiltonian, which required rather elaborate considerations. If one is willing to ignore most of the fine mathematical issues involved, however, the problem may to a certain extent be solved "by hand" starting from (11.1).

Self-Consistent Perturbation Method

The essential idea has been introduced by Matsumoto [37] in nonequilibrium thermo field dynamics and is easily adapted to equilibrium representations. Namely, one rewrites (11.1) as

$$iG^{(r)}(x_1 \cdots x_n) = \langle T[\Phi^{r_1}(x_1) \cdots \Phi^{r_n}(x_n)\, \hat{U}_{II}(\infty, -\infty)] \rangle. \tag{11.2}$$

Here

$$\hat{U}_{II} = T \exp\left(-i \int d^4x\, [\hat{\mathscr{H}}_I(x) + S(x)] \right), \tag{11.3}$$

where $\hat{\mathscr{H}}_I = \mathscr{H}_I - \tilde{\mathscr{H}}_I$, with $\mathscr{H}_I$ the interaction Hamiltonian density of the vacuum theory, while S is given by

$$S(x) := -\tfrac{1}{2}\Phi^r(x)\, \Pi^{(rs)}(i\partial)\, \Phi^s(x) \tag{11.4}$$

for a certain 2×2-matrix Π, to be specified below. The average in (11.2) is given for any time-ordered operator product $T\mathcal{O}$ by

$$\langle T\mathcal{O} \rangle = \langle 0, \beta | T\left[\mathcal{O} \exp i \int d^4x\, S(x) \right] | 0, \beta \rangle / \langle 0, \beta |\, T\left[\exp i \int d^4x S(x) \right] | 0,\ \beta \rangle. \tag{11.5}$$

Because of the presence of the time-ordering in (11.1) and (11.2), the latter is actually an identity if the factors U in (11.1) can be ignored. As remarked above, these factors ascertain the identical fulfillment of the KMS condition. We must, therefore, choose the matrix Π in (11.4) in such a way that the perturbation expansion derived from (11.2) identically satisfies the KMS condition. It should be remarked that in the absence of the factors U in (11.1) the denominator is equal to unity, because the closed diagrams corresponding to the nontilde operators in $\hat{U}$ are canceled by their tilde counterparts. In (11.4), on the other hand, the matrix Π will in general be nondiagonal so that tilde ($\mu = 2$) and nontilde ($\mu = 1$) fields are mixed. The normalization factor in (11.5) is therefore explicitly present.

It is easily shown that the averaging (11.5) satisfies the Wick theorem. To wit, the generating functional

$$Z_0[j^r] := \left\langle T \exp i \int d^4x\, j^r(x)\, \Phi^r(x) \right\rangle \tag{11.6}$$

satisfies the differential equation

$$\left\{ [\partial^2 + m^2)\, \tau_3^{(rs)} + \Pi^{(rs)}(i\partial)] \frac{\delta}{i\delta j^s(x)} - j^r(x) \right\} Z_0 = 0, \tag{11.7}$$

where m is the mass in the free Hamiltonian; cf. the text below (11.1). The solution is

$$Z_0[j^r] = \exp\left(-\frac{i}{2} \int d^4x\, d^4x'\, j^r(x)\, D^{(rs)}(x, x')\, j^s(x') \right), \tag{11.8}$$

where the propagator satisfies

$$[(\partial_x^2 + m^2)\, \tau_3^{(rt)} + \Pi^{(rt)}(i\partial)]\, D^{(ts)}(x, x') = -\delta^{(rs)} \delta^4(x - x'). \tag{11.9}$$

Functional differentiation of (11.6) and comparison with (11.5), (11.8) then proves the Wick theorem.

Consistent Choice of Π

If $\Pi^{(rs)}$ is replaced by the thermal self-energy $\Sigma^{(rs)}$, Eq. (11.9) becomes formally identical to the equation for the full two-point function $G^{(rs)}$. This allows us to further specify $\Pi^{(rs)}$ by the demand that the free propagator of the improved perturbation expansion (11.2) satisfy the KMS condition. Namely, it is known [3] that

the Green function $G^{(rs)}$ satisfies the KMS condition if and only if the self-energy has the form (Eq. (3.2.17) of Ref. [3])

$$\Sigma^{(rs)}(p) = M^{-1} \begin{pmatrix} \Sigma(p) & 0 \\ 0 & -\bar{\Sigma}(p) \end{pmatrix} M^{-1}, \tag{11.10}$$

where $M = M(|\, p_0 \,|)$ is given by (10.5), and $\Sigma(p)$ is some scalar function. We infer that $\Pi^{(rs)}(p)$ must have the same matrix form. Any choice of the function $\Pi(p)$, analogous to $\Sigma(p)$ in (11.10), then gives rise to a perturbation theory in which the free propagator $D^{(rs)}$, and thereby [3] the full Green functions, satisfy the KMS condition. The only restriction is that $\Pi(p)$ should have a nonvanishing, negative imaginary part in order that (11.5) exists. For example, the usual real-time propagator is obtained by choosing $\Pi = -i\varepsilon$.

A simple specific choice for $\Pi(p)$ is

$$\Pi(p) = \varepsilon(\mathbf{p})^2 - \mathbf{p}^2 - m^2 - \kappa^2 - 2i\kappa\varepsilon(\mathbf{p}) \tag{11.11}$$

for certain $\varepsilon(\mathbf{p})$ and κ. We may then determine $\Pi^{(rs)}$ by the analogue of (11.10), and subsequently find $D^{(rs)}$ by solving (11.9). The Fourier transform $\check{D}_0^{(rs)}(p)$ then comes out to be exactly identical to the function $\check{G}_0^{(rs)}(p)$ in (10.17). The presence of the operator S in (11.3) also induces a new two-point counterterm. If the choice (11.11) is combined with the mass counterterm $\delta m^2 \tau_3$ coming from $\hat{\mathscr{H}}_I$ in (11.3), on use of the identity (10.19) the total two-point counterterm just reproduces (10.18).

In the approach of this section, the detailed matrix forms of (10.17) and (10.18) are a consequence of the KMS condition (which is equivalent to (11.10) that has been imposed on the choice of $\Pi^{(rs)}$. In the procedure of Section 9 and 10, on the other hand, the correct matrix structure came out automatically on choosing (10.16) for $Z(E, \sigma^2)$. In any case, the Feynman rules (10.17) and (10.18), and thereby the specific form (10.16) of Z, have now been motivated from a rather different point of view. Effectively, the "improved" Gell-Mann/Low formula amounts to a resummation of naive thermal perturbation theory to infinite order. By the above results, the same may be said of the approach in the previous sections for the special case (10.16). We should stress, however, that being tied to Green functions the formalism of this section cannot determine the actual form of the function $\Pi(p)$, whereas the function $Z(E, \sigma^2)$ is in principle fixed by the canonical relations, as we have seen.

Dissipative Field Theory

The most significant characteristic of (10.17) is the presence of dissipation already at the tree level. The emergence of the complex counterterm (10.18) is necessarily associated with this feature. Dissipative propagators and imaginary counterterms were first introduced in nonequilibrium thermo field dynamics [34–36] *at the operator level.* The free Hamiltonian determining the free propagators was there taken to be non-hermitian from the outset. This is definitely different in our

approach, where both the free and the full Hamiltonians are formally self-adjoint, and the emergence of imaginary quantities is due to the integration (in (9.5) and (10.11)) over the continuous variable E. The continuity of this parameter is a consequence of the fact that in interacting thermal representations the energy spectrum is continuous even at fixed momentum.

The appearance of dissipative terms in the bare Feynman rules even in interacting *equilibrium* representations should be no surprise. The representation space $\mathscr{H}_\omega$ contains not only the equilibrium state $|0(\beta)\rangle$ but in addition all states $A|0(\beta)\rangle$, $A \in \pi_\omega(\mathfrak{A})$. These states describe fluctuations around and excitations of the equilibrium state, which by the fluctuation–dissipation theorem are directly related to the dissipative part of the full propagator. A heuristic perspective on the emergence of dissipation in quantum field theory is given in Ref. [36].

12. Thermal Renormalization

The final ingredient in any calculational scheme based on propagators and vertices is the expression of bare parameters in terms of physical ones, and the associated choice of renormalization conditions. The physical parameters should be chosen to be directly observable quantities. The point we make is that the choice of these quantities is representation-dependent. That is, apart from being observable in principle, the physical parameters must be immediately relevant to the context in which the calculations are performed.

Renormalization in Vacuum Representations

For example, in a scattering-theoretic context one studies (approximate) vacuum representations in which the Haag–Ruelle theory is realized; then the relevant observables are obviously the masses m of the asymptotic particles and the scattering amplitudes $\lambda(\mu)$ at a given energy scale μ. The process of translating the bare parameters m_0, λ_0 into the physical and contextually relevant ones m, $\lambda(\mu)$ is manageable in diagrammatic perturbation theory if λ_0 is a formal power series in $\lambda(\mu)$ (usually with divergent coefficients). In that case the bare vertex (10.12) is replaced by an infinite series of vertices proportional to $\lambda^n(\mu)$ for $n = 1, 2, \ldots$. The first of these is already, by definition, the complete scattering amplitude at energy scale μ, while the other vertices are counterterms whose presence ascertains that at energy scale μ there are indeed no further contributions. The explicit form of $\lambda(\mu)$ may be determined by the renormalization group equation $d\lambda_0/d\mu = 0$. Also, the physical mass already appears directly in the Feynman rules (i.e., in the free propagator), and the two-point counterterms $i\delta m^2 = i(m^2 - Zm_0^2)$ and $ip^2(Z-1)$ are fixed by the requirement that m actually be the physical mass.

This translation process is relatively easy in (Haag–Ruelle) vacuum representations because the relevant physical observables have a simple and direct diagrammatic interpretation: a two-particle scattering amplitude is just a four-leg diagram,

while masses are identified in terms of the one-particle irreducible self-energy. In fact, the whole process in practice boils down to a set of momentum-space subtractions.

Thermal Complications

In thermal representations the situation is much more complicated. The contextually relevant observable numbers are now thermodynamic response functions and dissipative transport coefficients rather than masses and cross sections. Therefore, the renormalization procedure will by necessity be qualitatively different from the one in vacuum representations. Indeed, since thermal and vacuum representations have just about nothing in common, it would be asinine to first express λ_0 and m_0 in terms of the vacuum parameters $\lambda(\mu)$ and m and then calculate the thermodynamic observables as functions of these. Reasoning physically rather than mathematically, one should try to parametrize experimental results in terms of other experimental results in the same class.

For concreteness' sake, we will discuss this issue for a $\lambda_0 A^4$-theory in which we adopt the model choice (10.16) for Z. The bare parameters are λ_0 and m_0, and these are to be expressed in terms of two thermally observable quantities A, B plus a set of renormalization conditions ascertaining the self-consistency of this expression. As in the vacuum case, a third, dependent quantity enters, namely the wavefunction renormalization Z. In our case (10.16) we have $Z = Z(\kappa, \varepsilon_{\mathbf{p}})$. If we now succeed in relating λ_0, κ, and $\varepsilon_{\mathbf{p}}$ to the observables A and B, we can ultimately express the Feynman rules in terms of A and B rather than λ_0, etc. In analogy with the counterterm δm^2 in a vacuum theory, the thermal counterterm (10.18) is completely fixed by specifying $\lambda_0 = \lambda_0(A, B)$, $\kappa = \kappa(A, B)$, and $\varepsilon_{\mathbf{p}} = \varepsilon_{\mathbf{p}}(A, B, \mathbf{p})$ once the meaning of A and B has been agreed upon. Clearly, only two of these three relations may be independently stipulated. Of course, unlike A and B, the bare parameters λ_0 and m_0 are representation-independent. As in the vacuum case, this independence may be expressed by a renormalization group equation. In thermal field theory A and B will be functions of temperature T, a state of affairs that may be compared with the energy-scale dependence $\lambda = \lambda(\mu)$. (Of course, T is fixed in a *given* representation!) We thus have the renormalization group equation $d\lambda_0(T, A(T), B(T))/dT = 0$, and similarly for m_0, asserting the autonomy of the bare parameters.

Choice of Thermal Observables

The crucial step in the thermal renormalization procedure, then, is the appropriate choice of the relevant observables $A(T)$ and $B(T)$. An obvious practical requirement is that A and B have a relatively simple diagrammatic interpretation. Although the situation is definitely not as straightforward as that in vacuum field theory, this demand is to a certain extent met by observables that may be represented in terms of correlation functions of the energy-momentum tensor $t^{\mu\nu}$ (or of conserved currents in theories with internal symmetries). For a first attempt, we choose

the specific heat c_V and a static transport coefficient, e.g., the shear viscosity η, as the basic thermal observables.

For vanishing chemical potential, it is straightforward to express the specific heat in terms of the time-ordered energy-density autocorrelation function, viz.

$$c_V(T, \mu = 0) = T\frac{\partial^2 P}{\partial T^2} = -i\beta^2 \lim_{\delta \downarrow 0} \sum_{\substack{\text{res} \\ \text{Im } z < 0}} e^{-iz\delta} \check{T}^{00,00}(z, \mathbf{0}), \tag{12.1}$$

where

$$\check{T}^{\mu\nu,\rho\sigma}(p) = \int d^4x\, e^{ipx} \langle 0(\beta)|\, T[t^{\mu\nu}(x)\, t^{\rho\sigma}(0)]\,|0(\beta)\rangle_c \tag{12.2}$$

is the connected part of the Fourier transform of the full Green function of the energy-momentum tensor $t^{\mu\nu} = t^{\mu\nu}[A]$. The sum in (12.1) is over the residues of all poles in the lower-half complex plane.

The (static)) shear viscosity η may be written as [57]

$$\eta = \frac{i}{5}\left(\frac{d}{dp_0}\check{\Pi}_R\right)(0), \tag{12.3}$$

with the retarded function

$$i\check{\Pi}_R(p) = \int d^4x\, e^{ipx}\theta(t)\langle 0(\beta)|\,[\pi^{ij}(x), \pi_{ij}(0)]\,|0(\beta)\rangle, \tag{12.4}$$

which contains the traceless irreversibles stress tensor $(i, j, k = 1 \cdots 3)$

$$\pi^{ij} := t^{ij} - \tfrac{1}{3}\delta^{ij}t^{kk}. \tag{12.5}$$

In real-time thermal field theory any retarded two-point function G_R may be rewritten in terms of time-ordered ones $G^{(rs)}$ according to [58]

$$\check{G}_R(p) = \check{G}^{(11)}(p) - e^{-(1/2)\beta p_0}\check{G}^{(12)}(p), \tag{12.6}$$

where 1 and 2 are thermal indices [3]; cf. (9.10). This formula follows from the thermal state condition (5.25). In addition, $\check{G}^{(12)}$ is related to $\check{G}^{(11)}$ by [3]

$$\check{G}^{(12)}(p) = (e^{(1/2)\beta p_0} + e^{-(1/2)\beta p_0})^{-1}\,(\check{G}^{(11)}(p) - \overline{\check{G}^{(11)}(p)}). \tag{12.7}$$

We now use (12.2)–(12.7) and the fact that the real part of $G^{(11)}$ is an even function for $p_0 \to 0$ [3] to find the absorptive expression

$$\eta = -\tfrac{1}{10}\beta\, \mathrm{Im}\, \check{\Pi}^{ij,ij}(0). \tag{12.8}$$

Here Π is given by (12.2) with t^{ij} replaced by π^{ij} (12.5).

Nonrenormalization

It should be remarked that (12.2), and thereby (12.1) and (12.8), contain the bare energy-momentum tensor, i.e., the one given by its naive definition in terms of the bare Heisenberg field $A(x)$. However, this composite operator is automatically renormalized because its connected Green functions are finite [3] (the disconnected version of (12.2) contains a divergence coming from the product of the vacuum expectation values of $t^{\mu\nu}$ and $t^{\rho\sigma}$; this term is by definition subtracted in the connected Green function (12.2)). A similar statement holds for (most) other conserved currents [3]. It is a remarkable feature of thermal observables that they are expressed in Green functions of the bare fields of the theory. In contrast, vacuum observables (S-matrix elements) are given by Green functions of the "renormalized" fields $Z^{-1/2}A$ (cf. (3.11)). This distinction is crucial for our whole programme, because, as we have seen, the renormalization *constant* Z plays no role in thermal representations. Indeed, in the absence of particle poles with unit residue it would lack a natural renormalization prescription.

Example

A specific example of a renormalization prescription fixing the parameters λ_0, κ, and $\varepsilon_{\mathbf{p}}$ is the following: κ and $\varepsilon_{\mathbf{p}}$ are defined by demanding that the lowest-order (one-loop) contribution to c_V and η yields the exact result. Since the propagator (10.17) contains κ and $\varepsilon_{\mathbf{p}}$, these are directly fixed by this prescription. The bare coupling constant λ_0 is then regarded as a pure counterterm whose value, like that of ε_0 in (10.18), is determined by computing higher-order contributions to c_V and η and demanding that they vanish. This prescription may be surprising at first sight, but in fact it is the direct analogue of the prescription defining m and $\lambda(\mu)$ in vacuum field theory: m is defined to be the physical mass; i.e., the pole in the full propagator is required to be at $p^2 = m^2$, where it indeed is at tree-level. Higher-order corrections to this pole, i.e., $\Sigma(p^2 = m^2)$, are ordered to vanish, which fixes Z and δm^2. The treatment of $\lambda(\mu)$ is analogous (cf. the beginning of this section).

Using the Feynman rules for Green functions of the energy-momentum tensor (e.g., [3]) it is a simple matter to compute the one-loop contributions. As $t^{\mu\nu}$ and $t^{\rho\sigma}$ in (12.2) are both ordinary (nontilde) operators, i.e., of type 1, the one-loop diagrams just contain the "free" propagator $\check{G}_0^{(11)}$ given in (10.17). By the renormalization prescription above only the "free"-field contribution to $t^{\mu\nu}$ participates. If we parametrize $\varepsilon_{\mathbf{p}}^2 = \mathbf{p}^2 + \mu^2$ and assume for simplicity T and μ much larger than κ, we find

$$c_V = \beta^2 \int \frac{d^3p}{(2\pi)^3} \varepsilon_{\mathbf{p}}^2 N(1+N); \tag{12.9}$$

$$\eta = \frac{\beta}{15} \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}^4| \frac{N(1+N)}{\kappa(\varepsilon_{\mathbf{p}}^2 + \kappa^2)}, \tag{12.10}$$

where $N = (\exp(\beta\varepsilon_{\mathbf{p}}) - 1)^{-1}$.

Remarks

The expression for the specific heat looks like that for a free gas of particles of mass μ. This is a consequence of the condition $\kappa \ll \mu$ and the renormalization prescription, which now implicitly determines μ as a function of c_V by (12.9). Similarly, (12.10) is formally identical to a result obtained by Hosoya *et al.* [57] in the Matsubara formalism, but has a very different interpretation. In Ref. [57], κ (Γ in their notation) represents a particular approximation to the imaginary part of the self-energy proportional to the vacuum coupling constant squared. In our formalism, on the other hand, (12.10) is a renormalization prescription that shows a particular way of choosing the parameters of perturbation theory.

As stated before, the next step consists in calculating higher-order corrections to η and c_V as functions of λ_0, ε_0, $\kappa(c_V, \eta)$, and $\mu(c_V)$ and thereby determining $\lambda_0 = \lambda_0(T, c_V, \eta)$ and $\varepsilon_0(T, c_V, \eta)$ in such a way that these corrections vanish. How this is to be done is unfortunately unclear, because the functional form of λ_0 and ε_0 is not known (recall that in vacuum renormalization theory $\lambda_0 = \lambda_0(\mu, \lambda(\mu), m)$ is a power series in $\lambda(\mu)$). In principle, however, the solution gives rise to an infinite series of two- and four-point vertices, analogous to the usual counterterms at $T = 0$, with Feynman rules depending on η and c_V (or κ and μ). It is from these renormalized Feynman rules that other thermal observables, like the equation of state, dynamic transport coefficients, screening lengths, etc., are to be calculated. So in the end one finds the pressure $P = P(c_V, \eta)$ expressed in the specific heat and the shear viscosity, rather than the contextually meaningless vacuum parameters λ and m. Of course, different choices of the basic thermal observables should be considered, while with a given choice an infinite number of renormalization conditions is possible. A given renormalization condition, like that leading to (12.9) and (12.10), may turn out to be nonimplementable, i.e., inconsistent, in higher orders. A vacuum example is the condition $\Sigma(p^2 = M^2) = 0$ on the self-energy of a field describing an unstable particle with approximate mass $M > 2m$; cf. Section 4. In those cases one should just try again; in the present situation a modification of the Ansatz (10.16) for Z may also be contemplated.

The above is an idealized scenario of what, in our opinion, one should aim at in perturbative thermal field theory. The practical difficulties may well turn out to be insurmountale, in which case the whole idea of diagrammatic perturbation theory at finite temperature may have to be abandoned. The ultimate reason for this failure would be the unfortunate truth that thermal observables, in contrast with vacuum observables, are not themselves primitive building blocks of Feynman diagrams.

13. Summary and Conclusions

In the early years of relativistic thermal field theory, Weinberg [59] wrote "The aim of quantum field theory is to calculate S-matrix elements. However, at a finite

temperature there is no such thing as an S-matrix; a finite temperature means that space is filled with debris like black-body radiation, so any particle that participates in a collision is scattered again and again before it gets out to infinity. Instead of S-matrix elements, one wants to calculate... ."

In our opinion this remark is both deep and true (although in an infinite thermodynamic system a particle does not "get out" at all), but regrettably enough, it apparently has not significantly influenced the daily practice of the subject. Conventional thermal perturbation theory [20, 3] is strongly based on the belief that the presence of a finite temperature does *not* modify the particle structure of quantum field theory in an essential way. This attitude is most directly reflected in the choice of the free Hamiltonian defining perturbation theory: it is basically taken to be same as that in the vacuum theory.

We have made an effort to show that this point of view is untenable in an interacting system; this paper is an attempt to start all over again. Combining operator-algebraic, group-theoretic, and physical arguments, we have been led to a description and an interpretation of interacting thermal field theories in terms of so-called non-shell unstable particles. These are elementary unstable particle-like objects without any dispersion relation. The fact that states containing these peculiar objects, rather than ordinary on-shell particles, form the basic building blocks of the representation space of a thermal field theory may be interesting in itself; it certainly has a profound impact on the problem of constructing a consistent and tractable diagrammatic perturbation method.

As in vacuum field theory with a unitary S-matrix, there is a unique self-consistent choice of the free Hamiltonian and thereby of the free propagator that leads to a correct perturbation theory in the interaction picture; this feature is intimately related to the existence of the asymptotic time limits in the Gell-Mann/Low formula. However, in thermal perturbation theory the propagator is given by specifying an entire continuous function $Z(E, \sigma^2)$ rather than a single particle mass m. The functional form of Z is in principle given by a rather intractable self-consistent integral equation. In simple cases Z will contain two free parameters which are to be fixed by suitable renormalization prescriptions.

We have shown how this can be done, in principle, on the basis of a simple model choice of the function Z. This model choice, which we justified in a poor man's way, happens to lead to propagators and vertices that had been postulated before in the formalism of dissipative thermo field dynamics [34–37]. These dissipative Feynman rules illustrate the essential differences between thermal and vacuum renormalization; renormalized Feynman rules should, ideally, contain only thermodynamic observables rather than vacuum masses and coupling constants. It is, however, questionable whether a completely self-consistent "perturbation" theory, as constructed in this paper, is of much practical use. In fact, our results strongly indicate that not only thermal QCD (see below) but also all thermal field theories are inherently nonperturbative in nature.

We briefly return to the infrared problem in thermal QCD mentioned in the Introduction. Infrared problems emerging in naive perturbation theory are in

general a consequence of an incorrect selection of both the free Hamiltonian and the representation of the operator algebra [23]; as such they are completely circumvented in our approach. Indeed, either the thermal gluons are sharp thermal quasiparticles, in which case they do not interact (cf. Section 6) and are absent in loop diagrams where they might cause divergences, or the gluon states are rearranged in terms of "helicity" $n = \pm 1$ non-shell particle states, in which case their propagator is regular at $k^2 = 0$. Alternatively, it may be said that the dissipative parameter κ in the propagator (10.17) automatically shields all would-be infrared singularities.

Finally, it may be tempting to immediately generalize our results to non-relativistic field theories also. Although, in view of the physical arguments given, such an extension is not inevitably incorrect, it should be mentioned that the notion of mass, and thereby of a particle, is distinctly more involved in nonrelativistic physics. This is due to the mass superselection rule generated by the central extension of the Galilei-group [60, 51]. In relativistic theories there exists a deep connection between superselection rules and the representation theory of algebras of local observables [25]. In order to generalize our observations to the nonrelativistic case it is imperative to first investigate the emergence of superselection sectors in the representation theory of nonrelativistic field algebras.

APPENDIX: GROUP THEORY FOR KMS REPRESENTATIONS

In this appendix we review the representation theory of the boostless subgroup $S = SO(3)$ ($\times T_4$ of the Poincaré-group $P = L(\times T_4$, where L is the proper connected Lorentz group. In addition we will derive the Clebsch–Gordan series for the tensor product of two irreducible unitary ("canonical") representations.

Canonical Representations

The group S is a regular type I semidirect product [40] of the rotation group $SO(3)$, with elements R, and the abelian group T_4 of translations in four dimensions, with elements $a = (a_0, \mathbf{a})$. $SO(3)$ has a natural action on T_4 given by $Ra = (a_0, R\mathbf{a})$, which defines the semidirect product structure. Elements of S are labeled by the pair $s = (R, a)$. We will determine the canonical representations of S [60] by the method of induced representations [40]. The dual $\hat{T}_4$ of T_4 is isomorphic to the latter; we denote elements of $\hat{T}_4$ by $p = (p_0, \mathbf{p})$. The action of p on $a \in T_4$, which produces a one-dimensional unitary representative of a, is

$$p(a) = e^{ipa}, \tag{A.1}$$

where $pa = p_0 a_0 - \mathbf{p} \cdot \mathbf{a} \cdot S$ acts on the dual $\hat{T}_4$ according to

$$(sp)(a) := p(R^{-1}a) = \exp i(Rp)\, a, \tag{A.2}$$

where $Rp = (p_0, R\mathbf{p})$. The dual $\hat{T}_4$ then splits into orbits under the action of S; clearly the orbits $\mathcal{O}_{E,\sigma^2}$ are labeled by the pair (E, σ^2), so that a given orbit contains the points p satisfying $p_0 = E$, $\mathbf{p}^2 = \sigma^2$. In each orbit we choose a fiducial point, e.g., $\mathring{p} = (E, 0, 0, \sigma)$. The little group H is the subgroup of R satisfying $h\mathring{p} = \mathring{p}$, $h \in H$. Obviously $H = SO(2)$ for $\sigma^2 > 0$ and $H = SO(3)$ for $\sigma^2 = 0$.

The canonical representations of S are, therefore [40], labeled by the triples $\hat{s}_> = (E, \sigma^2 > 0, n)$, where $n \in \mathbb{Z}$ specifies the canonical representations of $H = SO(2)$, and $\hat{s}_0 = (E, \sigma^2 = 0, j)$, where $j \in \mathbb{N}$, specifies the canonical representations of $SO(3)$. Hence $\hat{S}$ as a set is

$$\hat{S} = (\mathbb{R} \times \mathbb{R}^+ \times \mathbb{Z}) \cup (\mathbb{R} \times \mathbb{N}). \tag{A.3}$$

The carrier space $\mathcal{H}(\hat{s}_>)$ of the representation $\hat{s}_>$ is the L^2-space of functions $\Phi_{E,\sigma^2}(\cdot)$ on the orbit $\mathcal{O}_{E,\sigma^2}$. The argument $\cdot$ is the constrained four-vector $p\,|\,(p_0 = E, \mathbf{p}^2 = \sigma^2)$ or, more explicitly, the pair of angles θ, ϕ. We choose the S-invariant inner product in $\mathcal{H}(E, \sigma^2 > 0, n)$ to be

$$\begin{aligned}
&\langle \Psi_{E,\sigma^2}(\cdot) | \Phi_{E,\sigma^2}(\cdot) \rangle_{\mathcal{H}(E,\sigma^2,n)} \\
&\quad = (2\pi)^{-3} \int d\Omega\, \bar{\Psi}_{E,\sigma^2}(\theta, \phi)\, \Phi_{E,\sigma^2}(\theta, \phi) \\
&\quad = \sigma^{-2}(2\pi)^{-3} \int d^4p \delta(p_0 - E)\, \delta(|\mathbf{p}| - \sigma)\, \bar{\Psi}(p)\, \Phi(p),
\end{aligned} \tag{A.4}$$

where $d\Omega = d\phi\, d\theta \sin\theta$, and Φ denotes the extension of Φ_{E,σ^2} to arbitrary values of its argument.

The unitary representative $U_{\hat{s}_>}(s)$ of an element $s = (R, a)$ of S in the representation $\hat{s}$ is defined by its action on the constrained function Φ_{E,σ^2} according to

$$(U_{\hat{s}_>}(s)\, \Phi_{E,\sigma^2})(p) = \exp i(Ea_0 - \mathbf{p} \cdot \mathbf{a} + n\alpha(R, p))\, \Phi_{E,\sigma^2}(R^{-1}p). \tag{A.5}$$

Here $\alpha(R, p)$ is the rotation angle around the z-axis specified by the Wigner rotation $\rho(p)^{-1}\, R\rho(R^{-1}p) \in SO(2) \subset SO(3)$. In this expression $\rho(p)$ is a particlar element of the coset $SO(3)/SO(2)$ that rotates $\mathring{p}$ into p, i.e., $\rho(p)\, \mathring{p} = p$. We see from (A.5) that the representations $\hat{s}_>$ of S are similar in structure to the massless representations of the Poincaré-group [50], with the helicity λ corresponding to the integer n in (A.5).

The representations $\hat{s}_0 = (E, 0, j)$ are simpler: the representation space $\mathcal{H}(\hat{s}_0)$ is identical to the $(2j+1)$-dimensional carrier space of the canonical representation $\mathcal{D}^j$ of $SO(3)$, and we have

$$(U_{\hat{s}_0}(s)\, \psi)_a = e^{iEa_0} \mathcal{D}^j(R)_{ab}\, \psi_b, \tag{A.6}$$

instead of (A.5). A special case is $\hat{s}_0 = (0, 0, 0)$ which gives the identity representation of S. In thermal field theory the carrier space $\mathcal{H}(0, 0, 0)$ is spanned by the thermal vacuum $|0(\beta)\rangle$.

In the main text an important role is played by the reducible direct integral representation

$$U_n^{(1)} := \int_0^\infty \frac{dE}{2\pi} \int_0^\infty d\sigma\, \sigma^2 [U_{(E,\sigma^2,n)} \oplus U_{(-E,\sigma^2,-n)}], \tag{A.7}$$

which acts on the carrier space

$$\hat{\mathscr{H}}_n^{(1)} := \int_0^\infty \frac{dE}{2\pi} d\sigma\, \sigma^2 [\mathscr{H}(E, \sigma^2, n) \oplus \mathscr{H}(-E, \sigma^2, -n)]. \tag{A.8}$$

This representation space has been described in Section 7 for $n = 0$; by the special structure of (A.5), however, this description, including the inner product (7.4), (7.5), is valid for any $n \in \mathbb{Z}$. By definition of a direct integral of representations [40] the action of $U_n^{(1)}(s)$ on the elements $\Phi(\cdot)$ of $\hat{\mathscr{H}}_n^{(1)}$ is given by

$$(U_n^{(1)}(s)\, \Phi)(p) = (U_{(p_0,\, \mathbf{p}^2, n_\pm)}(s)\, \Phi)(p), \tag{A.9}$$

with $n_\pm = \pm n$ determined by the sign of p_0. The corresponding expression has already been given in (A.5) for $p_0 = E$, $\mathbf{p}^2 = \sigma^2$. Note that $\sigma^2 = 0$ in (A.7) is of measure zero.

Covariant Representations

Apart from the canonical representations of semidirect products $A (\times B$, a second class of representations of such groups is of physical interest, namely the so-called covariant representations [52]. These are carried by the space of L^2-functions on B with values in the carrier space of a canonical representation of A. In our case ($A = SO(3)$, $B = T_4$) this means that we have functions $\phi_\alpha(\cdot)$ of the argument x, on which the covariant representation T^j acts according to ($s = (R, a)$)

$$(T^j(s)\, \phi)_a(x) = \mathscr{D}^j(R)_{ab}\, \phi_b(R^{-1}(x-a)). \tag{A.10}$$

To relate the covariant representations to the canonical ones [50, 52] we Fourier transform (A.10), yielding

$$\begin{aligned} (\check{T}^j(s)\, \check{\phi})_a(p) &:= \int d^4x\, e^{ipx} (T^j(s)\, \phi)_a(x) \\ &= e^{ipa} \mathscr{D}^j(R)_{ab}\, \check{\phi}_b(R^{-1}p). \end{aligned} \tag{A.11}$$

In general, covariant representations are nonunitary and reducible. To construct irreducible ones we adapt the method developed by Asorey *et al.* [52] (also cf. [50]) to the group S. We start by restricting the argument p in (A.11) to a specified orbit $\mathscr{O}_{E,\sigma^2}$ in $\hat{T}_4$, with fiducial point $\mathring{p} = (E, 0, 0, \sigma)$ and little group $H = SO(2)$ ($\sigma^2 > 0$). Subsequently, the representation $\mathscr{D}^j$ of $SO(3)$ is subduced (restricted) to H, which for $j > 0$ gives a reducible representation of H. Since H is compact, this

can be reduced, so that the carrier space $\mathscr{H}^j = \mathbb{C}^{2j+1}$ decomposes as a direct sum of one-dimensional irreducible subspaces $\mathscr{H}_\mu = \mathbb{C}$, $\mu = 1, ..., 2j+1$. Let $\{e_a\}$ be the basis with respect to which $\mathscr{D}^j_{ab}$ is defined, and let e_μ be a basis vector for $\mathscr{H}_\mu$; there exists a unitary matrix $u_{a\mu}$ such that $e_\mu = e_a u_{a\mu}$. Each subrepresentation $\mathscr{D}^j(H)$ restricted to $\mathscr{H}_\mu$ is specified by an integer n_μ. Therefore, if $R_z(\alpha) \in SO(3)$ is a rotation around the z-axis by angle α we have

$$(u^{-1})_{\mu a}\, \mathscr{D}^j(R_z(\alpha))_{ab}\, u_{b\nu} = \delta_{\mu\nu} \exp in_\mu \alpha. \tag{A.12}$$

Finally, let the function $\Phi^{(n_\mu)}_{E,\sigma^2}(\cdot)$ with argument p restricted to $\mathscr{O}_{E,\sigma^2}$ transform canonically by (A.5) with $n = n_\mu$. Then for any value of n_μ occurring in the reduction of $\mathscr{D}^{(j)}(H)$ sketched above, the function

$$\phi^{\hat{s}}_a(p) := \mathscr{D}^j(\rho(p))_{ab}\, u_{b\mu} \Phi^{(n_\mu)}_{E,\sigma^2}(p) \tag{A.13}$$

transforms covariantly, i.e., by (A.11). Here $\rho(p)$ has been defined below (A.5). As we see, the left-hand side, albeit transforming covariantly, carries the label $\hat{s} = (E, \sigma^2, n)$ specifying the canonical representation. This formula is used in Section 8. For scalar representations ($j = n_\mu = 0$) we call $\phi^{(E,\sigma^2,0)}(p)$ simply $\phi_{E,\sigma}(p)$.

A physically relevant illustration of the above construction is provided by the adjoint ($j = 1$) representation of $SO(3)$. The subrepresentation of $H = SO(2)$ (rotations around the z-axis) reduces into an $n = 0$ part with basis vector $(0, 0, 1)$, as $n = 1$ part with basis vector $(1, -i, 0)$ and an $n = -1$ part with basis vector $(1, i, 0)$. Any of these may be used to build a covariant three-vector field by (A.13) (also cf. Section 8).

Clebsch–Gordan Series

In Section 7 we needed the Clebsch–Gordan series for the decomposition of a product of two canonical representations of $S = SO(3)\,(\times T_4$. We close this paper by deriving this series. Since the subgroup T_1 of time translations factorizes in the semidirect product, hence in its Clebsch–Gordan series, the problem can be reduced to the determination of the Clebsch–Gordan series of the Euclidean subgroup $E(3) = SO(3)\,(\times T_3$.

By the reasoning in the first subsection above, the canonical representations of $E(3)$ are labeled by the pair $\hat{e} = (\sigma^2, n)$, $n \in \mathbb{Z}$. The carrier space $\mathscr{H}(\hat{e})$ consists of L^2-functions $\Phi_{\sigma^2}(\cdot)$ on the orbit $\mathscr{O}_{\sigma^2} = \{\mathbf{p} \mid \mathbf{p}^2 = \sigma^2\}$. Up to (A.20) below we assume $\sigma > 0$.

Let $E(3) \ni e = (R, \mathbf{a})$. The unitary representative $U_{\hat{e}}(e)$ acts on $\Phi_{\sigma^2}(\cdot)$ according to

$$(U_{\hat{e}}(e)\, \Phi_{\sigma^2})(\mathbf{p}) = \exp i(-\mathbf{p}\cdot\mathbf{a} + n\alpha(R, \mathbf{p}))\, \Phi_{\sigma^2}(R^{-1}\mathbf{p}); \tag{A.14}$$

cf. (A.5) and text below it. The Hilbert space $\mathscr{H}(\hat{e}_1) \otimes \mathscr{H}(\hat{e}_2)$ carrying the product representation $\hat{e}_1 \otimes \hat{e}_2$ consists of L^2-functions $\Phi_{\sigma_1^2, \sigma_2^2}(\cdot, \cdot)$ on the orbit $\mathscr{O}_{\sigma_1^2} \times \mathscr{O}_{\sigma_2^2}$; i.e.,

the argument $\cdot,\cdot$ is the pair $\mathbf{p}_1, \mathbf{p}_2$ with $\mathbf{p}_i^2 = \sigma_i^2$, $i = 1, 2$. By (A.14) the product representation is given by

$$(U_{\hat{e}_1} \otimes U_{\hat{e}_2}(e)\, \Phi_{\sigma_1^2, \sigma_2^2})(\mathbf{p}_1, \mathbf{p}_2)$$
$$= \exp i(-(\mathbf{p}_1 + \mathbf{p}_2)\cdot \mathbf{a} + n_1 \alpha(R, \mathbf{p}_1) + n_2 \alpha(R, \mathbf{p}_2))\, \Phi_{\sigma_1^2, \sigma_2^2}(R^{-1}\mathbf{p}_1, R^{-1}\mathbf{p}_2). \tag{A.15}$$

We are now going to reduce this product representation applying a geometric method given by Moussa and Stora [61] for general regular semidirect products.

In order to decompose $\mathscr{H}(\hat{e}_1) \otimes \mathscr{H}(\hat{e}_2)$ into irreducible subspaces, we first observe that the subset of $\mathscr{O}_{\sigma_1^2} \times \mathscr{O}_{\sigma_2^2}$,

$$M_{\mathbf{k}_1, \mathbf{k}_2} := \{(R'\mathbf{k}_1, R'\mathbf{k}_2) \mid R' \in SO(3)\}, \tag{A.16}$$

where $\mathbf{k}_1 \in \mathscr{O}_{\sigma_1^2}$ and $\mathbf{k}_2 \in \mathscr{O}_{\sigma_2^2}$ are fixed and R' varies in $SO(3)$, is closed (i.e., invariant, but not pointwisely so) under the natural action $R(\mathbf{p}_1, \mathbf{p}_2) = (R\mathbf{p}_1, R\mathbf{p}_2)$ of $SO(3)$ on the product orbit (cf. (A.2)). A pair $(\mathbf{p}_1, \mathbf{p}_2) \in \mathscr{O}_{\sigma_1^2} \times \mathscr{O}_{\sigma_2^2}$ clearly is in $M_{\mathbf{k}_1, \mathbf{k}_2}$ if and only if $\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{k}_1 \cdot \mathbf{k}_2$. However, any two pairs $(\mathbf{p}_1, \mathbf{p}_2)$, $(\mathbf{l}_1, \mathbf{l}_2)$ satisfying $\mathbf{p}_i^2 = \mathbf{l}_i^2$, $i = 1, 2$, and $\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{l}_1 \cdot \mathbf{l}_2$ are connected by a rotation R such that $R(\mathbf{p}_1, \mathbf{p}_2) = (R\mathbf{p}_1, R\mathbf{p}_2) = (\mathbf{l}_1, \mathbf{l}_2)$. This shows that $M_{\mathbf{k}_1, \mathbf{k}_2}$ is not only closed but also irreducible under the action of $SO(3)$. By the same argument we see that, σ_1 and σ_2 being given, the manifold $M_{\mathbf{k}_1, \mathbf{k}_2}$ is completely identified by specifying the angle θ in $\mathbf{k}_1 \cdot \mathbf{k}_2 = \sigma_1 \sigma_2 \cos\theta$. We therefore rename the manifold (A.16) as M_θ.

Returning to (A.15), it follows that the subspace $\mathscr{H}_\theta$ of functions taking values on M_θ is irreducible under the representation $\hat{e}_1 \otimes \hat{e}_2$. Hence $\mathscr{H}(\hat{e}_1) \otimes \mathscr{H}(\hat{e}_2)$ can be reduced as a direct integral over these subspaces. Because each submanifold M_θ occurs only once in the product orbit, it follows that each Hilbert subspace $\mathscr{H}_\theta$ occurs with multiplicity one in this reduction. We can easily relate $\mathscr{H}_\theta$ to the canonical carrier spaces $\mathscr{H}(\sigma^2, n)$ by comparing (A.15), with $\Phi_{\sigma_1^2 \sigma_2^2}$ restricted to $\mathscr{H}_\theta$, with (A.14). This shows that

$$\mathscr{H}_\theta = \mathscr{H}(\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 \cos\theta, n_1 + n_2). \tag{A.17}$$

Thus the tensor product decomposes as

$$\mathscr{H}(\sigma_1^2, n_1) \otimes \mathscr{H}(\sigma_2^2, n_2) = \int_{|\sigma_1 - \sigma_2|}^{\sigma_1 + \sigma_2} d\sigma\, \mathscr{H}(\sigma^2, n_1 + n_2). \tag{A.18}$$

Combining (A.18) with the trivial Clebsch–Gordan series for the group T_1, we find the reduction of a product of two canonical representations $\hat{s}_i = (E_i, \sigma_i^2, n_i)$ of $SO(3)\,(\times T_4$ to be

$$\mathscr{H}(\hat{s}_i) \otimes \mathscr{H}(\hat{s}_2) = \int_{|\sigma_1 - \sigma_2|}^{\sigma_1 + \sigma_2} d\sigma\, \mathscr{H}(E_1 + E_2, \sigma^2, n_1 + n_2), \tag{A.19}$$

for $\sigma_1 \neq 0 \neq \sigma_2$. The same equation holds, of course, if $\mathscr{H}$ is replaced by U. In particular, with $\hat{\mathscr{H}}^{(1)}$ given by (7.2) it follows that

$$\hat{\mathscr{H}}^{(1)} \otimes \hat{\mathscr{H}}^{(1)} = \infty \hat{\mathscr{H}}^{(1)}. \tag{A.20}$$

Together with (7.1) and the obvious property $m \cdot \infty = \infty$, $m \in \mathbb{N}$, the Fock space nature of $\hat{\mathscr{H}}_\omega$ for the scalar case, i.e., Eq. (7.3), ensues. The analogy between (A.19) and the Clebsch–Gordan series for the decomposition of the product of two massless representations of the Poincaré-group [61] is illuminating. In this analogy the integer n corresponds to the helicity of a massless particle.

To be complete we also give the results for $\sigma_1 = \sigma_2 = 0$ and $\sigma_1 = 0 \neq \sigma_2$. The first case reduces to giving the Clebsch–Gordan series series for $SO(3)$ (cf. the text above (A.3)), because the carrier spaces $\mathscr{H}(E, 0, j)$ are those of the representation $\mathscr{D}^j$ of the rotation group. Hence

$$\mathscr{H}(E_1, 0, j_1) \otimes \mathscr{H}(E_2, 0, j_2) = \sum_{k=|j_1 - j_2|}^{j_1 + j_2} \mathscr{H}(E_1 + E_2, 0, k). \tag{A.21}$$

The second case is similar to the reduction of covariant representations discussed earlier. Specializing to $E(3)$ as above, the carrier space $\mathscr{H}(0, j) \otimes \mathscr{H}(\sigma^2, n)$ consists of functions $\psi_a(\,\cdot\,)$ on the orbit $\mathscr{O}_{\sigma^2}$ taking values in $\mathbb{C}^{2j+1}$. By (A.6) and (A.14) the product representation acts by

$$\begin{aligned}(U_{(0,j)} \otimes U_{(\sigma^2,n)}(e)\,\psi)_a\,(\mathbf{p}) \\ = \exp i(-\mathbf{p}\cdot\mathbf{a} + n\alpha(R, \mathbf{p}))\, \mathscr{D}^j(R)_{ab} \psi_b(R^{-1}\mathbf{p});\end{aligned} \tag{A.22}$$

cf. (A.11). The reasoning in the previous subsection may then be imitated to derive the Clebsch–Gordan series

$$\mathscr{H}(E_1, 0, j) \otimes \mathscr{H}(E_2, \sigma^2, n) = \sum_m \mathscr{H}(E_1 + E_2, \sigma^2, m + n). \tag{A.23}$$

The values of m occurring in this sum are those labeling the irreducible components in the representation $\mathscr{D}^j(SO(2))$ obtained by subducing $\mathscr{D}^j$ from $SO(3)$ to $(SO(2)$; cf. the text after (A.11).

Fermions

We finally remark that all results in this appendix and, indeed, *mutatis mutandis*, in the rest of the paper, may be extended to the fermionic case. To do this one simply replaces S by its covering group $\bar{S} = SU(2)$ $(\times T_4$. The little group $SO(2)$ in the above considerations is then to be replaced by its lift in $SU(2)$, namely $U(1) \times Z_2$ (and not by its own covering group $\overline{SO(2)} = \mathbb{R}$!). The net effect of these substitutions is that the integers n and j above are now allowed to be half-integers as well.

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REFERENCES

1. L. McLerran, *Rev. Mod. Phys.* **58** (1986), 1021.
2. R. J. Brandenberger, *Rev. Mod. Phys.* **57** (1985), 1.
3. N. P. Landsman and Ch. G. van Weert, *Phys Rep.* **145** (1987), 141, and references therein.
4. F. Karsch, *in* "Proceedings, Sixth International Conference on Ultra Relativistic Nucleus–Nucleus Collisions, *Z. Phys. C* **38** (1988), 147.
5. H. Matsumoto, I. Ojima, and H. Umezawa, *Ann. Phys. (N.Y)* **152** (1984), 348.
6. K. Funakubo and M. Sakamoto, *Prog. Theor. Phys.* **76** (1986), 490; *Phys. Lett. B* **186** (1987), 205.
7. H. Nakkagawa and A. Niegawa, *Phys. Lett. B* **193** (1987), 263; *Phys. Lett. B* **196** (1987), 571 (E); also cf. Y. Fujimoto and H. Yamada, *Phys. Lett. B* **195** (1987), 231; **200** (1988), 167.
8. J. M. Luttinger and J. C. Ward, *Phys. Rev.* **118** (1960), 1417.
9. K. Hepp, *in* "Axiomatic Field Theory" (M. Chretien and S. Deser, Eds.), Gordon & Breach, New York, 1966.
10. M. Reed and B. Simon, "Scattering Theory, Methods of Modern Mathematical Physics," Vol. III, Academic Press, New York, 1979.
11. J. Glimm and A. Jaffe, "Quantum Physics: A Functional Integral Point of View," 2nd ed., Springer-Verlag, Berlin, 1987.
12. G. Gallavotti, *Rev. Mod. Phys.* **57** (1985), 471.
13. K. Hepp, "Lecture Notes in Physics," Vol. 2, Springer-Verlag, Berlin, 1969.
14. M. Veltman, *Physica* **29** (1963), 186.
15. G. 't Hooft and M. Veltman, "Diagrammar," CERN Yellow Report 73–9, 1973.
16. N. P. Landsman, *Phys. Rev. Lett.* **60** (1988), 1909.
17. A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, "Methods of Quantum Field Theory in Statistical Physics," Prentice–Hall, Englewood Cliffs, NJ, 1963.
18. H. Umezawa, H. Matsumoto, and M. Tachiki, "Thermo Field Dynamics and Condensed States," North-Holland, Amsterdam, 1982.
19. A. D. Linde, *Phys. Lett. B* **96** (1980), 289.
20. D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *Rev. Mod. Phys.* **53** (1981), 43.
21. J. Polonyi, *Nucl. Phys. A* **461** (1987), C279.
22. T. A. DeGrand and C. E. De Tar, *Phys. Rev. D* **34** (1986), 2469, **35** ((1987), 742.
23. G. Morchio and F. Strocchi, *in* "Fundamental Problems of Gauge Field Theory" (G. Velo and A. S. Wightman, Eds.), Plenum, New York, 1986.
24. H. Narnhofer, M. Requardt, and W. Thirring, *Commun. Math. Phys.* **92** (1983), 247.
25. R. Haag, *in* "Lectures on Elementary Particles and Quantum Field Theory" (S. Deser *et al.*, Eds.), MIT Press, Cambridge, MA, 1970; *in* "Progress in Quantum Field Theory" (H. Ezawa and S. Kamefuchi, Eds.), North-Holland, Amsterdam, 1986; *in* "Fields and Geometry" (A. Jadczyk, Ed.), World Scientific, Singapore, 1986.
26. O. Bratteli and D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanics," Vol. I, 2nd ed., and Vol. II, Springer-Verlag, New York, 1987 and 1981.

27. N. P. LANDSMAN, *in* "Proceedings of the CAP-NSERC Summer Institute in Theoretical Physics" (F. C. Khanna, H. Umezawa, G. Kunstatter, and H. C. Lee, Eds.), Vol. 1, World Scientific, Singapore, 1988.
28. J. LUKIERSKI, *Nuovo Cimento A* **60** (1969), 353.
29. J. LUKIERSKI AND M. OZIEWICZ, *Acta Phys. Pol. B* **3** (1972), 231.
30. J. LUKIERSKI, *Fortschr. Phys.* **21** (1973), 85.
31. A. L. LICHT, *Ann. Phys. (N. Y.)* **34** (1965), 161.
32. A. S. WIGHTMAN, *in* "Cargèse Lectures in Theoretical Physics: High Energy Electromagnetic Interactions and Field Theory" (M. Lévy, Ed.), Gordon & Breach, New York, 1967.
33. V. GLASER, H. LEHMANN, AND W. ZIMMERMANN, *Nuovo Cimento* **6** (1957), 1122.
34. T. ARIMITSU, H. UMEZAWA, Y. YAMANAKA, AND N. J. PAPASTAMATIOU, *Physica A* **148** (1988), 27.
35. T. ARIMITSU, H. UMEZAWA, AND Y. YAMANAKA, *J. Math. Phys.* **28** (1987), 2741.
36. H. UMEZAWA AND Y. YAMANAKA, "Micro, Macro and Thermal Concepts in Quantum Field Theory," Univ. of Alberta preprint, 1987 (unpublished).
37. H. MATSUMOTO, *Fortschr. Phys.* **35** (1987), 475.
38. H. MATSUMOTO, *in* "Progress in Quantum Field Theory" (H. Ezawa and S. Kamefuchi, Eds.), North-Holland, Amsterdam, 1986.
39. G. G. EMCH, "Algebraic Methods in Statistical Mechanics and Quantum Field Theory," Wiley, New York, 1972.
40. K. MAURIN, "General Eigenfunction Expansions and Unitary Representations of Topological Groups," PWN, Warszawa, 1968.
41. K. FREDENHAGEN, *in* "Fundamental Problems of Gauge Field Theory" (G. Velo and A. S. Wightman, Eds.), Plenum, New York, 1986.
42. G. VALENT, *Nucl. Phys. B* **65** (1973), 445.
43. G. L. SEWELL, "Quantum Theory of Collective Phenomena," Oxford Univ. Press, Oxford, 1986.
44. G. TEN BRINKE AND M. WINNINK, *Commun. Math. Phys* **51** (1976), 135.
45. G. K. PEDERSEN, "C^*-Algebras and their Automorphism Groups," Academic Press, New York, 1979.
46. I. OJIMA, *Ann. Phys. (N.Y.)* **137** (1981), 1.
47. I. OJIMA, *Lett. Math. Phys.* **11** (1986), 73.
48. H. MATSUMOTO, H. UMEZAWA, N. YAMAMOTO, AND N. J. PAPASTAMATIOU, *Phys. Rev. D* **34** (1986), 3217.
49. H. AOYAMA, *Phys. Lett. B* **171** (1986), 420.
50. S. WEINBERG, *Phys. Rev.* **138B** (1965), 988.
51. H. J. BORCHERS AND R. N. SEN, *Commun. Math. Phys.* **42** (1975), 101.
52. M. ASOREY, L. J. BOYA, AND J. F. CARIÑENA, *Rep. Math. Phys.* **21** (1985), 391.
53. A. BRZESKI AND J. LUKIERSKI, *Acta Phys. Pol. B* **6** (1975), 577.
54. K.-C. CHOU, Z.-B. SU, B.-L. HAO, AND L. YU, *Phys. Rep.* **118** (1985), 1.
55. H. MATSUMOTO, H. UMEZAWA, AND J. P. WHITEHEAD, *Prog. Theor. Phys.* **76** (1986), 260.
56. Y. NAKAWAKI, Univ. of Alberta preprint, 1987 (unpublished).
57. A. HOSOYA, M. SAKAGAMI, AND M. TAKAO, *Ann. Phys. (N. Y.)* **154** (1984), 229.
58. H. MATSUMOTO, Y. NAKANO, AND H. UMEZAWA, *Phys. Lett. A* **100** (1984), 125.
59. S. WEINBERG, *in* "Understanding the Fundamental Constituents of Matter (Erice, 1976)" (A. Zichici, Ed.), Plenum, New York, 1978.
60. J. M. LÉVY-LEBLOND, *in* "Group Theory and its Applications" (E. M. Loebl, Ed.), Vol. II, Academic Press, New York, 1971.
61. P. MOUSSA AND R. STORA, *in* "Lectures in Theoretical Physics" (W. E. Brittin and A. O. Barut, Eds.), Vol. VIIa, Univ. of Colorado Press, Boulder, 1965.