Is mathematics like a game?

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Abstract

We re-examine the old question to what extent mathematics may be compared with a game. Mainly inspired by Hilbert and Wittgenstein, our answer is that mathematics is something like a "rhododendron of language games", where the rules are inferential. The pure side of mathematics is essentially formalist, where we propose that truth is not carried by theorems correspondencing to whatever independent reality and *arrived at* through proof, but is *defined by* correctness of rule-following (and as such is objective given these rules). Gödel's theorems, which are often seen as a threat to formalist philosophies of mathematics, actually strengthen our concept of truth. The applied side of mathematics arises from two practices: first, the dual nature of axiomatization as *taking* from heuristic practices like physics and informal mathematics whilst *giving* proofs and logical analysis; and second, the ability of using the inferential role of theorems to make "surrogative" inferences about natural phenomena. Our framework is pluralist, combining various (non-referential) philosophies of mathematics.

Contents

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1 Introduction

The aim of this paper is to re-examine the old question to what extent mathematics may be compared with a game (like chess).[1] It is no accident that the idea of such a comparison originated in the late nineteenth century, since that was the time of the "modernist transformation" in which mathematics lost its connection with physical reality and visualizability, which were replaced by abstraction and rigorous proof.[2] Indeed, serious analysis of the analogy between mathematics and games like chess started with Frege's criticisms of Thomae (1898), Heine (1872) and Illigens (1893).[3] Frege rejected the analogy; he was primarily unable to comprehend how a mere game could describe any "thought",[4] and his secondary, more specific reasons for disapproving of the analogy reduce to this idiosyncratic inability (see §2 below). Frege's discussion remains of considerable interest for the general philosophy of mathematics, whose questions concern:[5]

1. *Ontology:* What is mathematics *about*?[6] What (and where) are mathematical objects?

2. *Truth:* What is the nature of mathematical truth?

3. *Epistemology:* How can we know about mathematics? Do we *discover* or *invent* it?

4. *Applicability:* What makes applied mathematics possible?

We cannot even begin to summarize the huge literature about these questions, which go back to Plato and Aristotle. But there is one position–incorporating all kinds of platonism and other forms of realism–we would like to highlight as a foil, since it seems to be very widely shared among both "working mathematicians" and philosophers. Here is a typical expressions of it by Hardy:[7]

> 'It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and

[1] The review by Epple (1994) provides an excellent historical and philosophical introduction to this question; see also Detlefsen (2005) and Weir (2022). We will not discuss the the mathematics of games, cf. du Sautoy (2023).

[2] See, for example, Mehrtens (1990) and Gray (2008).

[3] See Frege (1903), §§86–137, translated in Geach and Black (1960).

[4] This was the central ingredient of his philosophy of both mathematics and language. The word 'thought' (*Gedanke*) is ambiguous in both English and German. Frege (1892) clarifies that for him, thoughts do not refer to the subjective act of thinking, but to the objective content thereof, and thus are possible truth carriers. Frege (1918) moves to an uncompromising platonism, e.g., 'Without wishing to give a definition, I call a thought something for which the question of truth arises. (...) A thought is something immaterial and everything material and perceptible is excluded from this sphere of that for which the question of truth arises.' (Frege, 1918, p. 292). He even uses the (no longer usable) term 'dritte Reich' (best translated as: third Realm) as the sphere beyond the material and perceptible.

[5] For a somewhat different list see Linnebo (2017), §1.1. See also Tait (2001).

[6] Anticipating a Wittgensteinian setting, a more precise form of this question would be: *are mathematical objects given in advance?*, where we adopt the following definition: 'An object is *given in advance* iff the criteria of identity for the object which the language game is about are *not* completely stated or presented by the language game itself; and it is *not given in advance* iff the criteria of identity for the object are completely stated or presented in the language game – [so] that this identity is given by the language game alone and by nothing else.' (Mühlhölzer, 2012, p. 114).

[7] See also Hardy (1940), pp. 63–64, or more recently: 'The assumption that an arithmetical statement is true is not an assumption about what can be proved in any formal system, or about what can be "seen to be true," and nor is it an assumption presupposing any dubious metaphysics. Rather, the assumption that Goldbach's conjecture is true is exactly equivalent to the assumption that every even number greater than 2 is the sum of two primes. Similarly, the assumption that the twin prime conjecture is true means no more and no less than the assumption that there are infinitely many primes p such that $p+2$ is also a prime, and so on. In other words "the twin prime conjecture is true" is simply another way of saying exactly what the twin prime conjecture says. It is a mathematical statement, not a statement about what can be known or proved, or about any relation between language and a mathematical reality.' (Franzén, 2005, p. 30)

independent of our knowledge of them. In some sense, mathematical truth is part of objective reality. "Any number is the sum of 4 squares"; "any number is the sum of 3 squares"; "any even number is the sum of 2 primes". These are not convenient working hypotheses, or half-truths about the Absolute, or collections of marks on paper, or classes of noises summarising reactions of laryngeal glands. They are, in one sense or another, however elusive and sophisticated that sense may be, theorems concerning reality, of which the first is true, the second is false, and the third is either true or false, though which we do not know. They are not creations of our minds; Lagrange discovered the first in 1774; when he discovered it he discovered some- thing; and to that something Lagrange, and the year 1774, are equally indifferent.' (Hardy, 1929, p. 4).

This belief that any proposition about at least natural numbers is either true or false *full stop*, i.e., irrespective of axioms, deductions, etc., has been called *arithmetical determinacy* (Warren, 2020) or *truth-completeness of arithmetic* (Paseau and Pregel, 2023). The latter also call it a '*fundamental commitment of mathematics*'. It is debatable whether this commitment applies just to arithmetic or to further or even all parts of mathematics (as suggested by Hardy's introductory prose but not by his examples); but this difference is irrelevant to us, since we reject it even in its–most basic and most convincing– arithmetical form. As we will explain, this rejection and the alternative view we replace it by are a direct consequence of our analysis of the title question, which we initially follow up through Frege and his great adversaries Hilbert and Wittgenstein.

Via this route we arrive at an answer to the effect that although comparing mathematics to a *game* is far too simple, it may be favourably compared with a certain combination of *language games* whose rules are inferential and whose interlocking is well described by a *rhododendron*, having both multiple roots and branches.[8] At its roots one finds foundational theories or overall formal and logical frameworks for mathematics like ZFC set theory, *as well as its serious competitors*.[9] Each of these roots branches out into one or even various forms of metamathematics, as well as into (typically interwoven) individual areas of mathematics (like number theory or group theory), which in turn brach out into increasingly specialized areas thereof (like algebraic number theory or Lie groups). Within all of these areas (as well as within the foundational theories at the bottom) one has a language game of pure mathematics with its formalized proofs; in many of them, one has a second language game of applied mathematics (including mathematical physics). Our picture is pluralist and provides a coat rack onto which various (originally partly hostile) philosophies of mathematics may be attached and may even peacefully support each other, e.g.:[10]

- *Formalism*, and its close relatives *deductivism* and *coventionalism* as the language game governing pure mathematics grounded in proofs and defining the meaning of truth.

- *Intuitionism* comes in twice: (i) as one of the possible language games giving a foundation of pure mathematics; and (ii): even if the latter is classical, as a possible logic of the metamathematics used to analyze the framework (and similarly for *finitism*, as Hilbert tried).

- *Inferentialism* also enters twice: first, in pure mathematics as the source of its conventionalism, and second, in applied mathematics through its use in the concept of *surrogative inference* in relating mathematical models to empirical phenomena (see below).

One advantage of our approach is that it describes both the *practice* and the *results* of mathematics. For example, each step in a proof is seen as a *move* in a specific language game, which is a practice:

[8]See §3 for the connection of this idea with Wittgenstein's famous 'motley' of language games.

[9]Their axioms are what Feferman (1999) calls *foundational*. The others are *structural*. See also Schlimm (2013).

[10]The comparison with Structuralism is more subtle and is postponed to Landsman (2025).

the final *score* of the game is the result. For this reason we also incorporate the *philosophy of mathematical practice*, which aligns with our aim (which was also Wittgenstein's) to describe mathematics *as it is*, seeing it as a human practice grounded in history.[11] More generally, rather than contradicting each other all these philosophies in fact complement and reinforce each other.[12]

As an introduction to our proposal, let us summarize our answers to the four questions above.

1. *Ontology*. We follow Wittgenstein in warning against the confusion that mathematical expressions are (primarily) referential (see §2). Numerous problems are created by assuming the ethereal "existence" of mathematical objects (not just in the Platonic sense). The things mathematics *talks* about are similar to chess pieces, whose physical or spiritual embodiment is irrelevant. Yet mathematics *is* not about these pieces themselves in whatever incarnation. It is about the rules they are subject to, and their consequences. As we see it, this makes mathematics intersubjective on the verge of objectivity in the following sense:[13]

   > The constitutive property of mathematical items is not existence, but identity. (...) It is painful to abandon the age-old prejudice that identity must presuppose existence. The permanence of the identity of a mathematical item through space and history, and across civilizations, is an extraordinary phenomenon for which there is no easy explanation, and which is shared by few objects of the world. (Rota, 2000, pp. 93)

   Husserl (1954) explained an interesting aspect of this permanence, which we endorse. Historically, mathematics originated in experience and applications, but subsequently underwent a process of 'idealization', in that for example mathematicians idealize different drawings of a circle into identical circles. After this first step of 'idealization' (which is relevant in the specific context of Euclidean geometry that he discusses but can be replaced by formalizing any piece of would-be mathematics), the 'objectification' or 'permanence' (*Immerfort-Sein*) inherent in the concept of 'identity' takes place within humanity seen as an 'emphatic and linguistic community' (*Einfühlungs- und Sprachgemeinschaft*) or 'communication community' (*Mitteilungsgemeinschaft*). Permanence then arises via written or other communication, followed by 'reactivation' (*Reaktivierung*). This seems a valid description of the sense in which both pure mathematics and (tournament) chess are shared.[14]

2. *Truth*. Theorems lack essential properties of things that have a truth value. What may be true (or a matter of fact) is the claim *that some sentence is a theorem within a specific formal system*: it is not the *content of a theorem* that is true but its "theoremhood" (i.e., that the result follows from the premisses according to the rules; that the game has correctly been played). Although this kind of truth is remote from the Platonist one, it shares the advantage that we should all agree about it: even Brouwer should admit that Hilbert's theorems are correct *by his own standards*, and *vice versa*. We extensively argue this point in §4.

3. *Epistemology*. We *know* about mathematical items because we *invented* them. If mathematics is primarily seen as a human practice, the epistemological question hardly arises. For example, if one believes with Russell that (mathematical) logic reveals the logical structure

[11]For introductions to the philosophy of mathematical practice we refer to e.g. Mancosu (2008), Hamami & Morris (2020). Pérez-Escobar (2022) argues that this philosophy resonates well with the later Wittgenstein.

[12]In his dispute with Brouwer, Hilbert came to see this at least for the first and the second points (Mancosu, 1998).

[13]Rota, a mathematician, here summarizes a key element of Husserl's philosophy of mathematics as he saw it shortly before his death in 1999, after 40 years of study. An important primary source is Husserl (1954). From the vast secondary literature we just refer to Hacking (2009) for the work just cited, and to Hartimo (2021) more generally.

[14]Dawkins's well-known concept of a *meme* also seems to describe the nature of mathematics as we just described.

of reality, the question is how we know this structure. For us, any kind of logic is an inferential language game invented by logicians on the basis of studying actual and practical linguistic and mathematical reasoning, which may subsequently be studied by itself, or may be held against language or mathematics as an object of comparison, perhaps even in a normative way (as in prescribing rules of proof).[15] The epistemology of this is unproblematic.

4. *Applicability.* Though sometimes ignored, this is one of the "hard problems" of philosophy:

   > To an unappreciated degree, the history of Western Philosophy is the history of attempts to understand why mathematics is applicable to Nature, despite apparently good reasons to believe that it should not be. (Steiner, 2005, p. 625)

   For us, the main problem is to match our non-referential account of pure mathematics with its apparent representational role in describing the natural world. Our answer will be presented in historical and philosophical detail in a successor paper (Landsman, 2025).[16]

We elaborate our proposal in three sections. In §2 we summarize Frege's objections to the analogy between mathematics and games, and introduce our main protagonists Hilbert and Wittgenstein through their replies to Frege, followed by a summary of (late) Wittgenstein, including the Brandomian turn to *inferential* rule-following, and a review of Hilbert's views on the foundations of mathematics in so far as these are relevant. §3 presents our picture of mathematics as a "rhododendron of language games", followed by an analysis of truth in §4. We conclude the main body of the paper in §5, followed by an appendix that relates our proposal to four related philosophies of mathematics, viz. formalism, pluralism, conventionalism, and deductivism.

2 From Frege to Hilbert and Wittgenstein

Our proposal that mathematics is like a rhododendron of inferential language games originated in an analysis of Frege's arguments against an analogy between mathematics and chess, which we compared with the pertinent positions of Hilbert and Wittgenstein. Predicated on his idea that mathematics is about "thoughts" which games are allegedly too poor to carry, Frege argued that:

- Mathematics is *meaningful*, since it refers to thoughts. But games are meaningless.
- Thus the rules of mathematics originate in reality, whereas for games they are arbitrary.
- Grounded in reality, there is *truth* in mathematical theorems, which games lack.
- On Frege's conceptualization, logical inference must take us from truth to truth. Without truth (which games allegedly lack) there is no logical deduction and hence no mathematics.

[15] We follow late Wittgenstein here. See for example *Philosophical Investigations*, §§130–131, as well as Bangu (2018), Kuusela (2019), chapter 6, and Peregrin (2019) for expositions. More generally, as we shall see in §2, mathematical theorems are held against practices like someone computing 25×25 or counting apples, but this time as *norms*.

[16] Here is already a sketch: We just saw that logical language games are both extracted from natural language and held against it. Similarly, in Hilbert-style mathematical physics axioms are inspired by (heuristic) theoretical physics so as to define theories of of pure mathematics (again seen as inferential language games), which are subsequently held against the relevant natural phenomena as yardsticks or objects of comparison. But how is this comparison or measurement actually made? This is done via what is called *surrogative inference*, in which inferences made from a mathematical model mirror inferences made about natural (or artificial) phenomena (Suárez, 2024). This obviously squares with our inferential view of pure mathematics, and (perhaps less obviously) also matches our anti-realism about (pure) mathematics with a corresponding empiricist philosophy of science as developed notably by van Fraassen (2008).

- The applicability of mathematics would be incomprehensible if it were merely a game, and also leads to irresolvable ambiguities between the formal and the applied sides.
- Even if mathematics initially were just a game, it also incorporates *the theory of the game* instead of only *being the game*. Similarly, there are theorems *about* chess.[17] This is something an allegedly meaningless game by itself (i.e. mathematics) could not accomplish.

The following exchange arguably contains the essence of the debate between Frege and Thomae:[18]

> Wer eine Arithmetik auf eine formale Zahlenlehre aufbauen will, eine Lehre, die nicht fragt, was sind und was sollen die Zahlen, sondern die fragt, was brauchen wir von den Zahlen in der Arithmetik, dem wird es erwünscht sein, auf ein anderes Beispiel einer rein formalen Schöpfung des menschlichen Verstandes hinweisen zu können. Ein solches Beispiel glaubte ich im Schachspiel gefunden zu haben. Die Schachfiguren sind Zeichen, denen im Spiel kein anderer Inhalt zukommt, als der ihnen durch die Spielregeln zuerteilt wird. Die Redeweise, die Zeichen sind leere, kann zu Mißverständnissen da führen, wo der gute Wille zum Verständnis fehlt. So glaubte ich auch die Zahlen in der Arithmetik, im Rechenspiel als Zeichen ansehen zu dürfen, denen eben im Spiel kein anderer Inhalt zukommt, als der ihnen durch die Rechnungs- oder Spielregeln zugeteilt wird. Das Zeichensystem des Rechenspiels wird aus den Zeichen 0 1 2 3 4 5 6 7 8 9 in bekannter Weise hergestellt. (Thomae, 1906a, pp. 434–435)
>
> Herr Thomae schreibt: "Das Zeichensystem des Rechenspiels wird aus den Zeichen
>
> 0 1 2 3 4 5 6 7 8 9
>
> in bekannter Weise hergestellt." Wenn er einfach sagte, das Rechenspiel habe als Spielgegenstände jene Ziffern, so wären wir zufrieden. Nun scheint er aber sagen zu wollen, daß die Spielgegenstände aus diesen Ziffern hergestellt werden, und zwar in bekannter Weise.
>
> Wie soll uns die Sache bekannt sein, da wir das Rechenspiel doch erst kennen lernen wollen? Herr Thomae macht hier den immer bei ihm wiederkehrenden Fehler, das als bekannt vorauszusetzen, wozu er erst den Grund legen will. (Frege, 1908a, p. 52)

About a decade earlier, Frege had had a more balanced exchange with Hilbert on similar themes. Their correspondence between 1895 and 1903 is a jewel in the history and philosophy of mathematics, and although many readers will be familiar with it we now quote two passages whose theme is the same as in the exchange just cited (although chess is not mentioned explicitly):[19]

> Meine Meinung ist eben die, dass ein Begriff nur durch seine Beziehungen zu anderen Begriffen logisch festgelegt werden kann. Diese Beziehungen, in bestimmten Aussagen formulirt, nenne ich Axiome und komme so dazu, dass die Axiome (ev. mit Hinzunahme der Namengebungen für die Begriffe) die Definitionen der Begriffe sind. Diese Auffassung habe ich mir nicht etwa zur Kurzweil ausgedacht, sondern ich sah mich zu derselben gedrängt durch die Forderung der Strenge beim logischen Schliessen und beim logischen Aufbau einer Theorie. Ich bin zu der Überzeugung gekommen, dass man in der Mathematik und den Naturwissenschaften subtilere Dinge nur so mit Sicherheit behandeln kann, anderenfalls sich bloss im Kreise dreht. (Hilbert to Frege, 22 September 1900)

[17] For example, a Bishop cannot move to a square of a different color: this is not a rule but a consequence of the rules (albeit a trivial one). A deeper example is the fact that chess games must end after finitely many moves.

[18] This debate (which started in a friendly way but eventually degenerated into an acrimonious personal polemic) consists of Frege (1903), §§86–103, Thomae (1906ab, 1908), and Frege (1906, 1908ab), so that our main text only gives a very small (literally quoted) excerpt. See also Frege (1899) in response to Schubert, written in a similar style.

[19] The letters may be found in the original German in Gabriel, Kambartel, and Thiel (1980), with English translations in Gabriel *et al.* (1980). See also Blanchette (2018) and Rohr (2023).

Here, following his famous *Grundlagen der Geometrie* from 1899, Hilbert replaced the traditional understanding of mathematical objects (according to which they are always defined explicitly prior to appearing in axioms) by what was later called *implicit definition*, according to which such objects are defined by the axiom systems in which they occur.[20] Modern mathematics would be unthinkable without this idea, which in particular put an end to the struggles by Cantor, Frege, and others to define sets more explicitly–a struggle that also failed for the concept of number, even for the number one, as Frege fatefully found out. In our view, this should also have put an end to arithmetical determinacy (cf. the Introduction), but it hasn't; we take this up in §4.

Yet Hilbert was no philosopher;[21] in that direction we turn to Wittgenstein.[22] In this section we merely discuss Wittgenstein's most acute comments on Frege and Hardy:

> Consider [Hardy (1929)] and his remark that "to mathematical propositions there corresponds—in some sense, however sophisticated–a reality". (...) We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses that are more and more remote. For instance, if I say the word 'picture', you would think first and foremost of something drawn and painted and, say, hung up on the wall. You would not think of Mercator's projection of the globe; still less of the sense in which a man's handwriting is a picture of his character. A word has one or more nuclei of uses which come into every body's mind first; so that if one says so-and-so is also a picture—a map or *Darstellung* in mathematics—in this lies a comparison, as it were, "Look at this as a continuation of that." So if you forget where the expression "a reality corresponds to" is really at home— What is "reality"? We think of "reality" of something we can *point* to. It is *this, that*.' (Wittgenstein, *Lectures on the Foundations of Mathematics*, pp. 239–240).

This is one of the many expressions of Wittgenstein's non-referential approach to the philosophy of mathematics,[23] which he extended to the philosophy of language. In particular, Wittgenstein's

[20]See Peckhaus (1996), Pollard (2010), Schlimmm (2011, 2013), Giovannini & Schiemer (2021), Biagioli (2024), and Sereni (2024) for various aspects of the history of implicit definitions. Briefly, there was a French line of development starting with Gergonne, a German one in which Pasch and to some extent Schröder were key predecessors of Hilbert, and an Italian line involving Burali-Forti, Peano, and Enriques (who introduced the term 'implicit definition' in the axiomatic context, which is different from Gergonne's who also used this term). Giovannini & Schiemer (2021) call implicit definitions *structural*, since the words 'implicit' and 'explicit' are adjectives for certain technical definitions in logic (which Beth's definability theorem identifies). In fact, even an implicit definition à la Hilbert and Enriques may be seen as an explicit definition of an n-place relation, where n is the number of symbols implicitly defined by the axioms (e.g. Blanchette, 2018, §2.1). Poincaré's conventionalism is also based on implicit definitions (which he called 'definitions in disguise' given by axioms), and despite some differences his debate with Russell mirrored the one between Hilbert and Frege. See e.g. Ben-Menahem (2006).

[21]This is not to say that Hilbert was a novice to philosophy; he had clearly read Kant and knew for example Husserl in person. Hilbert's student Weyl went well beyond this, but as explained by Toader (2011), he never overcame the tension between: (i) the pull of Hilbert's formalism, which Weyl saw as necessary for both objectivity (in the sense of mind-independence) and free concept formation by 'symbolic construction' of the kind needed for theoretical physics, yet at the cost of intelligibility; (ii) Husserl's phenomenology, which required contentual reasoning and concept formation by abstraction from immediate experience; and (iii) Brouwer's intuitionism, which emphasized individual mathematical understanding at the cost of objectivity and formalism. See also Da Silva (2017).

[22]For comparisons of Hilbert and Wittgenstein see e.g. Muller (2004), Mühlhölzer (2006, 2008, 2010, 2012), and Friederich (2011, 2014). The closest links are between Hilbert's views of mathematics around 1900 as reviewed above and those of "middle" Wittgenstein (i.e., 1929–1936), where they both opposed Frege such that Hilbert's mathematics matched Wittgenstein's philosophy. In his later period Wittgenstein became quite critical of Hilbert's metamathematics.

[23]The philosophy of mathematics occupied Wittgenstein throughout his career but it was never completed or even prepared for publication by Wittgenstein himself. The primary sources are the posthumous *Bemerkungen über die Grundlagen der Mathematik* (BGM), written 1937–1944 (Wittgenstein, 1984c) and the *Lectures on the Foundations of Mathematics, Cambridge 1939* (LFM; Diamond, 1975). Recent secondary literature includes e.g. Mühlhölzer (2006, 2008, 2010, 2012), Schroeder (2020), Floyd (2021), Scheppers (2023), and Bangu (2025).

reflections on the analogy between mathematics and chess during his middle period were pivotal in arriving at this late philosophy, as is especially clear from the following comment on Frege:[24]

> Frege ridiculed the formalist conception of mathematics by saying that the formalists confused the unimportant thing, the sign, with the important, the meaning. Surely, one wishes to say, mathematics does not treat of dashes on a bit of paper. Frege's idea could be expressed thus: the propositions of mathematics, if they were just complexes of dashes, would be dead and utterly uninteresting, whereas they obviously have a kind of life. And the same, of course, could be said of any proposition: Without a sense, or without the thought, a proposition would be an utterly dead and trivial thing. And further it seems clear that no adding of inorganic signs can make the proposition live. And the conclusion which one draws from this is that what must be added to the dead signs in order to make a live proposition is something immaterial, with properties different from all mere signs.
>
> But if we had to name anything which is the life of the sign, we should have to say that it was its *use*. (Wittgenstein, *Blue Book*, p. 4).

In other words, Frege (allegedly) just saw two possibilities: either symbols refer to something in reality, in which case the game is meaningful (which, in his view, doesn't apply to chess since it lacks "thoughts", whose absence supposedly blocks any analogy with mathematics), or they don't (which for Frege applies to chess but not to mathematics), in which case the game is meaningless. Wittgenstein's point, then, is that Frege overlooked the possibility that even *a priori* meaningless symbols might "come alive" by their use, as governed by the rules they are subject to.[25] In sum:

- For Frege, the use of symbols *follows from their meaning*, given by their external referents;
- For Wittgenstein, the use of symbols, as determined by certain rules, *is their meaning*.

Wittgenstein also echoed our last quote from Hilbert to Frege, now in explicit reference to chess:

> Es ist übrigens sehr wichtig, daß ich den Holzklötzchen auch nicht ansehen kann, ob sie Bauer, Läufer, Turm, etc. sind. Ich kann nicht sagen: Das ist ein Bauer und für diese Figur gelten die und die Spielregeln. Sondern die Spielregeln bestimmen erst diese Figur:
>
> Der Bauer ist die Summe der Regeln, nach welchen er bewegt wird (auch das Feld ist eine Figur), so wie in der Sprache die Regeln der Syntax das Logische im Wort bestimmen. (*Wittgenstein und der Wiener Kreis*, p. 134).

During the 1930s Wittgenstein moved from what has been called a *calculus conception* of mathematics, partly inspired by the analogy with chess, to a *language-game conception*.[26] This accompanied (and probably even induced) a similar move in his philosophy of language,[27] which in fact is a more important source for us than his (unfinished) philosophy of mathematics. Very briefly and with hindsight, in his *Philosophical Investigations* he replaced *referential* theories of meaning

[24] Secondary literature may be traced back from Lawrence (2023). Kienzler (1997) remains irreplaceable.

[25] As noted by Kienzler (1997), §4a, Wittgenstein himself overlooks or ignores the fact that Frege (1903) *does* note this 'other possibility', for in footnote 1 on page 83 (which is part of §71) he says: 'Es besteht freilich auch eine Meinung, nach der die Zahlen weder Zeichen sind, die etwas bedeuten, noch auch unsinnliche Bedeutungen solcher Zeichen, sondern Figuren, die nach gewissen Regeln gehandhabt werden, etwa wie Schachfiguren. Danach sind die Zahlen weder Hülfsmittel der Forschung noch Gegenstände der Betrachtung, sondern Gegenstände der Handhabung. Das werd später zu prüfen sein.' Indeed, in §95 Frege (1903) complains that the rules of chess do not endow the chess pieces with any *content* that would be the consequence of these rules, 'like the name "Sirius" designates a certain fixed star.' This suggests a stubborn inability or refusal to see that the rules themselves comprise the meaning of chess (even though the pieces are meaningless); which was Wittgenstein's point.

[26] See Gerrard (1991), p. 127.

[27] See Kienzler (1997) and Kuusela (2019) for his move in the philosophy of both mathematics and language.

by what we reconstruct as *inferential* ones, via the intermediate device of *language games*. Although Wittgenstein himself refrains from giving a definition of those–and would surely regard any such definition as misguided, since games, languages, and language games are among his main examples of *family resemblances*, which somehow defy definition–we try it nonetheless:[28]

1. A language game is a *practice* where certain words and symbols are *used*.
2. The *meaning* of (most) words and symbols is given by their *use* within such a practice.
3. This *use* is determined by specific *rules* (forming the *grammar* of the language game).[29]
4. These rules are *inferential*: that is, the meaning of sentences lies in their inferential role.

Though it seems compatible with (late) Wittgenstein, the last point was made more explicitly by Brandom (1995, 2001) concerning language and social practices as a whole.[30] Whatever the value of inferentialism in (natural) language, it does seem appropriate to logic (where the original inspiration came from),[31] and, though less easily implementable, to mathematics—though surely, considerable work remains to be done in developing a full and satisfactory inferentialist account of pure mathematics.[32] For the moment we take this possibility for granted and proceed. The question also arises *which* language games in the above sense correspond to some form of mathematics; we let this question be answered by mathematical practice and ideas of family resemblance. The answer has to be fluid, if only because the rules of mathematics–and hence what came to be accepted as mathematics–have regularly been subject to change; and despite a century of apparent stability they will surely change again (see also the end of §4 below).

Furthermore, the key point found in both (late) Wittgenstein and Brandom is that language–or mathematics–is primarily a (rule-governed) *practice*; if one looks for a "foundation" of language–or of mathematics–then this practice (rather than some logical formalism) *is* its foundation.[33] As already mentioned, such an attitude towards mathematics seems to make it difficult to explain how mathematics can be such a powerful tool for describing the physical world. Though we defer a detailed analysis to a successor paper (Landsman, 2025), as a first step towards understanding applications of at least basic arithmetic we again turn to Wittgenstein. Among the coherent fragments of his philosophy of mathematics is the idea that mathematics does not provide *representations* of "reality" (at least not primarily), but *yardsticks* to *measure* reality:

[28]Wittgenstein introduced language games as a tool of his analysis of language in his middle period (notably in the *Blue and Brown Books* from 1933–1935), generalizing this concept in the *Philosophische Untersuchungen*. The closest Wittgenstein himself comes to at least a characterization of language games is his list of examples in §23 of the PU.

[29]See especially §§185–242 of the PI. For a brief discussion we recommend Mühlhölzer, (2010), §I.5. Kuusela (2019), Chapter 6, holds that language games *need* not be based on rules; but those in mathematics are.

[30]This is the subject of an immense literature. Brandom (2007), Peregrin (2014) and Beran, Kolman, and Koreň (2018) are good points to start. See also Haaparanta (2019) and Wischin (2019) comparing Brandom and Wittgenstein.

[31]The original sources of Brandom's inferentialism was Gentzen's proof system of *Natural Deduction* in logic (see e.g. Von Plato, 2013), where each logical symbol has an introduction rule and an elimination rule. These are seen as rules of inference for its use, from which its 'usual' meaning is supposed to follow. This was in fact also suggested by Wittgenstein, for example in BGM VII.30: 'Das logische Schließen ist ein Teil eines Sprachspiels. (. . .) Man kann es so auffassen – will ich sagen –, daß die Schlußregeln den Zeichen ihre Bedeutung geben, weil sie Regeln der Verwendung der Zeichen sind.' Garson (2013), Peregrin (2019), and Warren (2020) are inferentialist accounts of logic.

[32]Some steps were taken by Warren (2020) in support of his Conventionalism, but already our account of truth in §4 differs from his, and also the role of (notably implicit) definitions as well as the origin of axioms, both à la Hilbert, need to be clarified. See our appendix below for further details and its relationship with Hilbert's formalism.

[33]For Wittgenstein's philosophy of mathematics this view is developed in detail in Mühlhölzer (2010).

> Ich will doch sagen: Die Mathematik ist als solche immer Maß und nicht Gemessenes.[34]
> (Wittgenstein, BGM §III.75h)

Conversely, although mathematical results typically *originate* in experience, they are not empirical themselves, but should rather be seen as 'empirical propositions hardened into a rule':

> Es ist, als hätten wir den Erfahrungssatz zur Regel verhärtet. Und wir haben nun nicht eine Hypothese, die durch die Erfahrung geprüft wird, sondern ein Paradigma, womit die Erfahrung verglichen und beurteilt wird. Also eine neue Art von Urteil.
>
> Ein Utrteil nämlich ist: 'Er hat 25×25 gerechnet, war dabei aufmerksam und gewissenhaft and hat 615 erhalten'; und ein anderes 'Er hat 25×25 gerechnet und statt 626 615 herausgebracht'. Aber kommen beide Urteile nicht zu demselben hinaus?
>
> Der arithmetische Satz ist nicht der Erfahrungssatz: 'Wenn ich *das* tue, so erhalte ich *das*' – wo das Kriterium dafür, daß ich *das* tue, nicht sein darf was dabei herauskommt.[35]
> (Wittgenstein, BGM, §VI.22bcd)

Once the result has become a rule, it has become a piece of mathematics that as such is no longer subject to checks by experience: it is 'put into the archives'.[36] This resolves the tension between mathematics as being timeless, non-spatial, acausal, etc., and yet applicable to our causal world in space and time.[37] This tension is due to a confusion between two different language games, namely pure mathematics, within which theorems are proved or looked up in some reliable book, and applied mathematics, in which theorems are compared with reality. As Wittgenstein warned:

> seltsam erscheint der Satz nur, wenn man sich zu ihm ein anderes Sprachspiel vorstellt als das, worin wir ihn tatsächlich verwenden. (Wittgenstein, PU, §195).

On the other hand, for all his earlier sympathy for the analogy between mathematics and chess and his emphasis on rules, later Wittgenstein saw mathematics as applied *by definition*:[38]

> Ich will sagen: Es ist der Mathematik wesentlich, daß ihre Zeichen auch im *Zivil* gebraucht werden. Es ist der Gebrauch außerhalb der Mathematik, also die Bedeutung der Zeichen, was das Zeichenspiel zur Mathematik macht. (Wittgenstein, BGM, §V.2)

This view is problematic in modern mathematics, which is grounded in the autonomous development of mathematics from the 19th century onwards (see footnote 2). Likewise, his comments on the transition from the empirical origins of mathematics to a rule-governed activity reviewed above have an extremely limited scope in modern mathematics and mathematical physics; and even within his elementary context Wittgenstein had no acceptable theory of applied mathematics.

The relationship between pure and applied mathematics, which is a central question for us, was a major theme for Hilbert, who (unlike Wittgenstein) was not just familiar with most or all of the mathematics and mathematical physics of his time; he had created or inspired much of it.

Was Hilbert a "formalist"? He emphasized the importance of axiomatization throughout his career; in the context of what is now called "Hilbert's program" this even came to include the rules of proof. As such, on a par with Euclid he was a leading contributor to the idea that mathematics

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[34] 'What I want to say is: mathematics as such is always measure, not the thing measured.' See also Kuusela (2019), §4.4, for similar views on the philosophy of language during Wittgenstein's middle period (notably in the *Blue Book*).

[35] See Schroeder (2020), Chapter 7, and Bangu (2025), Chapter 3, for detailed analysis of this passage.

[36] See LFM, Lecture IX, p. 107, further discussed in §4 below.

[37] The timeless of mathematics seems denied by Brouwer's intuitionism, and perhaps some forms of constructivism.

[38] On this topic see also Gerrard (1991), Mühlhölzer (2010), and Dawson (2014).

is governed by (inferential) rules.[39] But it was one of his deepest conceptual insights that *both the rigour and the applicability of mathematics originate in axiomatization:*[40]

> Der Mathematik kommt hierbei eine zweifache Aufgabe zu: Einerseits gilt es, die Systeme von Relationen zu entwickeln und auf ihre logischen Konsequenzen zu untersuchen, wie dies ja in den rein mathematischen Disziplinen geschieht. Dies ist die *progressive Aufgabe* der Mathematik. Andererseits kommt es darauf an, den an Hand der Erfahrung gebildeten Theorien ein festeres Gefüge und eine möglichst einfache Grundlage zu geben.
>
> Hierzu ist es nötig, die Voraussetzungen deutlich herauszuarbeiten, und überall genau zu unterscheiden, was Annahme und was logische Folgerung ist. Dadurch gewinnt man insbesondere auch Klarheit über alle unbewußt gemachten Voraussetzungen, und man erkennt die Tragweite der verschiedenen Annahmen, so daß man übersehen kann, was für Modifikationen sich ergeben, falls eine oder die andere von diesen Annahmen aufgehoben werden muß. Dies ist die *regressieve Aufgabe* der Mathematik. (Hilbert, 1919/1992, pp. 17–18).

By axiomatization, Hilbert meant the identification of certain sentences (becoming axioms) that form the foundation of a specific field in the sense that its theoretical structure (*Fachwerk*) can be (re)constructed from the axioms via logical principles. The epistemological status of the axioms differs between various fields of mathematics. For example, Hilbert considered geometry initially a natural science that emerged from the observation of nature (i.e. experience), which then turned into a mathematical science through axiomatization (Corry, 2004, p. 90). This does not mean that he treated the axioms of geometry as definitive, let alone as "true" (as Euclid *cum suis* had done).[41] Especially in physics Hilbert often stressed the tentative and malleable nature of axiom systems:

> Wie man aus dem bisher Gesagten ersieht, wird in den physikalischen Theorien die Beseitigung sich einstellender Widersprüche stets durch veränderte Wahl der Axiome erfolgen müssen und die Schwierigkeit besteht darin, die Auswahl so zu treffen, daß alle beobachteten physikalischen Gesetze logische Folgen der ausgewällten Axiome sind.(Hilbert, 1918, p. 411)

For Hilbert, the axiomatization of physical theories is therefore never a static process: it moves on as physics itself moves on (Corry, 2004; Majer, 2014). Axiomatization may lead to the exposure of contradictions via a purely logical analysis, whose removal is then an important step forward. Indeed, as Majer powerfully summarized Hilbert's view on the axiomatization of physics:

> physical theories live, as it were, on the border of inconsistency (Majer, 2014, p. 72)

As Majer explains Hilbert, this is a consequence of an important difference between mathematics and physics in so far as axiomatization is concerned: the former usually considers single disciplines in what he calls 'maximal conceptual purity', whereas the latter often combines and intertwines a number of mathematical theories into a single highly complicated physical theory.

Axiomatization, then, contributes in two very different ways to the *rigour* of mathematics:

1. via syntactic proofs from the axioms (whose symbols remains uninterpreted);
2. via the axiomatization of sufficiently mature informal theories of mathematics.[42]

[39] See Mancosu, Zach, and Badesa (2009), and Ewald (2018).

[40] Pasch, whose work foreshadowed Hilbert's in various ways, had similar ideas (Pollard, 2010; Schlimm, 2010).

[41] See Pulte (2005) for a history of the interpretation of mathematical axioms related to physics.

[42] Even in mathematics itself Hilbert acknowledged the appearance of contradictions as a historical phenomenon. But unlike physics, he apparently found contradictions unacceptable in mathematics, give his obsession of proving the consistency of classical mathematics. This marks a major difference with Wittgenstein, whose cheerful acceptance of inconsistent theories, repeated comments on the indeterminateness of decimal expansions (e.g. of π), relaxed attitude towards the possibility of rejecting a correct proof, and whose insisting that proofs change or even define the nature of what was proved, sound out of touch with modern mathematics and hence are inappropriate for our program.

Similarly, axiomatization is also the key to the *applicability* of mathematics, namely:

3. via the axiomatization of sufficiently mature theories of physics, space, quantity, etc.

In fact, it seems neither possible nor necessary to sharply distinguish between the second and third activities: for example, are Euclid's axioms (more precisely: his so-called postulates and common notions–whatever their clarity and worth from a modern point of view) attempts to axiomatize earlier informal geometry, or some physical theory of space? Even the axiomatization of set theory in the early twentieth century brought rigour into both the informal set theories of Riemann, Dedekind, and Cantor, and the genuine efforts by Frege, Russell, and others to understand sets as ingredients of the physical universe or at least the human mind (Ferreirós, 2008).

Thus the key difference is between numbers 1 on the one hand and 2–3 on the other: the first is formal and focuses on proofs, whereas numbers 2 and 3 both take us outside (formal) mathematics.[43] Thus it would reflect the spirit of Hilbert's 'zweifache Aufgabe' (two-fold task) of mathematics to only list two sources of rigour in mathematics and the mathematical sciences:

(i) Defining mathematical theories by *finding* appropriate axioms from heuristic considerations;

(ii) Proving theorems, *given* these axioms (including deduction rules, themselves axiomatized).

3 Mathematics as a rhododendron of language games

Despite his insights into the empirical sources of axioms and his impressive record in mathematical physics, even Hilbert hardly bridged the gap between his formalist views of pure mathematics (even if this was inspired by physics or other applications) and the mathematical description of natural phenomena: he relied on the notion of "*pre-established harmony*", a philosophical or even theological doctrine with roots in the monadology of Leibniz.[44] Wittgenstein, on the other hand, left us both the impressive but far too narrow view that mathematics consists of 'empirical propositions hardened into a rule' (see §2), and his famous description of what mathematics is:

> Die Mathematik ist ein BUNTES *Gemisch* von Beweistechniken. – Und darauf beruht ihre mannigfache Anwendbarkeit und ihre Wichtigkeit.[45] (Wittgenstein, BGM, §III.46a)

This juxtaposition of mathematical proof and applicability seems bizarre–unless one understands what Wittgenstein means by 'Beweistechniken' (proof techniques); the passage goes on as follows:

> Und das kommt doch auf das Gleiche hinaus, wie zu sagen: Wer ein System, wie das R[ussell].sche, besäße und aus diesem durch entsprechende Definitionen Systeme, wie den Differential-kalkül, erzeugte, der erfände ein neues Stück Mathematik.
>
> Nun, man könnte doch einfach sagen: Wenn ein Mensch das Rechnen im Dezimalsystem erfunden hätte – der hätte doch eine mathematische Erfindung gemacht! – Auch wenn ihm Russell's Principia Mathematica bereits vorgelegen wären. (Wittgenstein, BGM, §III.46bc)

In other words, by 'Beweistechniken' Wittgenstein means systems of definitions that, together with the logical deduction rules in *Principia Mathematica* form the basis of new pieces of mathematics and hence (potentially) of new applications. If Wittgenstein hadn't disliked set theory

[43] We follow Tait (1986) in seeing models in set theory as internal to mathematics, and hence the distinction between syntax and their interpretation in set theory is irrelevant for our theme.

[44] See Pyenson (1982), Kragh (2015), and Corry (2004).

[45] Anscombe famously translated Wittgenstein's 'buntes Gemisch' as 'motley', which is the traditional costume of the court jester or fool. See Mühlhölzer (2005), p. 66, footnote 15, for a critique of this translation.

so much, the opening quote of this section could simply be that mathematics is a motley of its various branches, formalized within set theory. And this richness is indeed the key to its manifold applications and importance. Moreover, we may combine the first quote with a later one:

> Das logische Schließen ist Teil eines Sprachspiels. (Wittgenstein, BGM VII. 30)

The suggestion then arises that mathematics is a motley of language games; although Wittgenstein himself never seems to have claimed this in general, the spirit of the idea is noticeable in both his *Remarks on the Foundations of Mathematics* and the *Philosophical Investigations*, and he does identify a few special cases as such. But the setting in which he does so remains very limited.

For a successful version of this idea, one should incorporate all that mathematics involves:

1. *A long history:* from numerical tables in Mesopotamia almost 4000 years ago to the rigorous concept of a function in the 19th and 20th centuries; from quantitative methods of surveying to Riemannian geometry; from counting to class field theory, *et cetera*. It has thereby led to:

2. A number of different *formal foundations of mathematics*, like ZF or ZFC or BNG set theory, intuitionistic set theory, λ-calculus, topos theory, homotopy type theory, *et cetera*.

3. Within each of these foundational systems: a wide collection of mathematical theories (also called areas, branches, disciplines, or fields),[46] each with its own community, goals, and standards of proof. One may also think of Peano arithmetic or (Hilbert-style) Euclidean geometry. These areas typically also overlap (e.g. Lie groups combine group theory and differential geometry; functional analysis combines linear algebra and topology, etc.). Following Hilbert (see §2) we find it hard to maintain the traditional distinction between "pure" and "applied" mathematics (although many mathematics departments do!).

4. Associated *notions of proof* ranging from the informal reasoning of ancient Babylonian and Chinese mathematicians to the pseudo-axiomatic setting of Euclid (which lacked explicit rules of deduction) to the advanced logical apparatus of Frege, Russell, Hilbert, and Gödel. But even the logic differs not only between the formal foundational systems just mentioned (and others), but also includes considerable diversity in what is being tolerated within each of them, from informal rigour to bending the rules.[47] See also Wittgenstein's quote above.

5. The meta-theory of the axiomatized theories (i.e. Frege's "theory of the game"), including *both* formal aspects like proof theory *and* informal aspects like "the strategy of the game".[48]

6. Applications of individual branches of mathematics to physics and other disciplines.

Expanding the notion of a game, we answer our title question '*Is mathematics like a game?*' by:

Mathematics is a rhododendron of language games of a very specific (formalized) kind.

[46] See the *Mathematics Subject Classification* (MSC) at `https://mathscinet.ams.org/mathscinet/msc/msc2020.html`, or the 'Branches of Mathematics' listed in Gowers (2008).

[47] See Mühlhölzer (2006) and Floyd (2023) for Wittgenstein's notion of 'surveyability' of a proof, which includes the criterion that its reproduction must be 'an easy task'. This may be held against the proof of $1+1=2$ in *Principia Mathematica* *54.43, which including all preparation takes hundreds of pages. Fully written Hilbert-style formal proofs of more complicated theorems would share this fate–though Dutilh Novaes (2011) identifies eight different ways in which formality and rules can be interpreted. An informal proof is more likely to be understandable but is not rigorous, whereas a formal proof will hardly be understandable if only because of its length, which also increases the probability of error (Avigad, 2021). In some cases informal proofs are pointless, as in the four-colour theorem; we suggest that computer-assisted proofs and computer-verified proofs have their own language game (as defined below).

[48] Developing the formals aspect of this, i.e., metamathematics, was of course Hilbert's achievement.

Here our word ‘rhododendron’ replaces Wittgenstein’s ‘motley’ in order to emphasize that the structure of mathematics we propose has both multiple roots and various branches above these. The roots correspond to the various possible formal foundations, as just listed.[49] From each such root, such as ZFC set theory, further pieces of formalized mathematics branch out, again as listed.

Most of mathematics takes the form of three language games played on *the same* theories:

1. The cleanest mathematical language game is the development of theorems and proofs—where different proof systems may be used, defining different language games (cf. §4).

2. The second one is Hilbert-style metamathematics, yet with one version for each root.[50]

3. The third is applied mathematics in the specific form summarized in the Introduction.

The first is “mathematicians’s mathematics”. The second is played by logicians and philosophers. The third, which practiced by applied mathematicians and mathematical physicists, is obviously restricted to ‘applicable” branches of mathematics (which of course expand in the course of time).

Hilbert’s formalist emphasis on the meaninglessness of mathematical symbols applies to the first, but only *in so far as proofs and other formal aspects of axiom systems are concerned* (such as consistency and completeness). The analogy between mathematics and chess applies here. This analogy is not uniquely definable, but an attractive version is the one proposed by Weyl (1926):[51]

- The *axioms* of some theory are analogous to the starting position of a game of chess;

- The *deduction rules* (à la Natural Deduction) are analogous to the possible moves;[52]

- A *sentence* (as defined in logic) is analogous to *some* position on a chess board;

- A *theorem* is like a *legal* position in a correctly played chess game;

- A *proof* is like a game leading to that position, played according to the rules;

- A *definition* resembles the idea that chess pieces are defined by the rules of chess.

Given the formal notion of proof developed by by Frege, Russell, and Hilbert, in which (unlike in Euclid–let alone 17th and 18th century mathematics) not only the axioms but also the rules of deduction are formalized and stated, the rules of this language game are clearly inferential.

[49] Wittgenstein would already regard this starting point as misguided! Cf. Mühlhölzer (2010) and Scheppers (2023).

[50] If (against our advice) mathematics is seen as a game, then metamathematics is Frege’s “theory of the game”. Whatever one’s philosophy of mathematics, Frege was right that the “theory of the game” cannot be about meaningless symbols since it is about mathematical proofs; and this interprets the symbols. Nonetheless, metamathematics is determined by inferential rules (like mathematics itself) and the corresponding “meaning is use” semantics returns this interpretation. Hence the metamathematical language games share this semantics with all the other mathematical language games and in that sense they are all on the same par. A similar comment applies to the theory of any game with a mathematical structure (which includes the metamathematics of some given root of the rhododendron).

[51] The last point, which is the idea of implicit definition, was not mentioned by Weyl (1926) and should be attributed to Hilbert and others; see footnote 20. What is admittedly missing in the analogy between mathematics and chess is a translation of the goal of *winning* in chess: there seems to be no analogue of checkmate in mathematics (although there is an emotional analogue of resigning, i.e., “giving up”, after repeated failure to prove some theorem). Indeed, in the latter the goal is to establish the counterpart not of a winning position but of an arbitrary legal position (i.e. a theorem). Perhaps the shared aspect of beauty in both games of chess and proofs somewhat compensates for this discrepancy.

[52] What we have in mind here is that the axioms are supposed to describe some specific mathematical theory (such as set theory, or arithmetic, or Euclidean geometry) whereas all deduction rules are logical in character and, perhaps with a few exceptions, are universal for all fields of mathematics (like Euclid’s common notions). See e.g. von Plato (2017). In contrast, in a Hilbert-style calculus (Hilbert & Ackermann, 1928) *modus ponens* is the only deduction rule whilst the other deduction rules in Natural Deduction are seen as axioms. This calculus does not fit our metaphor.

The second language game is one level above the previous one but it also squarely lies on the formal and inferential side: although the object of investigation is a mathematical theory, its symbols remain uninterpreted. It is played on the same theories as the previous one (for example, arithmetic, as in "Hilbert's program"), but it need not follow the same logic as the original game.[53]

The third one, which is crucial in understanding the relationship between mathematics and the physical world, is based on rules that are inferential in a much less obvious way, for which we again refer to the Introduction for a summary and to Landsman (2025) for a full development.

And of course there are numerous other language games that are especially relevant to mathematical practice; for example, in either trying to find proofs or in creating new mathematics even the most stubborn formalist will look for interpretations and perhaps visualizations in both applied and non-applied (or not-yet-applied) fields, for links with different fields of mathematics, etc. Similarly, learning mathematics is a language game by itself (much as learning a language is, as analyzed in the early parts of the *Philosophical Investigations*). And so on and so forth.

4 Truth

> The truth predicate then preserves his contact with the world, where his heart is. (Quine, 1986, p. 35)

What does the "pure" or "formalist" language game imply for the concept of *truth* in mathematics? On the one hand this is a difficult question, since we see mathematics as a human practice, for whose rules there were always many different possibilities and choices, even when these rules were inspired by empirical phenomena. On the other hand, mathematical practice suggests that "truth" is merely a *façon de parler*, in that for most "working mathematicians" "p is true" simply means that p is a theorem. Anything beyond this meets the scathing comment of Bourbaki:

> Mathematicians have always been sure that they prove "truths" or "true propositions"; such a conviction can obviously only be sentimental or metaphysical. (Bourbaki, 1994, p. 11)

We agree, but some place for "truth" remains in mathematics; it just needs to be relocated.

Let us return to chess for inspiration. It seems meaningless to say that a position p in chess is "true". But it does make sense to claim that p is *legal*, in that it arose from a game played according to the rules R. This claim, call it $R \vdash p$, rather than p itself, could be said to be true or false, and this can be established by a proof in the form of an actual (legal) chess game leading to p, cf. §3. In normal games p even arises in this way; in so-called retrogade chess problems one has to reconstruct p. Similarly, in our non-referential ideology mathematical theorems cannot be true either, since there is no objective state of affairs they could describe correctly.[54] Moreover, the kind of truth conventionalists aspire to is, in our view, covered much better by our proposal below than by declaring theorems of propositions themselves to be true. Like in chess, truth in mathematics cannot lie in sentences φ (such as closed formulae in first-order logic), but only in claims $T \vdash \varphi$ stating *that φ is a theorem within an ambient theory T* (which is supposed to include rules of inference). And this is the case (by definition) iff there exists a proof of φ according to the rules of T. Thus the only thing we can say about mathematical truth in our framework is this:

Mathematical truth resides not in *theorems but in claims* that *some sentence is a theorem.*

[53]Here one need not think of the full scope of Hilbert's program; Gödel's completeness theorem is already metamathematical, as is its special case for propositional logic (Zach, 1999).

[54]We repeat the point already made in footnote 43: Tarski's concept of truth as defined in model theory is internal to pure mathematics and has little or nothing to do with the notion of truth sought by the Platonists or naturalists. It will play a minor role in the discussion of Gödel's theorems below.

This makes a proof of φ in T the truth-maker of the truth-bearer $T \vdash \varphi$. Our only compromise towards realism is our belief that such truth (or falsehood) is a matter of *fact*, whether or not it is *known*.[55] But this is not the truth of platonism (or of naturalistic views of mathematics), which concerns φ rather than $T \vdash \varphi$, backed by a correspondence theory of truth. We also reject other approaches which argue that a sentence φ *itself* (as opposed to $T \vdash \varphi$) is true iff φ has a proof.[56]

First, unless one believes that there is a single "true" foundational system for mathematics (such as ZFC set theory with additional cardinality axioms, as proposed by Gödel), such proposals endorse a *coherence theory of truth* (Young, 2018), in which each such system would come with its own set of truths. As explained in §3, we reject this (also cf. Landsman, 2025). On our proposal, although people may differ about the virtues of different foundational systems, given unambiguous concepts of inference and proof they cannot rationally differ about the theorems in each of these.

Second,[57] according to Gödel's first incompleteness theorem, for any T (satisfying the usual assumptions) there are sentences φ such that neither φ nor $\neg\varphi$ is provable in T. Yet in classical logic $\varphi \vee \neg\varphi$ is *provable* for every φ. If this implies that $\varphi \vee \neg\varphi$ is *true*, then for undecidable φ this would be the case without either φ or $\neg\varphi$ being true, which is awkward.[58] But since the claim '$T \vdash (\varphi \vee \neg\varphi)$' is clearly different from '$(T \vdash \varphi)$ or $T \vdash (\neg\varphi)$', there is no argument to conclude from the truth of the former that the latter is true, and so even on TND and the everyday understanding of 'or' we are not forced to (wrongly) conclude that either $T \vdash \varphi$ or $T \vdash \neg\varphi$ is true,

Thus the possibility of assigning truth to φ is challenged by Gödel's incompleteness theorems, whereas $T \vdash \varphi$ faces no such problems.[59] If Dummett's (1963/1978) famous (but controversial) argument for the 'vagueness' of the concept natural number is correct,[60] then the stance of arithmetical determinacy mentioned in the Introduction (and in its wake the more general 'fundamental commitment of mathematics') is also *weakened* by Gödel's incompleteness theorems, although these are usually seen as a *threat* to philosophies like ours.[61] The following preamble is uncontroversial. Take $T = PA$ (i.e., Peano Arithmetic with first-order logic) to be specific, and let G_T be its Gödel sentence, which expresses its own unprovability in T.[62] Then it is well known that G_T is undecidable (i.e., neither $T \vdash G_T$ nor $T \vdash \neg G_T$) *if and only if* T *is consistent*.[63] Then:

- Arithmetical determinists take the "existence" of the natural numbers $\mathbb{N}$ and their satisfaction of the PA axioms as given, whence PA is consistent. From this, they are entitled to conclude that the interpretation $[[G_T]]_{\mathbb{N}}$ of G_T in $\mathbb{N}$ is true in their absolute sense (cf. §1).

- Without the commitment to arithmetical determinacy, $[[G_T]]_{\mathbb{N}}$ is merely true in the formal

[55] This might be varied by defining $T \vdash \varphi$ to be true if a proof of φ *is* known, as in intuitionistic mathematics.

[56] See e.g. Dieudonné (1971) and Tait (1986). See also Appendix A.1 for truth in formalism and deductivism. The so-called BHK (Brouwer–Heyting–Kolmogorov) interpretation (or semantics) of intuitionistic logic is also often taken to mean that a sentence is true iff it has a proof (Artemov & Fitting, 2021). We reject this, too, but even so one may still support the more modest BHK interpretation of the logical connectives in terms of proofs (van Atten, 2023).

[57] See also Paseau & Pregel (2023), §9, and references therein.

[58] This problem obviously does not arise in intuitionistic logic, which Gödel (1931) actually incorporated.

[59] As brought to our attention by Barteld Kooi, incompleteness does lead us to an asymmetry between truth and falsehood the naive approach (in which φ itself is true or false) does not have. Namely, if $T \vdash \varphi$ is true then it has a *truth*-maker in the form of a proof of φ from T; but if $T \vdash \varphi$ is false, i.e., not true, then there is a *false*-maker for it just in case φ is decidable in T (in which case $T \vdash \neg\varphi$ and hence the false-maker is a proof of $\neg\varphi$ from T). But this seems a lesser evil (for us) than some assumption of mathematical realism.

[60] See e.g. Engler (2025) and references therein. See also Parsons (1990) for arguments similar to Dummett's.

[61] See Weir (2010), §4.III, for formalism; Warren (2020), §11.VII, for conventionalism; and Paseau and Pregel (2023), §9, for deductivism. See also §A.1 below for our relationship to these philosophies or attitudes.

[62] In very naive discussions this sentence is claimed to be true full stop, and since it cannot be proved (in T) this is supposedly an argument against formalism and for platonims and/or the superiority of the brain over any formal system. See Franzén (2005) for a critical discussion of this and many other misunderstandings of Gödel's theorems.

[63] See e.g. Franzén (2005) or Raatikainen (2022).

sense of Tarski for model theory, where $\mathbb{N}$ is a construction within ZFC set theory.[64]

Thus the claim that $[[G_T]]_{\mathbb{N}}$ or G_T is true in an "absolute" sense, i.e., beyond derivability in some axiomatized theory (such as ZFC in the second case), must already *assume* arithmetical determinacy: the incompleteness theorems cannot be used to *derive* it or even argue for it.[65] Indeed:

> The [arithmetical determinist] however, operates with the notion of a model as if it were something that could be given to us independently of any description: as a kind of intuitive conception which we can survey in its entirety in our mind's eye, even though we can find no description which determines it uniquely. This has nothing to do with the concept of a model as that concept is legitimately used in mathematics. There is no way in which we can be 'given' a model save by being given a description of that model. If we cannot be given a complete characterisation of a model for number theory, then there is not any other way in which, in the absence of such a complete description, we could nevertheless somehow gain a complete conception of its structure. (Dummett, 1978, p. 191)

This confirms the circularity in arguing that Gödel's theorems enforce arithmetical determinacy. Our account of this situation is as follows, *assuming* PA is consistent. No sentence in PA = T, including G_T, has anything like a truth value. The claims $PA \vdash G_T$ and $PA \vdash \neg G_T$ are both false in our sense (i.e., not true). The "truth" of $[[G_T]]_{\mathbb{N}}$ is just the truth of $\mathbb{N} \vDash G_T$, seen as a theorem in ZFC or in some weaker system, such as the proof system one obtains by adding the so-called ω-rule to PA.[66] In view of our definition of truth we here effectively replace "truth-talk" (in the usual sense) by "proof-talk", with the crucial feature that by changing the proof system in passing from $PA \vdash G_T$ (which is false) to $\mathbb{N} \vDash G_T$ (which is true) *we switched to a different language game*.[67]

Still assuming consistency of PA, there is a non-standard model $\mathbb{N}'$ of PA in which the interpretation $[[G_T]]_{\mathbb{N}'}$ of G_T is false.[68] This falsehood indicates that the concept of a natural number is not sufficiently captured by PA, and since this argument is independent of the choice of PA arithmetical determinists must agree that 'we have a certain, quite definite, concept, which cannot be fully characterised just by the fact that we make certain assertions about it' and that we 'cannot characterise completely the meaning of "natural number" by specifying which arithmetical statements we are prepared to assert and which forms of inference within arithmetic we are prepared to accept.'[69] *We* conclude that since PA–or some similar system, facing similar problems—is the only intuition we have about natural numbers, one cannot possibly claim that the natural numbers are "defined" or "exist" in some absolute sense.[70] But *arithmetical determinists*, instead of giving up their position, conclude from this that at least in this case *meaning* (namely the absolute concept of natural numbers they have in mind) cannot be given by *use* (according to the axioms and rules of inference). It is this way out that the controversial remainder of Dummett's argument tries to undermine, so that Dummett's higher goal lies in defending a "meaning = use" semantics.

[64] Or in some fragment thereof in which the construction of $\mathbb{N}$ can be carried out and in which the consistency of *PA* can be proved. This only makes sense if ZFC (or the fragment just alluded to) is consistent, which of course is a big if.

[65] As does for example Connes in support of his Platonism (Connes, Lichnerowicz, and Schützenberger, 2001).

[66] This rule, which goes back to Hilbert, states that $\varphi(n)$ for all n defined in PA as $0 = 0$, $1 = S(0)$, $2 = S(S(0))$, etc., implies $\forall_x \varphi(x)$. This rule allows one to prove all true statements $\forall_x \varphi(x)$ provided $PA \vdash \varphi(n)$ for each $n = 0, 1, 2, \ldots$, at the expense of using an infinite number of assumptions. As such it distinguishes the standard model $\mathbb{N}$ of PA from all other (i.e. non-standard) models, in that the ω-rule holds for all $\varphi(x)$ precisely in $\mathbb{N}$. The Gödel sentence G_T is of the form $G_T = \forall_x \varphi(x)$, where each $\varphi(n)$ for $n = 0, 1, 2, \ldots$ is a theorem of PA. The gap between the inability to prove G_T in PA and the ability to prove it in $\mathbb{N}$ is therefore precisely bridged by the ω-rule. See e.g. Warren (2020), §10.VII.

[67] See also Kolman (2014, 2016), which makes a similar point in a different context.

[68] Continuing footnote 66: each sentence $\varphi(n)$ appearing in $G_T = \forall_x \varphi(x)$ is true on $\mathbb{N}'$, yet G_T is false in $\mathbb{N}'$.

[69] Dummett (1978), p. 186, 187.

[70] We recall the sad fact that Frege's life work of defining the natural numbers failed even for the number one.

In any case, one does need to get used to the idea that say $7+5=12$ is not true (or false; it is just not the kind of mathematical statement that has a truth value),[71] whereas the claim

$$\mathrm{PA} \vdash (7+5=12) \tag{4.1}$$

is true, i.e., $7+5=12$ is a theorem of Peano arithmetic, or briefly: $7+5=12$ is true *in PA*.[72]

Don't seven apples add up with five apples to yield twelve apples? They do. But this expresses neither $7+5=12$ nor $\mathrm{PA} \vdash (7+5=12)$: the former is *held against the apples as a yardstick*, justified by the latter. This yardstick even seems to yields perfect results (which is rare and may be restricted to counting and elementary arithmetic), but already Aristotle realized how much is involved in this: one must regard each apple as a unit, which deliberately overlooks that firstly each apple is divisible, and secondly that all apples are different. Even granting these idealizations, what we have is a match between empirical data and some mathematical theorem, viz. $7+5=12$. Following Wittgenstein, the latter is an 'empirical proposition hardened into a rule'. But this rule cannot inherit any kind of truth from the empirical propositions that originally inspired it (such as the counting of objects like apples), since that would confuse the physical world with the role of mathematics as a set of yardsticks invented by humans to understand it:

> I am trying to show in a very general way how the misunderstanding of supposing a mathematical proposition to be like an experiential proposition leads to the misunderstanding of supposing that a mathematical proposition is about scratches on the blackboard.
>
> Take "$20+15=35$". We say this is about numbers. Now is it about the symbols, the scratches? That is absurd. It couldn't be called a statement or proposition about them; if we have to say that it is a so-and-so about them, we could say that it is a rule or convention about them.–One might say, "Could it not be a statement about how people use symbols?" I should reply that that is not in fact how it is used–any more than as a declaration of love. (Wittgenstein, LFM, Lecture XII, p. 112)

The unity of mathematics emphasized by Hilbert provides an additional argument for the lack of truth of $7+5=12$. If this theorem were true, then every theorem in mathematics should be true. This leads to a problem discussed earlier: since different foundational systems may yield contradictory results, just one of these systems could be "true". The history of mathematics suggests this is dubious. Even if only one of them ultimately comes out be correct (e.g. since the others unexpectedly are inconsistent), putting esoteric result in ZFC set theory about inaccessible cardinals on a par with $7+5=12$ as both being "true" sounds equally wrong. The only way to get around these problems seems to be to treat all theorems from all foundational systems on a par; but instead of declaring them all true, the ensuing notion of truth is expressed much better by saying that the claim $T \vdash \varphi$ that φ can be deduced from T is true, rather than φ itself.

We see that even the simplest theorems in arithmetic, involving very small integers, are no threat to our notion of truth. Our case against realism is even stronger if large integers are involved; and still stronger for Euclidean geometry; and stronger again if we use the advanced theories of mathematics physics, about which only very naive physicist would say that their theorems are "true"; see Landsman (2025) for further analysis. But our argument is uniform for all these cases.

[71] An anonymous referee highlighted the radical nature of our concept of truth by mentioning the Sylow theorems for finite groups. Here the temptation to relate a purely mathematical claim to real things like apples seems absent, but this *weakens* the pull to attach any truth label to such theorems. Like all mathematical objects, finite groups and their properties have a 'permanence of identity through space and history' (see Rota quoted in the Introduction), but like numbers this identity is given by specific definitions and other rules, as opposed to things that actually have the said properties. This *increases* the pull in the opposite direction of believing that theorems are statements about rules.

[72] In some crazy theory T where $T \vdash (7+5=10)$, this would still be true on our criterion (as long as the proof in T is correct!). This theory might be an interesting game but it would be a useless yardstick in applied mathematics.

Wittgenstein's (disquotational) concept of truth differs from ours: in §136 of the *Philosophical Investigations* he identifies *'p' is true* with p itself (and *'p' is false* with not-p). But in a marked difference with the 'fundamental commitment of mathematics' he then adds that *this concept of truth belongs to the rules of the language game in question*.[73] Similarly, Wittgenstein wrote:

> Mathematical truth isn't established by their all agreeing that it's true–as if they were witnesses to it. *Because* they all agree in what they do, we lay it down as a rule, and put it in the archives. (LFM, Lecture IX, p. 107)
>
> Es bricht kein Streit darüber aus (etwa zwischen Mathematikern) ob der Regel gemäß vorgegangen wurde oder nicht. Es kommt darüber z.B. nicht zu Tätlichkeiten. (PU §240)

5 Conclusion

> What is the conclusion to which we come? Modern mathematics and physics may seem to move in thin air. But they rest on a quite manifest and familiar foundation, namely the concrete existence of man in his world. (Weyl, 2009, p. 188)

Our aim was to re-examine the question to what extent mathematics may be compared with a game. Trying to answer this question led us essentially to Weyl's conclusion just quoted. To get there, we combined certain insights by Hilbert and Wittgenstein (as "inferentialised" by Brandom). From Hilbert, we took the idea that axiomatization enables both pure and applied mathematics (including mathematical physics). From Wittgenstein, our main lessons is that pure and applied mathematics correspond to different language games, both of which are non-referential. The "applied" one also relies on his remarkable idea of using mathematical theorems or theories as yardsticks.

Mathematics also incorporates various other language games, together forming a structure we like to call a "rhododendron" (rather than Wittgenstein's "motley", which seems too flat). Various mainstream philosophies of mathematics find a peaceful place within this structure, except Platonism: Platonists and other realists will like little of our proposal, since they already reject our (and Wittgenstein's) starting point of ultimately grounding mathematics in human practice. Formalists may have more sympathy for our view, since we not only defend what should be the deductivist concept of truth, but also moved the problem of understanding applied mathematics within a formalist framework a step forward. Logicism, formalism, and intuitionism may no longer exist in their original form, but what is left of them is also welcomed within the rhododendron. We take this peaceful coexistence to be a major advantage of our proposal. We propose this rhododendron of language games as an object of comparison itself: we invite readers to compare the mathematics they have in mind with this picture, to see where it agrees and where it deviates.

Finally, although our paper is by no means meant as an analysis of Wittgenstein's philosophy of mathematics (nor of Hilbert's), we hope to have implicitly answered certain objections to it by firstly doing some cherry-picking (i.e. tacitly removing some of his more extreme and outdated views) and secondly integrating these marbles with modern ideas appropriate to contemporary mathematics (especially ideas of Hilbert's, hopefully without falling into *his* traps either).

A Formalism, deductivism, conventionalism, and pluralism

In tis appendix we briefly compare our approach with certain interpretations (or even philosophies) of mathematics that are related to ours and of which we have tried to incorporate some parts.

[73] Similarly in §6 of Anhang II of BGM.

A.1 Formalism and deductivism

Though often associated with Hilbert,[74] there isn't a canonical notion of "formalism". Here are two caricatures from leading textbooks in the philosophy of mathematics:

> The various philosophies that go by the name of 'formalism' pursue a claim that the *essence* of mathematics is the manipulation of characters. A list of the characters and allowed rules all but exhausts what there is to say about a given branch of mathematics. (Shapiro, 2000, p. 140).
>
> *Formalism* is the view that mathematics has no need for semantic notions, or at least none that cannot be reduced to syntactic ones. (Linnebø, 2017, p. 39)

This is the version attacked by Frege; cf. his correspondence with Hilbert quoted in §2. But Hilbert is a straw man; for him it is only *in the context of proofs and in the analysis of consistency of axiom systems etc.* that mathematics is a deductive enterprise in which symbols have no meaning (outside the rules they are subject to). A broader view of formalism is described by Detlefsen (2005). Our own approach incorporates formalism on the side of pure mathematics, but tries to balance it via Hilbert's emphasis on the informal meaning of symbols inherited from the heuristic theories of either physics or mathematics inspiring most axiomatizations. His implicit definitions, or Wittgenstein's corresponding ideas about the meaning of symboles given by their use ('the life of the sign') also gives meaning (or even 'life') to formalism in a way that many philosophical discussions seem to overlook.[75] We are not "formalists" if this is seen, as it historically has, in opposition to logicism or intuitionism; appropriately formalized (!) at least the latter is a valid mathematical language game on a par with classical mathematics as used by most formalists.

Our concept of truth within the formalist language game expounded in §4 is compatible with a philosophy of mathematics called *deductivism*, famously summarized by Russell (1903) as:

> Pure mathematics is the class of all propositions of the form "*p* implies *q*".

See Paseau and Pregel (2023). Different "deductivists" had different concepts of truth. For example, Russell saw the truth of logic and mathematics in the universe. Around 1900, Hilbert defined mathematical truth as flowing from axioms, which by themselves were deemed "true" if they were consistent (Paseau and Pregel, 2023, §3.1). Curry (1951)'s Chapter II is called 'The problem of mathematical truth' and starts with the statement that 'The central problem in the philosophy of mathematics is the definition of mathematical truth'. It may then come as a slight disappointment that he simply puts truth in the theorems themselves, justified by a verification procedure that comes down to checking their proofs. Lolli (1998) states that only *logical* truth matters, defined as 'truth under any interpretation whatsoever' which in turn he takes to mean 'true under any notion of truth' (p. 118), adding that 'the great success of mathematical logic is to have shown that all of logic is independent of a definition of truth. (p. 119). For Weir (2010), truth comes from the correctness of utterances, which in mathematics is guaranteed by provability. But unlike in our analysis it is still the utterance, i.e., the theorem, which is deemed true. *Et cetera*.

A.2 Pluralism and Conventionalism

Our proposal is meant to give room to various "philosophies" or "foundations" of mathematics that are often seen to be mutually incompatible, such as classical mathematics (based on set theory) and

[74] Indeed, the exposition of formalism by von Neumann (1931) is indeed entirely devoted to Hilbert's program.

[75] In addition, Hilbert's program of giving a finitist consistency proof of classical mathematics re-introduced content even in a purely formal setting, cf. Hilbert (1918). This is taken into account by the second language game in §3.

the formalism often associated with it, intuitionism, constructivism, etc. We see these as different language games. As such, it is clearly "pluralist" in character. But what does this mean?

Friend (2014) ends her monograph on mathematical pluralism with the following manifesto:

> One can be pluralist in different respects, at different levels and one's pluralism can be governed by different logical inclinations or hypotheses. In general, the pluralist aspires to the following virtues: unprejudiced observation of mathematical practice and a desire to encompass and accommodate as wide a variety of practices as is coherently possible. The inverse of these virtues are manifested when we insist on unique, simple, teleologically satisfying answers, beyond what the evidence will support. (...) The pluralist position is meant to give a philosophical theory to support what is already happening in the philosophy of mathematics. Thus the position is 'new' in the sense of not having yet been expressed this way in print, but it is 'old' in the sense of being already implemented and understood, at some level, by some mathematicians and philosophers of mathematics. (Friend, 2014, pp. 241–242)

In a more recent text, Priest (2024) argues for the coexistence of various foundations (or 'arenas') of mathematics (including even substructural logics), draws the analogy with games, and even ends with the same quote of Wittgenstein that appears in the opening of our §3.[76] Finally, a recent paper by Zalta (2024) opens with:

> Mathematical pluralism can take one of three forms: (1) every consistent mathematical theory consists of truths about its own domain of individuals and relations; (2) every mathematical theory, consistent or inconsistent, consists of truths about its own (possibly unintersting) domain of individuals and relations; and (3) the principal philosophies of mathematics are each based upon an insight or truth about the nature of mathematics that can be validated.

Our paper is written in the same spirit, contributing to mathematical pluralism. Priest (2024) is closest to us, but none of these works present a similar the analysis and architecture of mathematics (based on Hilbert and Wittgenstein) or has a comparable approach to applied mathematics.

Coventionalism is a philosophical position that is closely related to mathematical pluralism, as well as to our work. It has its roots in Poincaré, early Wittgenstein, and logical positivism (Ben-Menahem, 2006). Given our analysis in §2, it should be no surprise that the species of conventionalism of interest to us is inferentialism, in which the relevant conventions are (syntactic) inference rules. This is developed for natural language by Brandom and followers (see §2 and references therein), whereas for logic and mathematics Warren (2020) is a major source; see also Garson (2013) for logic.[77] All of this is also indebted to late Wittgenstein, as well as to Hilbert, and we largely align with it. As far is truth is concerned, there seem to be two main directions, of which we follow Ben-Menahem (2006) in thinking of conventions as hypothetical conditions rather than as freely postulated truths, as in Warren (2020). See Landsman (2025) for details.

[76]The first version of our article, still available as `https://arxiv.org/abs/2311.12478v1`, already contained this quote before the publication of Priest (2024), which we came to know only in 2025.

[77]Despite these books (and references therein), much work remains to be done in order to derive an inferentialist account of pure and applied mathematics; this will be taken up in the future.

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