Nuclear Physics B322 (1989) 498–530
North-Holland, Amsterdam

LIMITATIONS TO DIMENSIONAL REDUCTION AT HIGH TEMPERATURE

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Received 15 February 1989

The claim that renormalizable four-dimensional field theories in the infinite-temperature limit undergo a reduction to effective three-dimensional ones is analyzed in perturbation theory. An essential ingredient is the finite-temperature renormalization group in the imaginary-time formalism. A precise criterion for the occurrence of complete dimensional reduction is given. This is satisfied only in exceptional cases, and is violated e.g. by φ^4 and QCD. These theories dimensionally reduce only up to a given order in perturbation theory. Illustrative one-loop calculations are given on the basis of a novel summation technique. The perturbative structure of QED and QCD at high temperature is examined in detail. Ward identities as well as explicit computations are used to explain why QED dimensionally reduces to all orders, but QCD does not. In addition, a potential instability deriving from an anomalous diagram is identified and cured.

1. Introduction

The complicated dynamics of interacting thermal field theories invites the search for a regime where these theories simplify in some sense. Conventional wisdom has it that the infinite-temperature limit provides such a regime, the pertinent simplification being one from a four- to a three-dimensional field theory. Such a dimensional reduction (DR) would set in for static correlation functions at small momenta ($p/T \ll 1$, where T is the temperature).

Dimensional reduction at high temperature was first unambiguously claimed to take place by Appelquist and Pisarski [1], who based their reasoning on the zero-temperature Appelquist–Carazzone (AC) decoupling theorem [2–6]. The basic idea is that the imaginary-time formulation of thermal field theory [7] produces a perturbation theory with free propagators of the form $[\boldsymbol{k}^2+(2\pi nT)^2+m^2]^{-1}$, where $n \in \mathrm{Z}$ eventually has to be summed over. The term $2\pi nT$ acts like a mass, so that in the limit $T\to\infty$ the nonstatic ($n\neq 0$) modes ought to decouple. The zero modes obviously survive, and one is left with a three-dimensional theory whose parameters generically get renormalized by the decoupling process, as in the AC theorem [4].

0550-3213/89/$03.50

The realization of DR at high T has been checked in one-loop QCD by explicit computation [8], and its validity to all orders of perturbation theory is usually taken for granted [9–13]. However, as implicitly remarked already by Gross et al. [14], there is an important caveat: the AC-theorem holds up to terms of order p/T *and* m/T. Now the effective three-dimensional theory typically generates a dynamical mass $m(T) \sim g^n T$, where g is a coupling constant and $n > 0$. This implies that even at zero momentum, DR takes place only up to terms of order g^n. Hence in contrast to the decoupling of heavy particles at zero temperature, the correction terms to DR do not vanish in the infinite mass/temperature limit. The essential difference between the two cases is that at zero T a small mass m is an experimental input parameter, which does not grow with the mass of a decoupling heavy particle, whereas the light mass m at finite T acquires a T-dependent contribution. In a sense this is a thermal analogue of the hierarchy problem [4].

Both the pro- and contra-DR parts of the above reasoning are rather vague and heuristic. The purpose of this paper is to clarify the major aspects of the alleged dimensional reduction at infinite temperature. In particular, we wish to explain the precise meaning of the temperature-dependent "masses", and to state exactly when and to what extent DR indeed occurs. As we shall see, these goals can be achieved by exploiting recent technical advances in thermal field theory [15] and renormalization theory [16].

The present investigation is based on a simple question. An essential ingredient of the AC-theorem is that it holds only if a particular class of renormalization prescriptions is adopted [2, 4], the decoupling being optimal in the BPHZ scheme [5, 6]. So we ask: in which renormalization scheme would the decoupling of the nonzero modes at high T be maximal? As we shall see, the maximal decoupling scheme consists of BPHZ-like subtractions at zero momentum on the Feynman integral *at temperature T*. Thus we are led to a renormalization scheme with T-dependent counterterms [17], and an associated finite-temperature renormalization group equation [15]. The temperature-dependence of the renormalized "mass" $m_R = m_R(T)$, alluded to before, is governed by this equation. Indeed, the thermal mass derives its meaning from the thermal renormalization group, rather than having the usual interpretation of a mass. Dimensional reduction then holds up to corrections of order $(m_R(T)/T)^2$, which may or may not vanish in the infinite-temperature limit.

Of course, there is more to be said, and the plan of this paper is as follows. In sect. 2 we arrange the Matsubara formalism in a form allowing to regard thermal field theory as a vacuum theory with an infinite number of massive fields. We discuss general renormalization conditions, and adapt the finite-temperature renormalization group to the imaginary-time formalism. Sect. 3 presents one-loop sample calculations in a scalar theory. To separate the static and nonstatic modes explicitly, we use a summation technique which combines dimensional and zeta-function regularization, and whose technical details are explained in the appendix. These examples heuristically lead us to a maximal decoupling renormalization scheme, whose associated renormalization group equations are solved.

The core of the paper is sect. 4, in which we give a precise formulation of the Appelquist–Carazzone decoupling theorem, and its connection with DR at high temperature. We derive the exact criterion for DR to occur, and stress the essential difference between the zero- and the finite-T situation as far as decoupling is concerned.

In sects. 5 and 6 we present explicit one-loop computations for QED and QCD, respectively. Being gauge theories, these models exhibit the phenomenon that certain counterterms necessary for DR cannot be obtained by a redefinition of parameters in a bare lagrangian. In this respect, diagrams with four external electric photons c. q. gluons have a particularly interesting structure, and form a potential source of instability. This turns out to be harmless, while nevertheless leading to an interesting connection with anomalies. We will show that QED has special abelian features leading to DR to all orders. In contrast, QCD dimensionally reduces only up to a particular order in perturbation theory, which depends on the process under consideration.

Finally, in the conclusion we discuss some of the consequences of our results, in particular for QCD and for Kaluza–Klein theories.

2. General structure of thermal renormalization

2.1. COMPACTIFICATION AND RENORMALIZATION

We explain the basic setting for a theory of a bare scalar field $\varphi_{\rm B}(x)$ with quartic self-interaction. Its thermal bare Green functions are given by [7]

$$G_{\rm B}(x_1,\dots,x_N)=\int_{\rm B.C.}[d\varphi_{\rm B}]\varphi_{\rm B}(x_1)\dots\varphi_{\rm B}(x_N)\,{\rm e}^{-S_{\rm E}[\varphi_{\rm B}]}, \tag{2.1}$$

where the thermal euclidean action is given by

$$S_{\rm E}[\varphi_{\rm B}]=\int_0^\beta {\rm d}\tau\int {\rm d}^3x\left[\tfrac{1}{2}(\partial_\mu\varphi_{\rm B})^2+\tfrac{1}{2}m_{\rm B}^2\varphi_{\rm B}^2+(\lambda_B/4!)\varphi_{\rm B}^4\right], \tag{2.2}$$

with $\beta=1/T$. The boundary condition B.C. on the path integral prescribes that the fields be periodic in imaginary time. In the context of this paper, the path integral in eq. (2.1) is defined by a suitable renormalized perturbation expansion.

The basic message of a paper by Jourjine [18] on the present subject is that it helps very much to regard the analysis of DR as a problem in vacuum renormalization theory. We therefore rewrite eq. (2.2) as a three-dimensional zero-temperature action in terms of an infinite number of massive fields. The theory is multiplicatively renormalized at the same instant. Accordingly,

$$\varphi_B(\tau, x) = Z_3^{1/2} T^{1/2} \sum_{n=-\infty}^{\infty} \varphi_n(x) e^{i\omega_n \tau}, \tag{2.3}$$

with the well-known [7] Matsubara frequencies $\omega_n = 2\pi n T$. The mass and the coupling constant are renormalized by

$$m_B^2 = Z_1 Z_3^{-1} m^2, \qquad \lambda_B = Z_2 Z_3^{-2} \lambda. \tag{2.4}$$

We omit the subscript R on renormalized quantities; φ_n, m, and λ are understood to depend on the renormalization prescription R.

The action is now additively composed of six pieces

$$S_E[\varphi] = \int d^3x \left(\mathscr{L}_f^s + \mathscr{L}_b^s + \mathscr{L}_{ct}^s + \mathscr{L}_f^n + \mathscr{L}_b^n + \mathscr{L}_{ct}^n \right). \tag{2.5}$$

The first three terms describe the static (zero mode) sector. We here adopt some of Collins' terminology [4]: the free (static) lagrangian

$$\mathscr{L}_f^s = \tfrac{1}{2}(\partial_i \varphi_0)^2 + \tfrac{1}{2} m^2 \varphi_0^2 \tag{2.6}$$

generates the free static propagator, while the basic (static) interaction lagrangian

$$\mathscr{L}_b^s = \tfrac{1}{24} \lambda T \varphi_0^4 \tag{2.7}$$

generates the basic interaction vertex. The unsubtracted static Feynman amplitudes are determined by eqs. (2.6) and (2.7). Subtractions are implemented by the (static) counterterm lagrangian

$$\mathscr{L}_{ct}^s = \tfrac{1}{2}(Z_3 - 1)(\partial_i \varphi_0)^2 + \tfrac{1}{2}(Z_1 - 1) m^2 \varphi_0^2 + \tfrac{1}{24}(Z_2 - 1) \lambda T \varphi_0^4. \tag{2.8}$$

The three terms of the nonstatic action have an analogous meaning, and we have

$$\mathscr{L}_f^n = \tfrac{1}{2} \sum_n{}' \varphi_n \left(-\Delta + m^2 + \omega_n^2 \right) \varphi_{-n}, \tag{2.9}$$

$$\mathscr{L}_b^n = \tfrac{1}{24} \lambda T \sum_{n_1 \dots n_4}{}' \delta(n_1 + n_2 + n_3 + n_4) \varphi_{n_1} \varphi_{n_2} \varphi_{n_3} \varphi_{n_4}, \tag{2.10}$$

after which the reader may easily write down the expression for $\mathscr{L}_{ct}^n$. The prime in the sum means that the term with all n's equal to zero is to be omitted. Even so, $\mathscr{L}_b^n$ of course contains interactions of the static field with the nonstatic fields.

2.2. TEMPERATURE-DEPENDENT RENORMALIZATION CONDITIONS

The amputated momentum space N-point Green functions of the four- and the three-dimensional theory are obviously related by

$$G_{\mathrm{B}}^{(N)\mathrm{amp}}(p_1,\dots,p_{N-1},m_{\mathrm{B}},\lambda_{\mathrm{B}},T)$$
$$= Z_3^{-N/2}T^{1-N/2}G_{n_1\dots n_{N-1}}^{(N)\mathrm{amp}}(p_1,\dots,p_{N-1},m,\{\omega_n\},\lambda T), \qquad (2.11)$$

with $p_i^0 = 2\pi n_i T$. The temperature dependence of the three-dimensional theory enters via the coupling λT as well as via the "masses" ω_n in nonstatic loops. In addition, λ and m, as well as G itself, depend on the renormalization scheme, which might carry explicit T-dependence.

Each renormalization constant Z_i ($i = 1,2,3$) contains two essentially unrelated parts

$$Z_i - 1 = \delta Z_i^{\mathrm{s}} + \delta Z_i^{\mathrm{n}}. \qquad (2.12)$$

The first term on the r.h.s. takes care of the renormalization effects of the purely static sector, whereas the second term relates to the contributions of the nonstatic sector to both static and nonstatic Green functions. In order to have dimensional reduction the δZ_i^{n} should be chosen in such a way that they remove the nonnegligible, nonstatic loop contributions to static Green functions (cf. sect. 4), while the δZ_i^{s} are not determined by this demand. As we shall see, to have a chance for DR the counterterms and hence the Z_i have to be explicitly T-dependent. This motivates a general study of two-parameter renormalization prescriptions defined by subtractions of Feynman integrals at momentum scale μ and reference temperature T_0 (which eventually may coincide with the actual temperature T).

It is well-known that renormalization at zero temperature removes all ultraviolet divergences even at finite T [7]. Therefore, Green functions and parameters defined by renormalization conditions at temperature $T_0 > 0$ differ from their vacuum values by a *finite* renormalization [17] (as we shall see, these finite renormalizations are essential for DR). Since the renormalization constants Z_i have been chosen to be independent of n (cf. eqs. (2.3) and (2.4)), they can be defined by stating normalization conditions on the renormalized, static, one-particle irreducible vertex functions $\Gamma = -G_{\mathrm{1PI}}^{\mathrm{amp}}$. As we explained before, we may use a hybrid renormalization prescription, which treats completely static diagrams and diagrams with static external lines but nonstatic loops on a different footing. For the purpose of this paper the following scheme turns out to be useful: we renormalize diagrams with at least one nonstatic loop by subtractions at momentum scale μ and temperature T_0, while purely static diagrams are renormalized by minimal subtraction (MS) with arbitrary scale parameter ν (cf. ref. [4]). This scheme is equivalent to the normalization conditions

$$\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2,T_0)=\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2,T_0)_{\text{static, MS}},$$

$$\frac{\partial}{\partial \boldsymbol{p}^2}\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2,T_0)=\frac{\partial}{\partial \boldsymbol{p}^2}\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2,T_0)_{\text{static}},$$

$$\Gamma_{000}^{(4)}(s=t=u=\tfrac{4}{3}\mu^2,T_0)=\Gamma_{000}^{(4)}(s=t=u=\tfrac{4}{3}\mu^2,T_0)_{\text{static}},\qquad(2.13)$$

with $s=(\boldsymbol{p}_1+\boldsymbol{p}_2)^2$, $t=(\boldsymbol{p}_1-\boldsymbol{p}_3)^2$, $u=(\boldsymbol{p}_1-\boldsymbol{p}_4)^2$. Here we used the super-renormalizability of $(\varphi^4)_3$; hence (cf. eq. (2.12) $\delta Z_2^s=\delta Z_3^s=0$, while in dimensional regularization δZ_1^s receives contributions in two-loop order only.

2.3. FINITE TEMPERATURE RENORMALIZATION GROUP

The renormalization prescription independence of the bare Green functions in conjunction with eq. (2.11) and the definition of the 1PI functions Γ implies the renormalization group equations (RGE)

$$\left(\tfrac{1}{2}N\alpha_T+\beta_T\frac{\partial}{\partial\lambda}+\gamma_T m^2\frac{\partial}{\partial m^2}+T_0\frac{\partial}{\partial T_0}\right)\Gamma^{(N)}_{n_1\dots n_{N-1}}=0,\qquad(2.14)$$

$$\left(\tfrac{1}{2}N\alpha_\mu+\beta_\mu\frac{\partial}{\partial\lambda}+\gamma_\mu m^2\frac{\partial}{\partial m^2}+\mu\frac{\partial}{\partial\mu}\right)\Gamma^{(N)}_{n_1\dots n_{N-1}}=0.\qquad(2.15)$$

The coefficient functions are

$$\alpha_T=-T_0\frac{\partial}{\partial T_0}\log Z_3,\qquad \beta_T=T_0\frac{\partial\lambda}{\partial T_0},\qquad \gamma_T=m^{-2}T_0\frac{\partial}{\partial T_0}m^2,\qquad(2.16)$$

and similar definitions for α_μ, etc. As in the real-time case [15] the normalization conditions (2.13) imply certain identities between these functions, but we will not need them here. The explicit T_0, μ dependence of the $\Gamma^{(N)}$ comes from their explicit dependence on the renormalization scheme. In addition, of course, the $\Gamma^{(N)}$ depend on the actual temperature T via ω_n and λT, cf. eq. (2.11). Equations similar to (2.14) and (2.15) have been given in the real-time formalism [15, 19]. Note that in the zero-temperature limit eq. (2.15) reduces to the Georgi–Politzer RGE [20].

The RG functions α,β,γ may be evaluated by computing the static two- and four-point diagrams, and then determining the Z_i such that the conditions (2.13) are satisfied. This directly yields α_T and α_μ by (2.16), whereas β and γ follow from the invariance of the bare parameters m_B and λ_B in eq. (2.4), cf. sect. 3.

Solving eq. (2.16) gives the renormalized parameters m,λ,Z as functions $m=m(T_0,\mu)$ etc. We will find in the next sections that the choice $T_0=T$, $\mu=0$ leads to

DR in a suitable sense. It then follows from eqs. (2.14) and (2.15) (or, equivalently, from eq. (2.11)) that Green functions depending on arbitrary values of T_0, μ are related to Green functions in this "maximal decoupling" scheme by

$$\Gamma^{(N)}_{n_1 \dots n_{N-1}}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_{N-1}, m(T_0, \mu), \{\omega_n\}, T\lambda(T_0, \mu), T_0, \mu)$$

$$= \left(\frac{Z_3(m^2(T_0, \mu), \lambda(T_0, \mu), T_0, \mu)}{Z_3(m^2(T,0), \lambda(T,0), T, 0)} \right)^{N/2}$$

$$\times \Gamma^{(N)}_{n_1 \dots n_{N-1}}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_{N-1}, m(T,0), \{\omega_n\}, T\lambda(T,0), T, 0). \quad (2.17)$$

The decoupling theorem, to the extent that it holds, guarantees that $\Gamma^{(N)}$ on the r.h.s. is given by the purely static theory. A relation similar to eq. (2.17) holds for any choice of renormalization point $(\tilde{T}_0, \tilde{\mu})$ instead of $(T, 0)$. However, the statement of DR is only valid for a restricted class of choices for $(\tilde{T}_0, \tilde{\mu})$, of which $(T, 0)$ is the optimal one. The transition from the four-dimensional theory to the effective three-dimensional one is therefore renormalization-point dependent. Hence the full RG invariance (2.14) and (2.15) of the original theory is not inherited by the effective theory. One is left with the freedom of RG transformations induced by finite changes of δZ_i^s in eq. (2.12). This freedom is nontrivial, although it is not related to the removal of ultraviolet divergences.

3. Calculations in scalar field theory

3.1. ONE-LOOP DIAGRAMS

To see explicitly how dimensional reduction is supposed to work, we will now perform a one-loop analysis of the scalar φ^4 theory discussed above. The aim of this is to check whether a renormalization prescription exists in which DR holds, at least to this order. To do so, we will calculate the one-loop correction to static 1PI functions $\Gamma^{(N)}_{0 \dots 0}$, and try to determine the counterterms (2.8) in such a way that in the infinite-temperature limit the nonstatic loop is negligible compared to the static loop.

To achieve this goal in a clean fashion it is obviously desirable to explicitly separate the static modes from the nonstatic ones. The standard summation methods [7] based on analytic continuation do not realize this aim. Instead, we will perform the summation over the nonzero Matsubara frequencies by the following device: we first compute the momentum integrals using dimensional regularization, setting $D = 3 - 2\varepsilon$ ($\varepsilon > 0$) and using a mass parameter ν to give the Green functions their proper dimension. The mode sum is then automatically regularized in a way akin to zeta-function regularization [21]. (After completion of this work we found that a similar strategy has been employed in calculations of the Casimir effect [22, 23].) A Mellin transform subsequently allows systematic expansions for both low and high temperatures. Detailed formulae may be found in the appendix.

We start with the self-energy, given by a tadpole diagram. The nonstatic modes give (cf. eqs. (A.1), (A.7) and (A.8))

$$\Gamma_0^{(2)}(T)_{\text{nonstatic}} = \tfrac{1}{2}\lambda\nu^{2\varepsilon}T\sum_{n\neq 0}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\omega_n^2+m^2} = \lambda\left(\frac{T^2}{24} - \frac{m^2}{32\pi^2}\frac{1}{\hat{\varepsilon}_b} + \mathcal{O}(T^{-2})\right) \tag{3.1}$$

The static mode gives

$$\Gamma_0^{(2)}(T)_{\text{static}} = \tfrac{1}{2}\lambda\nu^{2\varepsilon}T\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+m^2} = -\frac{\lambda mT}{8\pi}. \tag{3.2}$$

Here and in the following, Γ_0 refers to a 1PI function in the three-dimensional theory having an infinite tower of masses. Static Green functions in the four-dimensional theory follow by eq. (2.11).

The four-point function with nonstatic loop gives

$$\begin{aligned}\Gamma_{000}^{(4)}(s,t,u,T)_{\text{nonstatic}} &= -\tfrac{1}{2}\lambda\nu^{2\varepsilon}T^2\sum_{n\neq 0}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+\omega_n^2+m^2}\\ &\quad\times\frac{1}{(k+p_1+p_2)^2+\omega_n^2+m^2} + (s\to t) + (s\to u)\\ &= -\frac{3\lambda^2T}{32\pi^2}\left[\frac{1}{\hat{\varepsilon}_b} + \frac{\zeta(3)}{16\pi^2T^2}\left(m^2+\tfrac{1}{18}(s+t+u)\right) + \mathcal{O}(T^{-4})\right].\end{aligned} \tag{3.3}$$

The zero mode integral equals

$$\Gamma_{000}^{(4)}(s,t,u,T)_{\text{static}} = -\frac{\lambda^2T^2}{8\pi\sqrt{s}}\arcsin\left[\left(1+4m^2/s\right)^{-1/2}\right] + (s\to t) + (s\to u). \tag{3.4}$$

For the nonstatic contribution to the six-point function we obtain by the same method

$$\Gamma^{(6)}_{00000}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_5,T)_{\text{nonstatic}} = \frac{\lambda^3}{128\pi^4}\left\{15\zeta(3) - \frac{\zeta(5)}{16\pi^2 T^2}\left[90m^2 + F(\boldsymbol{p}) + \mathcal{O}(T^{-4})\right]\right\} \tag{3.5}$$

with $F(\boldsymbol{p}) = 10\Sigma_{i=1}^{6}\boldsymbol{p}_i^2 + 8\Sigma_{i>j}^{6}\boldsymbol{p}_i\cdot\boldsymbol{p}_j$, and $\boldsymbol{p}_6 = -\Sigma_{i=1}^{5}\boldsymbol{p}_i$. For the zero mode we have an expansion in $\boldsymbol{p}/m$

$$\Gamma^{(6)}_{00000}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_5,T)_{\text{static}} = \frac{\lambda^3 T^3}{32\pi m^3}\left[15 - \frac{1}{4m^2}F(\boldsymbol{p}) + \mathcal{O}(m^{-4})\right]. \tag{3.6}$$

We will stop here, because the general form of convergent n-point functions is clear by now: the static contribution is proportional to $(\lambda T)^{n/2}/m^{n-3}$ in the zero-momentum limit, whereas the nonstatic modes in the loop give a Laurent series in T whose leading term is $\sim T^{3-n/2}$.

3.2. COUNTERTERMS FOR DIMENSIONAL REDUCTION

At this point we may write down the renormalized one-loop effective action Γ for the three-dimensional theory. By "effective" we here mean that the nonstatic modes have been integrated out; for simplicity we will not include the one-loop effects of the static modes in Γ. These are generated by the purely static three-dimensional theory, so that they do not affect the choice of counterterms leading to DR; moreover, they are finite to this order. Combining eqs. (2.6)–(2.8), (3.1), (3.3) and (3.6) we thus have

$$\begin{aligned}\Gamma[\varphi_0] = \int \mathrm{d}^3x \Bigg[& \mathscr{L}_f^s + \mathscr{L}_b^s + \tfrac{1}{2}(Z_3 - 1)\nabla\varphi_0\cdot\nabla\varphi_0 \\ & + \tfrac{1}{2}m^2\varphi_0^2\left(\frac{\lambda T^2}{24m^2} - \frac{\lambda}{32\pi^2\hat{\varepsilon}_b} + Z_1 - 1 + \ldots\right) \\ & + \tfrac{1}{24}\lambda T\varphi_0^4\left(-\frac{3\lambda}{32\pi^2\hat{\varepsilon}_b} + Z_2 - 1 + \ldots\right) + \frac{15\zeta(3)\lambda^3}{128\pi^4}\varphi_0^6 + \ldots\Bigg],\end{aligned} \tag{3.7}$$

where the singular term $1/\hat{\varepsilon}_b$ is given in eq. (A.8). The dots represent terms down by powers of m/T compared to the ones exhibited. Note that dim $\varphi_0 = \frac{1}{2}$ by eq. (2.3), so that eq. (3.7) is dimensionless.

The whole idea of dimensional reduction now consists in choosing the counterterms in eq. (3.7) in such a manner that the contributions from the nonstatic modes which are nonnegligible for $T \to \infty$ are cancelled. This implies

$$Z = 1 - \frac{\lambda T^2}{24m^2} + \frac{\lambda}{32\pi^2\hat{\varepsilon}_b}, \qquad Z_2 = 1 + \frac{3\lambda}{32\pi^2\hat{\varepsilon}_b}, \qquad Z_3 = 1, \tag{3.8}$$

after which Γ, up to negligible terms, reduces to

$$\Gamma[\varphi_0] = \int d^3x \left[\tfrac{1}{2}(\partial_i\varphi_0)^2 + \tfrac{1}{2}m^2\varphi_0^2 + \tfrac{1}{24}\lambda T\varphi_0^4 + \frac{15\zeta(3)}{128\pi^4}\lambda^3\varphi_0^6\right]. \tag{3.9}$$

It is remarkable that the theory defined by the action (3.9) is renormalizable. This is a general feature of dimensional reduction: nonrenormalizable interactions are suppressed by powers of the compactification radius. It is equally remarkable that the term with φ_0^6 is not already present in the original static action (2.5)–(2.7). It naively appears that this term is completely overpowered by the static one-loop term (3.6), but we will know better before long.

3.3. THERMAL RENORMALIZATION GROUP FLOW

As we already announced, the counterterms (3.8) which (naively) satisfy dimensional reduction imply the normalization conditions (2.13) with $T_0 = T$ and $\mu = 0$. Accordingly, the renormalization prescription, and thereby the renormalized parameters φ_0, m, and λ are temperature-dependent. Their T-dependence can be derived from eqs. (2.16) and (3.8), as described in sect. 2.

To this order we find

$$\beta_T = 3\lambda^2/16\pi^2; \qquad \gamma_T = \lambda T^2/12m^2 + \lambda/16\pi^2. \tag{3.10}$$

Defining $g = 3\lambda/32\pi^2$, and the RG invariant $\Lambda^2 = T_0^2\exp(1/g(T_0))$ it follows that

$$g(T) = 1/\log(\Lambda^2/T^2), \tag{3.11}$$

$$m^2(T) = g^{1/3}\Lambda^2\left[C + \Gamma(\tfrac{1}{3}) - \tfrac{4}{9}\pi^2\gamma(\tfrac{1}{3}, 1/g)\right]. \tag{3.12}$$

Here $g = g(T)$, and C is an RG invariant integration constant, which may be expressed in terms of the initial value $m^2(T_0)$. Use of the asymptotic expansion for the incomplete Gamma function γ in eq. (3.12) gives

$$m^2(T) = Cg^{1/3}\Lambda^2 + \left(\tfrac{4}{9}\pi^2 g + \mathcal{O}(g^2)\right)T^2. \tag{3.13}$$

It is clear from eq. (3.11) that the present calculation can be possibly meaningful only for $T \ll \Lambda$; the blow-up of the coupling constant at $T = \Lambda$ is a remnant of the Landau ghost in the vacuum theory, which hints at the inconsistency of self-interacting scalar field theories [24]. We also observe, a fortiori, that the preceding high-temperature expansions are meaningful only if $T^2 \gg Cg^{1/3}(T)\Lambda^2$. If triviality applies, the preceding is still sensible if Λ is regarded as a (huge) UV-cutoff, while $g(T)$ is small for $T \ll \Lambda$. Accordingly, the constant C should be tiny, so that the "window of validity" $Cg^{1/3}\Lambda^2 \ll T^2 \ll \Lambda^2$ of the preceding manipulations is large enough. Conditions of this kind would be absent in asymptotically free theories in which, on the other hand, the computational complexities would obscure our basic ideas (cf. sect. 5!).

In sect. 4 we will give a precise criterion for dimensional reduction to occur, and thence see to what extent this is realized for the scalar theory. Eqs. (3.12) and (3.13) will be particularly crucial in this study.

Finally, we wish to stress that eq. (3.11) exhibits the T-dependence implied by our normalization conditions (2.13). Replacing the right-hand sides in eqs. (2.13) by 0, 0, and λ, respectively, one would obtain a significantly different expression for $\lambda(T)$, cf. ref. [19]. It so happens that eq. (3.11) coincides with the formula for the running coupling in the MS renormalization scheme [4] with ν replaced by T. This, however, is an accidental one-loop feature.

4. General analysis of dimensional reduction at high temperature

4.1. APPELQUIST–CARAZZONE THEOREM AND HIERARCHY PROBLEM

To understand the essential difference between decoupling of heavy particles at zero temperature and dimensional reduction at high T we will start with a brief discussion of the AC decoupling theorem in vacuum field theory.

Consider a theory describing a heavy particle with mass M interacting with one or more light particles with mass m; these masses are here defined to be discrete eigenvalues of the operator P^2. Decoupling of the heavy particle takes place if observable quantities at low energy ($p \ll M$) can be reliably computed from the lagrangian of the light fields only. In vacuo the low-energy observables are cross sections for the scattering of the light particles. Since these can be computed from the renormalized Green functions $G^{\mathrm{R}}_{\mathrm{light}}$ with "light" external lines only, decoupling occurs if loop corrections to these functions involving the heavy particle can be neglected compared to those containing light particles only. The criterion for decoupling is therefore

$$\lim_{M\to\infty} \frac{G^{\mathrm{R}}_{\mathrm{light}}(p, m, M; \mathrm{heavy})}{G^{\mathrm{R}}_{\mathrm{light}}(p, m, M; \mathrm{light})} = 0, \tag{4.1}$$

when $p = \{p_1, \dots, p_N\}$ and m are fixed (i.e. do not grow with M). We use the notation $G(\dots, \mathrm{heavy})$ to indicate that at least one heavy-particle line is contained in the diagrams contributing to G; conversely, $G(\dots, \mathrm{light})$ means that only mutual interactions between the light particles are taken into account. The M-dependence of the latter may be there because some coupling constants might be M-dependent, as in the Higgs sector of the standard model (compare also the thermal situation (2.7)!).

The usual proof of the decoupling theorem [3, 5, 6] assumes that $G^{R}_{light}(\ldots, \text{light})$ is independent of M and nonvanishing. In that case, *and only in that case*, the theorem can be proved by showing that $G^{R}_{light}(M \to \infty; \text{heavy})$ vanishes. The essence of the AC theorem in this simplified situation is easily grasped. Suppose we regularize the theory with a Pauli–Villars mass Λ. A primitively divergent 1PI diagram Γ with degree of divergence δ then depends on Λ like Λ^{δ} ($\delta > 0$) or $\log \Lambda$ ($\delta = 0$) for large Λ. In renormalizable theories the divergent terms $\sim \Lambda^{\delta}$ (or $\log \Lambda$) can be removed by local counterterms, which thus depend on Λ, so that the renormalized diagram is finite. Convergent diagrams ($\delta < 0$) are independent of Λ for $\Lambda \to \infty$.

The situation involving the large mass M is quite similar. Divergent diagrams will be proportional to a positive power of M (or to $\log M$), but this dependence can be removed by choosing the local counterterms appropriately. Convergent diagrams are proportional to negative powers of M, and vanish for $M \to \infty$ without the need for a further renormalization (which would not be implementable by counterterms in any case!). This reasoning demonstrates that the validity of the decoupling theorem strongly depends on the renormalization scheme R. Mass-independent schemes, like minimal subtraction, do not remove the large M-dependence of divergent diagrams, and violate the AC theorem. In contrast, a scheme which is optimal with regard to the decoupling theorem is the BPHZ subtraction scheme [5], in which momentum-space subtractions are made at the origin. The on-shell scheme (subtraction at $p^2 = m^2$) also satisfies the decoupling theorem, because it is related to the BPHZ scheme by a "small" renormalization group transformation (in the sense that the subtraction point μ is shifted by an amount m which is small with respect to M).

Here we touch a deep aspect of the decoupling theorem, which we wish to stress because it is essential in order to understand the situation at high temperature. Any quantity P can be written as $P^{R} = L^{R} + H^{R}$, where L and H stand for the contribution to P coming from the light and the heavy sector, respectively; we have explicitly indicated the renormalization scheme (R) dependence of the objects involved. If now P is a physical observable, then it is obviously R-independent. Nevertheless, L^{R} and H^{R} still depend on R; only their sum does not. The whole point of the AC-theorem is that there exists a class of R-schemes in which H^{R} is negligible for low-energy observables P. Thus $P = L^{R}$, where L^{R} depends only negligibly on R if this is varied by small RG-transformations. For large RG-transformations one looses the previous equality, which explains why it can be semantically meaningful at all (compare this with the situation in which a gauge-invariant quantity in a particular class of gauges equals a gauge-dependent one). There also exists a purely light theory, which is not coupled to a heavy sector, and which carries P among its set of observables too. This light theory produces a contribution $\tilde{L}$ to

P, so that in this particular theory $P = \tilde{L}$ *exactly* and without any R-dependence. Hence $\tilde{L} = L^{\mathrm{R}}$ for R in the class in which the AC theorem holds, but *not* in arbitrary schemes! We see, therefore, that the light sector obtained by truncating a theory with heavy particles according to the decoupling theorem is distinct from a purely light theory: the former still remembers its origin by its behaviour under large RG-transformations. At low energy, on the other hand, they are indistinguishable.

In the BPHZ scheme (as well as in the on-shell scheme etc.) one can rigorously prove the following bound on a given amputated (not necessarily 1PI) diagram Γ contributing to a Green function with any type of external legs (cf. ref. [5] for a proof using k-space Feynman integrals, and ref. [16] for a proof employing their parametric representation):

$$\Gamma^{\mathrm{BPHZ}}(p, m, \kappa M, \text{heavy}) \leqslant F \cdot \kappa^{-q}(\log \kappa)^{l} \tag{4.2}$$

for $\kappa \to \infty$. Here $F = F(p, m, M)$ is independent of κ; l is the number of loops in Γ, and

$$q = \min_{\gamma \in \Gamma_M} \left[\max\{1, -\delta_\gamma\}\right]. \tag{4.3}$$

The minimum is taken among the subdiagrams $\gamma \in \Gamma$ which contain *all* internal heavy lines; δ_γ is the degree of divergence of γ. If Γ contains no superficial divergences, then no subtractions are made, and formula (4.3) reduces to $q = -\max_{\gamma \in \Gamma_M} \delta_\gamma$, which is a well-known result [6] (the maximum is reached by a γ consisting of a disjoint set of light-particle irreducible graphs containing all heavy lines, so that there is indeed complete equivalence with Caswell & Kennedy's bound). In theories where the dimension of all graphs is even (as in scalar and vector theories without derivative couplings) (4.3) can be sharpened by replacing 1 by 2 in the r.h.s. of the equation.

Since $q \geqslant 1$, eq. (4.2) indeed proves the decoupling theorem (4.1) in case that $G^{\mathrm{R}}_{\text{light}}(\dots; \text{light})$ is M-dependent and nonvanishing. No decoupling takes place if the contribution from the purely light sector to a given process happens to vanish; in that case virtual heavy-particle exchange contributes dominantly rather than negligibly to the process, giving corrections to zero rather than to an $\mathcal{O}(1)$ amplitude, as in certain weak decay processes at low energy [4].

In the former case we clearly have

$$G^{\mathrm{BPHZ}}_{\text{light}}(p, m, M) = G^{\mathrm{BPHZ}}_{\text{light}}(p, m, \text{light})\left(1 + \mathcal{O}\left(\frac{p^2}{M^2}, \frac{m^2}{M^2}\right)\right). \tag{4.4}$$

It follows from the structure of the BPHZ-scheme that the corrections to two- and four-point functions are $\mathcal{O}(p^2/M^2)$ only, while "heavy" contributions to higher-point Green functions are $\mathcal{O}(m^2/M^2)$ as well. The reason these corrections are indeed small is that m^2 is an input parameter of the theory, which is thereby M-independent and simply fixed to be small. To achieve this consistently the bare mass $m_{\rm B}$ must be M-dependent, and has to be fine-tuned to cancel M-dependent self-energy corrections to m. For example, in a theory with $\mathscr{L}_{\rm b} = \frac{1}{4}\lambda\varphi^2_{\rm light}\Phi^2_{\rm heavy}$ one has

$$m^2_{\rm BPHZ} = m^2_{\rm MS} - \frac{\lambda M^2}{32\pi^2}\log M^2/\nu^2, \tag{4.5}$$

where ν is the scale parameter in the MS formalism. At low energy ν must be chosen to be of order $m_{\overline{\rm MS}}$, and thereby $m^2_{\overline{\rm B}}$ must be fine-tuned so as to eliminate the large second term in the r.h.s. of eq. (4.5). This is the hierarchy problem [4]. The essential difference with the thermal case will be presented in subsect. 4.2.

4.2. WHAT IS DIMENSIONAL REDUCTION AT HIGH T?

Supported by the one-loop computations in sect. 3 one would like to use the AC theorem to show that the nonstatic modes decouple at high temperature, cf. (3.9). In the present application decoupling, or, equivalently, dimensional reduction, means that static thermal observables at low momenta can be reliably computed from the purely three-dimensional theory with the nonstatic modes deleted, and with possibly renormalized input parameters. Henceforth this will be called the 3D-theory.

The principal observables in thermal equilibrium, viz. the expectation values of composite operators like the energy–momentum tensor $T_{\mu\nu}$, are given by superficially divergent diagrams receiving contributions from all momentum scales. Therefore, even though these quantities are formally one-point Green functions at *zero* external momentum the nonstatic modes are essential for their evaluation. This point may partly be remedied by formally adding the nonstatic contributions to the equilibrium pressure $P = \frac{1}{3}\langle T_{ii}\rangle$ [7] to the 3D-action in the form of a cosmological constant. The energy density $E = \langle T_{00}\rangle$ is out of reach of the effective 3D theory in any case, though. Another important class of thermal observables consists of the hydrodynamic transport coefficients, which are given by integrals over time-correlation functions of the energy–momentum tensor and of the conserved currents. These objects are obviously totally beyond the range of the 3D-theory as well.

Next, we consider the equal-time two-point function of the canonical spatial (energy–)momentum tensor T_{ij}. Its Feynman diagrams have a four-dimensional degree of divergence four, so that the nonstatic modes do not even decouple naively (cf. eq. (4.7) below). On the other hand, their nonnegligible contributions can be absorbed into finite renormalization constants in passing from the 4D- to the 3D-theory. One therefore needs to connect T_{ij} in four dimensions to the energy–momentum tensor of the 3D-theory, and to the operators with which the latter mixes. This connection, then, involves a set of renormalization constants which have to be calculated from the mode sums in the full theory.

We thus see that a large number of thermal observables either cannot be defined or else cannot be calculated in the alleged effective 3D-theory.

Nevertheless, decoupling of the nonstatic modes (hence DR) may still be meaningfully defined in terms of the static Green functions $G_{0\ldots0}$ (cf. sect. 2). Repeating the reasoning in subsect. 4.1, it follows that the correct criterion for DR is

$$\lim_{T\to\infty} \frac{G^{\mathrm{R}}_{0\ldots0}(\boldsymbol{p}, m, T, \lambda T; \text{nonstatic})}{G^{\mathrm{R}}_{0\ldots0}(\boldsymbol{p}, m, \lambda T; \text{static})} = 0 . \tag{4.6}$$

Here "static" means that G^{R} is calculated from the 3D-theory of the zero modes. (As before, $\boldsymbol{p}$ stands for an arbitrary number of momenta.) The parameters m and λ are obviously the ones defined in the renormalization scheme R in which eq. (4.6) holds; we have seen that the decoupling theorem is very sensitive to the choice of R.

We know from the explicit computations in sect. 3 that the optimal renormalization scheme in which eq. (4.6) possibly holds is the one defined by the renormalization conditions (2.13). Other possible schemes differ by small renormalization group transformations in the sense that the choice of $\mu = 0$, $T_0 = T$ is shifted by $\Delta\mu$, ΔT_0 such that $\lim_{T\to\infty} \Delta\mu/T$, $\Delta T_0/T = 0$. Also, the renormalization prescription for the 3D-theory is arbitrary as far as DR is concerned (cf. eq. (2.12) and the text below). It is technically simplest to use a scheme in which the r.h.s. of the conditions (2.13) are replaced by $0, 0, \lambda$ respectively; this scheme differs from the previous one in that the 3D-theory is now renormalized by zero-momentum subtractions as well. We call this the MD-scheme (for Maximal Decoupling, cf. ref. [8]). In more general theories the MD-scheme is defined by the normalization condition that the first few terms in the Taylor expansion around zero momentum of a superficially divergent 1PI diagram at $T_0 = T$ vanish up to order $\tau = \delta - I^{\mathrm{s}}$, where δ is the superficial degree of divergence determined by *four*-dimensional power counting, and I^{s} is the number of static loops.

It can be shown [16] that this scheme removes all UV-divergences. This would not happen if τ were determined by three-dimensional power counting, because the mode summation increases the degree of divergence by one [18]. However, to determine the leading power of T for a given diagram à la (4.2) involves *three*-dimensional power counting, because for dimensional reasons the mode summation does not affect the leading power of T (assuming that trivial factors coming from λT have been factored out already). In this respect the situation differs from that in subsect. 4.1: in the zero-T decoupling theorem large-mass power counting coincides with UV-power counting. Another difference is that here we are dealing with an infinite set of heavy masses $2\pi nT$ going to infinity. This causes great technical complications, despite which the following bound can be rigorously proved [16] in analogy to eq. (4.2):

$$\Gamma^{\mathrm{MD}}(p, m, \kappa T, \lambda T; \text{nonstatic}) \leqslant F\kappa^{-r}(\log \kappa)^{l} \tag{4.7}$$

for $\kappa \to \infty$. Here Γ^{MD} is an arbitrary amputated diagram with l loops calculated in the MD renormalization scheme, and

$$r = \min_{\gamma \in \Gamma_T} \left[l_\gamma^{\mathrm{n}} + \max\{1, l_\gamma^{\mathrm{s}} - \delta_\gamma\} \right]. \tag{4.8}$$

The minimum is determined among the subdiagrams $\gamma \in \Gamma_{\mathrm{T}} \subset \Gamma$ which contain *all* internal nonstatic lines, cf. (4.3), and l_γ^{n} and l_γ^{s} are the number of nonstatic and static loops in γ, respectively. As we explained, δ_γ is the superficial degree of divergence of γ, determined by four-dimensional power counting. The hybrid structure of (4.8) disappears for diagrams without any divergences; we then simply obtain $r = -\max_{\gamma \in \Gamma_{\mathrm{T}}} \delta_\gamma^{(3)}$, where now $\delta_\gamma^{(3)}$ is evaluated by three-dimensional power counting. To avoid any confusion, we remark that the entire κ-dependence of Γ in eq. (4.7) comes from the masses $2\pi n \kappa T$ in nonstatic loops.

4.3. CRITERION FOR DIMENSIONAL REDUCTION

Does the bound (4.7) prove the criterion (4.6) for DR? In analogy to eq. (4.4), our bound implies that

$$G_{0\ldots0}^{\mathrm{MD}}(\boldsymbol{p}, m, T, \lambda T) = G_{0\ldots0}^{\mathrm{MD}}(\boldsymbol{p}, m, \lambda T; \text{static})(1 + \mathcal{O}(\boldsymbol{p}/T, m/T)) \tag{4.9}$$

for $T \to \infty$. As before, the correction terms are $\mathcal{O}(\boldsymbol{p}/T)$ for superficially divergent diagrams, whereas the nonstatic corrections to convergent diagrams are $\mathcal{O}(m/T)$ as well. The crucial indicator is therefore the value of $\Delta = \lim_{T\to\infty} m_{\mathrm{MD}}/T$. The parameter m_{MD} is T-dependent, because the MD-renormalization prescription is; this is the essential difference with the AC decoupling theorem where, due to fine-tuning of the bare parameters, eq. (4.4) already implied decoupling of the heavy particle.

For a moment let us go back to the φ^4 theory in sect. 3. As explained below eq. (3.13), the strict infinite-temperature limit cannot be taken in eqs. (3.12) and (3.13), but if we choose T very large yet within the domain of validity of the calculations, the indicator Δ approaches $\frac{4}{9}\pi^2 g + \mathcal{O}(g^2)$, which is practically constant in the given window. The correction terms in eq. (4.9) are therefore $\mathcal{O}(g)$ and will be felt in the next order of the loop expansion. We conclude that DR holds in lowest order (in agreement with ref. [10]), but breaks down in higher orders of perturbation theory. This can be clearly seen for the six-point function $G^{(6)}$. According to eqs. (3.5) and (3.6) the correction term in eq. (4.9) at zero momentum is $\zeta(3) m_{\mathrm{MD}}(T)^3/4\pi^3 T^3$.

This correction term multiplies $G^{(6)}_{0\dots0}(\dots;\text{static})$ which is $\sim\lambda^3$, so that to one-loop we may replace m_{MD} by a constant, cf. eq. (3.12), which makes the correction vanish for $T\to\infty$, and DR takes place. If we compute the $\mathcal{O}(\lambda^6)$ contribution to $G^{(6)}$, however, we have to cope with the $\mathcal{O}(\lambda^3)\cdot\mathcal{O}((m_{\mathrm{MD}}/T)^3)=\mathcal{O}(\lambda^6)$ correction from the nonstatic modes to the naive one-loop diagram. This contribution is not given by the 3D-theory, so that by definition DR then breaks down. The behaviour of higher-order Green functions is similar.

We clearly see that the generation of a T-dependent mass in the static theory is responsible for the breakdown of DR in higher orders of perturbation theory. The mass generation as such has, of course, well been recognized in the literature. It is usually argued that DR "obviously" takes place, after which the 3D-theory exhibits mass generation which then may lead to a further decoupling of certain fields. What has not been appreciated is that this mass generation, if it occurs, is actually a consequence of the renormalization prescription in which the reduction process should occur. The induced mass thereby affects the correction terms to DR, and, indeed, causes DR to fail in many cases. This feedback mechanism is automatically taken care of in our approach based on the finite-temperature renormalization group.

The essential difference between the present situation, in which $T\to\infty$ plays the role of a large mass, and the zero-temperature decoupling theorem, can be summarized as follows

$$\frac{\mathrm{d}m_{\mathrm{B}}}{\mathrm{d}M}\neq 0,\qquad \frac{\mathrm{d}m_{\mathrm{B}}}{\mathrm{d}T}=0,\qquad \frac{\mathrm{d}m_{\mathrm{MD}}}{\mathrm{d}M}=0,\qquad \frac{\mathrm{d}m_{\mathrm{MD}}}{\mathrm{d}T}\neq 0.$$

In other words, at finite T the large T^2-dependent term in eq. (3.12) is simply there, and cannot be cancelled by a fine-tuning of the bare parameters, since by definition these know nothing about a temperature.

4.4. QUALIFICATIONS

By the same token we deduce which theories actually do exhibit DR at high T, namely the ones not suffering from the hierarchy problem (cf. subsect. 4.1). Several such examples of fields escaping thermal (and vacuum!) mass generation are known.

(i) As shown by Appelquist and Pisarski [1] the magnetic part of the photon field A_i in $(\mathrm{QED})_4$ remains massless to all orders of perturbation theory (also cf. subsect. 5 below).

(ii) Theories with a dimensionful coupling constant g, like $(\varphi^3)_4$, $(\mathrm{QED})_2$ and $(\mathrm{QED})_3$ generate masses proportional to g rather than T, so that $\Delta=0$ in those cases. Indeed, the occurrence of DR in the QED-examples has been demonstrated by Alvarez–Estrada with totally different methods [25, 26].

(iii) In supersymmetric theories there is no mass generation because of cancellations between bosonic and fermionic loops in self-energy diagrams.

For those theories the bound (4.7) provides a rigorous proof of DR at finite temperature.

In vacuum field theory only scalar models exhibit the hierarchy problem. One might naively think that other theories, e.g. gauge theories, which in general generate no masses in four dimensions, would show complete DR at high T. This is not correct, because at finite temperature a field of any spin is rearranged into a set of fields carrying a generalized helicity parameter [27] as a consequence of the breakdown of Lorentz invariance. Different spin components of the original field are grouped into different irreducible multiplets of the unbroken symmetry group, and in a sense all behave like scalar fields. A familiar example of this phenomenon is the time-like component A_0 of the spin-1 vector field A_μ. However, even the spatial part A_i splits up (cf. the appendix of ref. [27]) and can in principle generate a mass, as is well-known in QCD. This argument shows that, in great contrast to vacuum field theory, at finite T mass generation is a rule rather than an exception.

In all the aforementioned, the notions of "mass" and "mass generation" are used from the perspective of the 3D-theory, which is formally a euclidean vacuum theory with T-dependent parameters. This qualification needs to be made, because an interacting thermal field theory does not admit the notion of mass in the usual sense of a real-axis pole in the full propagator, or, equivalently, a discrete eigenvalue of the operator P^2 [27]. From the point of view of the full 4D-theory, the temperature-dependent "mass" is *defined* by the finite T renormalization group.

4.5. HOW GOOD IS THE EFFECTIVE THEORY?

Finally, we wish to find out how good the effective 3D-theory still is in case that DR fails. In the first place, the 3D-theory is not particularly useful in the evaluation of correlation functions of composite operators, cf. subsect. 4.2. Instead, let us turn to the Green functions of the elementary fields of the theory. To estimate the $\mathcal{O}(m/T)$ correction terms in eq. (4.9), we should obtain a bound on the l.h.s. of (4.6). This bound depends on the renormalization prescription for the 3D-theory, and it is easily shown that it is optimal if the 3D-theory is renormalized by the BPHZ prescription based on three-dimensional power counting. Namely, the BPHZ scheme in general minimizes the large-mass contribution to renormalized Green functions (cf. sect. 4.1), which is precisely what we need in view of the fact that the T-dependent mass induced in the static 3D-theory caused the breakdown of DR to all orders.

Suppose, then, that due to the MD-renormalization prescription for the nonstatic modes a mass $\sim \lambda^{\alpha/2}T$ has been induced in the 3D-theory, λ being a generic coupling constant, and $\alpha > 0$. The total mass of the static modes is then $m^2 = m_0^2 + c\lambda^\alpha T^2$, where m_0^2 and c are T-independent. We regard both m_0^2/T^2 and λ^α as small parameters. The nonstatic modes have masses $M_n^2 = m_0^2 + c\lambda^\alpha T^2 + (2\pi nT)^2$. For simplicity we study the correction terms to DR at zero external momentum, so that the ratio in eq. (4.6) can depend on m/T only.

First we study a given convergent Green function G. In the BPHZ scheme no scale is introduced by the renormalization of divergent subdiagrams, so that $G(\dots;\text{static})$ as computed in the 3D-theory must be proportional to $m^{\delta^{(3)}}$, where $\delta^{(3)}$ is the dimension of G in 3D-power counting. We may extract the same factor from the nonstatic contributions to G. The remaining factor of $G(\dots;\text{nonstatic})$ must then be a function of m/T, whose leading form for $T \to \infty$ follows from our result (4.7) and (4.8). All in all, we find that the ratio $\Gamma(\dots;\text{nonstatic})/\Gamma(\dots;\text{static})$ of a given diagram Γ, contributing to $G(\dots;\text{nonstatic})/G(\dots;\text{static})$ in a given order of (naive) perturbation theory in the MD-scheme, to leading order in m/T is proportional to $(T/m)^{-r}(\log(T/m))^I$, with r given by eq. (4.8), and I the number of loops in Γ.

Now let $\{\Gamma_l\}_l$ be the set of lowest-order (one-loop, in general) nonstatic diagrams contributing to a given static Green function $G_{0\dots0}$, so that $\Gamma_l \sim \lambda^\beta$ naively (i.e. without taking into account the λ-dependence of the masses) for some $\beta \in \mathbb{N}$. Define r_l as in eq. (4.8), with $\Gamma = \Gamma_l$ (if Γ_l is not a one-loop diagram one should take the minimum along all diagrams with the given topological structure which contain at least one nonstatic loop), and let $\rho = \min\{r_l\}$. It then follows from the preceding that the correction terms $\mathcal{O}(m/T)$ (as defined in eq. (4.9)) to the lowest-order Green functions are of order $(m_0^2/T^2 + \lambda^\alpha)^{\rho/2}$ times possible logarithms. For $T \gg m_0^2\lambda^{-\alpha}$ the effective 3D-theory therefore gives the complete result for $G_{0\dots0}^{\text{MD}}$ up to and including order $\beta + \frac{1}{2}\alpha\rho - 1$ in λ. On the other hand, if $m_0^2 \ll T^2 < m_0^2\lambda^{-\alpha}$ then the effective theory is useful only up to lowest order, i.e. $\mathcal{O}(\lambda^\beta)$.

We may apply this result to the six-point function in the theory of sect. 3. We then have $\alpha = 2$, $\beta = 3$, $\rho = 3$, so that the dimensionally reduced theory generates the full theory up to fifth order. This can be improved, of course, by adding the final, sixth-order, term in the effective action (3.9) by hand. In general, one should add an infinite number of (nonrenormalizable) interactions to the original 3D-theory to reproduce the nonstatic corrections by hand.

What about the superficially divergent Green functions? Here the $\mathcal{O}(m/T)$ corrections to DR are removed by the zero-momentum subtractions, and the actual corrections to DR are $\mathcal{O}(\boldsymbol{p}/T)$. At zero momentum DR therefore holds exactly for Green functions with nonnegative dimension. At small but finite momenta the existence of an induced mass $\sim T$ obstructs DR in a similar way as in the convergent case: the 3D-theory gives finite-momentum corrections of order $\boldsymbol{p}/m$, whereas the nonstatic modes produce terms of order $\boldsymbol{p}/T$. The ratio of these corrections is of order λ^α, so that one can consistently use the 3D-theory to compute the $\mathcal{O}(\boldsymbol{p}^2)$ corrections to a given divergent Green function up to and including the order $\alpha + \beta - 1$ in λ, with β defined as above.

In conclusion, dimensional reduction holds to all orders for superficially divergent Green functions at strictly zero momentum. In this domain of validity the theory reduces to a set of constants, whose values must be determined from the full 4D-theory!

5. Quantum electrodynamics

5.1. EFFECTIVE ACTION

The basic reason why dimensional reduction at high temperature to a certain extent holds is that nonnegligible T-dependent terms coming from nonstatic loops in superficially divergent 1PI functions can be cancelled by local counterterms, which are necessarily present because they have to cancel UV-divergences from these diagrams as well. A very interesting situation arises if a theory has superficially divergent 1PI functions which are actually UV-finite by virtue of cancellations between different diagrams. It is by no means guaranteed that the nonnegligible high-T terms then cancel out, too, while at the same time there are no counterterms to subtract these terms. In those situations the nonstatic modes do not even decouple in lowest order. On the other hand, the effective interaction they induce in the 3D-theory must be local, and can simply be added to the effective lagrangian at high T by hand.

Gauge theories provide a spicy example of this phenomenon, featured already by QED. Apart from the nonstatic photon modes, one would naively expect the entire fermion field to decouple, because it has antiperiodic boundary conditions in the path integral (2.1) [7], so that all its Fourier modes are nonstatic. The decoupling modes will induce an effective photon "mass", for the photon self-energy is superficially quadratically divergent. The quadratic UV-divergence vanishes by gauge invariance, but the term proportional to T^2 does not. Similarly, the four-photon Green function superficially has a logarithmic divergence, which cancels out, whereas a T-independent contribution induced by the fermion modes remains.

The general form of the effective 3D-theory can be inferred without any calculation. The complete 4D-effective action $\Gamma[A^\mu, \psi, \bar{\psi}]$ in the Gupta–Bleuler formalism satisfies the Ward–Takahashi identity [28]

$$-\partial^\mu \frac{\delta \Gamma}{\delta A^\mu} + ie\left[\Gamma \frac{\overleftarrow{\delta}}{\delta \psi}\psi - \bar{\psi}\frac{\delta \Gamma}{\delta \bar{\psi}}\right] = \lambda \Box \partial \cdot A\,. \tag{5.1}$$

In the dimensionally reduced theory at high T time-derivatives are simply to be deleted, so that the effective action $\Gamma^{3\mathrm{D}}[A_0^\mu]$ of the static photon mode satisfies

$$\partial^i \frac{\delta \Gamma^{3\mathrm{D}}}{\delta A^i} = -\lambda \Delta\, \partial_i A^i\,, \tag{5.2}$$

with $A^i = A^i_0$. This shows that Γ^{3D} must be a gauge-invariant functional of the spatial static gauge field A^i up to the usual gauge-fixing terms, whereas its dependence on the timelike static gauge field A^4 is completely arbitrary. In particular, an induced "electric mass" term of the form $\frac{1}{2}m^2(A^4)^2$ in Γ^{3D} is to be expected. By spatial gauge invariance, an induced "magnetic mass" term would have to be of the form $\mu^2 F^{ij}(x)\Delta^{-1}F^{ij}(x)$, as in the Schwinger model [29]. It is easily seen, however, that $(\mathrm{QED})_4$ is not infrared-singular enough to produce the inverse laplacian in the photon polarization tensor, or in any other 1PI function. Hence all magnetic terms in Γ^{3D} are suppressed by powers of momenta $1/T$.

5.2. ONE-LOOP RESULTS AND THEIR GENERALIZATION

These expectations are borne out by explicit calculations. We computed the one-loop diagrams with two, four, and six external photon lines in the high-temperature limit, using the same techniques as in sect. 3 (i.e. based on the formulae in the appendix). This leads to an expression for the 3D-effective action similar to eq. (3.7). We use euclidean metric and fields $A^\mu = A^\mu_{\mathrm{E}} = (A^i_{\mathrm{M}}, -iA^0_{\mathrm{M}})$, and decompose them as in eq. (2.3), with the proviso that different wave function renormalization constants Z_{S}, Z_{T} are used for A^i and A^4, respectively (cf. ref. [8] for QCD). We find (omitting the subscript $_0$ on the r.h.s.)

$$\begin{aligned}\Gamma^{3D}[A^\mu_0] = \int \mathrm{d}^3x \Bigg\{ &\tfrac{1}{4}F^{ij}F^{ij}\left(1 + \frac{e^2}{12\pi^2}\frac{1}{\hat{\varepsilon}_f} + \frac{7\zeta(3)\,e^2}{48\pi^4T^2}M^2 + Z_{\mathrm{S}} - 1\right) + \frac{1}{2\xi}(\partial_i A^i)^2(Z_{\mathrm{S}} - 1)\\ &+ \frac{7\zeta(3)e^2}{960\pi^4T^2}F^{ij}\Delta F^{ij} + \tfrac{1}{2}(\partial_i A^4)^2\left[1 + \frac{e^2}{12\pi^2}\left(\frac{1}{\hat{\varepsilon}_f} - 1\right) + Z_{\mathrm{T}} - 1\right] + \tfrac{1}{2}m^2(T)(A^4)^2\\ &- \frac{e^4T}{12\pi^2}(A^4)^4 + \frac{889\zeta(3)e^6}{6144\pi^4}(A^4)^6 - \frac{7\zeta(3)e^2}{960\pi^4T^2}\left[(\Delta A^4)^2 - 20e^2T(A^4)^2(\partial_i A^4)^2\right]\\ &+ \frac{7\zeta(3)\,e^4}{96\pi^4T}(A^4)^2F^{ij}F^{ij} + \cdots \Bigg\}. \end{aligned} \tag{5.3}$$

Here M is the electron mass, and the fermionic pole is given in eq. (A.9).

The induced "electric mass" is given by (cf. ref. [30])

$$m^2(T) = e^2\left(\frac{T^2}{3} - \frac{M^2}{2\pi^2} + \frac{7\zeta(3)M^4}{16\pi^4T^2} + \dots\right). \tag{5.4}$$

We exhibit a few suppressed terms in (5.3) in order to show explicitly that the effective action satisfies the reduced WT-identity (5.2). The reduced gauge invariance of Γ^{3D} implies that the induced interactions between A^i and A^4 vanish at zero momentum, and are suppressed by powers of T^{-1} at finite momenta.

We clearly see that Z_T and Z_S can be chosen in such a way that dimensional reduction to one-loop order takes place. Doing so (cf. eqs. (3.7) and (3.8)) gives, up to negligible terms for $T \to \infty$,

$$\Gamma^{3D}[A_0^\mu] = \int d^3x \Bigg[\tfrac{1}{4} F^{ij} F^{ij} + \frac{1}{2\xi_R} (\partial_i A^i)^2 + \tfrac{1}{2} (\partial_i A^4)^2 + \tfrac{1}{2} m^2(T) (A^4)^2 - \frac{e^4 T}{12\pi^2} (A^4)^4 + \frac{889\zeta(3)\, e^6}{6144\pi^4} (A^4)^6 \Bigg]. \quad (5.5)$$

In accord with the above remark, there are no residual interactions between the spatial and the temporal gauge field.

This result is clearly not particular to the one-loop approximation: at finite T the electric sector decouples to all orders from the magnetic one. Another consequence of eq. (5.2) is that the magnetic field F^{ij} remains massless to all orders [1]. The general limitation to dimensional reduction explained in subsects. 4.3–4.5 thus pertains to the electric sector only. The ordinary Appelquist–Carazzone theorem (subsect. 4.1) implies that electricity itself becomes irrelevant at momenta $\boldsymbol{p} \ll eT$, so that in QED a complete dimensional reduction indeed takes place. This result has been obtained before by heuristic reasoning [1], and by an explicit scan of all diagrams [31]. We wish to stress, however, that the general arguments obstructive to an all-order DR could be circumvented by the interplay of two features, viz. the absence of a magnetic "mass" and of induced electro-magnetic interactions, which are highly peculiar to the abelian situation (cf. sect. 6).

5.3. BOX DIAGRAM AND ANOMALY

The negative coefficient of the quartic term in eq. (5.5) is quite remarkable. This term is not large enough to shift the absolute minimum of Γ^{3D} from $A^\mu = 0$ to some nonzero value, but nevertheless the minus sign in front of it has an interesting explanation which we would like to point out.

The four-photon diagram is dimensionless (note that the factor T in the quartic term in eq. (5.5) is cancelled by the one in eq. (2.11) in passing to the 4D-Green functions) and has a superficial logarithmic UV-divergence which is cancelled. In the infinite-temperature limit the diagram must therefore approach a finite constant (factors of $\log T$ are impossible, because the zero mass-momentum limit is smooth). To compute this constant the Feynman integral must be regularized, in spite of the final cancellation of infinities. To explain the minus sign, let us regularize the diagram by adding a Pauli–Villars field with mass Λ. The sum of six static one-loop diagram with four timelike external photon lines at zero momentum then gives

$$\Gamma^{(4)}_{000} = 6e^4 T \lim_{\Lambda\to\infty} \int \frac{d^3p}{(2\pi)^3} \left\{ \mathrm{tr} \frac{[\gamma^4(\not p - iM)]^4}{(p^2+M^2)^4} - \mathrm{tr} \frac{[\gamma^4(\not p - i\Lambda)]^4}{(p^2+\Lambda^2)^4} \right\}$$

$$= \frac{3e^4}{\pi^2} \lim_{\Lambda\to\infty} \left\{ \left(\frac{M}{2\pi T}\right)^4 S\left(\frac{M^2}{4\pi^2T^2}; \frac{5}{2}, \frac{1}{2}\right) - \left(\frac{\Lambda}{2\pi T}\right)^4 S\left(\frac{\Lambda^2}{4\pi^2T^2}; \frac{5}{2}, \frac{1}{2}\right) \right\}, \quad (5.6)$$

with S defined by eq. (A.3).

In the limit $T \to 0$, $\Lambda \to \infty$ there is only one scale (M/T and Λ/T both explode). Using eq. (A.5) it follows that the two terms in eq. (5.6) cancel, so that the vacuum light–light scattering at zero momentum vanishes, as it should. On the other hand, in the limit $T \to \infty$, $\Lambda \to \infty$ (in such a way that $\Lambda/T \to \infty$) there are two scales which decouple: one has $m/T \to 0$, and $\Lambda/T \to \infty$. We must then use the expansion (A.6) for the first term in (5.6), and (A.5) for the second. The first term then vanishes, while the second leaves the finite result $-2e^4/\pi^2$. The negative sign in (5.5) therefore derives from the fact that the entire amplitude comes from the regulator field, which always enters with the wrong sign.

The calculation may be repeated in the dimensional regulation scheme of sect. 3 and the appendix. One then finds

$$\Gamma^{(4)}_{000} = \frac{3e^4}{\pi^2} \left(\frac{M}{2\pi T}\right)^4 S\left(\frac{M^2}{4\pi^2 T}; \frac{5}{2}, \frac{1}{2}\right) - \frac{4e^4}{\pi^2} \lim_{\varepsilon\to 0} \varepsilon S\left(\frac{M^2}{4\pi^2T^2}; \varepsilon + \frac{1}{2}, \frac{1}{2}\right). \quad (5.7)$$

One again, for $T \to 0$ the expansion (A.5) should be used, upon which the two terms in eq. (5.7) cancel. For $T \to \infty$, however, the first term vanishes on use of (A.6), while the second leaves the finite contribuition $-2e^4/\pi^2$. The origin of the minus sign is again to be found in the way the regulator enters.

The appearance of the finite, regulator-independent terms of the form Λ/Λ or ε/ε is familiar from the theory of anomalies [29]. Their origin lies in a failure of the decoupling theorem caused by dimensionless superficially divergent diagrams which turn out to be finite, and which therefore may lack a counterterm in the lagrangian. Superheavy virtual particles running around in such diagrams, like nonstatic modes at high T, or Pauli–Villars fermions, do not decouple, and leave anomalies behind. This is particularly clear in the elegant description of Farhi and D'Hoker [32], also cf. ref. [33] and subsect. 1.5 of ref. [34]. The quartic term in A^4 is anomalous even from a technical point of view [28], because it violates the WT-identity (5.1) in the full theory. The reduced identity (5.2) is satisfied, though.

6. Quantum chromodynamics

6.1. ONE-LOOP EFFECTIVE ACTION

High-temperature QCD differs fundamentally from QED in three respects, all due to its nonabelian nature

(i) there are both direct and induced unsuppressed interactions between the timelike ("chromo-electric") and the spatial ("chromo-magnetic") sector;

(ii) there are unsuppressed self-interactions in the chromo-magnetic sector;

(iii) magnetic mass generation in the 3D-theory is not excluded and is in fact to be expected [13, 35].

Thus the special features allowing complete dimensional reduction in QED are absent in QCD, so that the general arguments in sect. 4 apply, prohibiting DR to all orders. The same arguments show that DR definitely does hold in one-loop order, where bare internal propagators may be used.

Let us first infer the general structure of the effective action Γ^{3D} induced by the nonstatic modes. An argument similar to the one in subsect. 5.1, but now based on the full BRS invariance of the QCD effective action [36, 37], shows that the reduced effective action of the static modes $\Gamma^{BPHZ}[A_0^\mu, \bar{C}_0, C_0]$ must be a BRS-invariant functional in the sense that A_0^i is an ordinary gauge field, whereas A_0^4 behaves like a scalar field in the adjoint representation of the gauge group. The static ghost fields $\bar{C}_0$ and C_0 maintain their usual role.

To fix the notation, we give the classical static lagrangian

$$\mathscr{L}_{cl}^{3D} = \tfrac{1}{4}F^{ij}F^{ij} + \tfrac{1}{2}\left(D_iA^4\right)^2 + gT^{1/2}\left[\left(\partial_iA^j\right)A^iA^j - \left(\partial_iA^4\right)A^4A^i\right]$$

$$+ \tfrac{1}{4}g^2T\left[A^iA^jA^iA^j + 2A^4A^iA^4A^i\right] + (1/2\xi)\left(\partial_iA^i\right)^2 - \partial_i\bar{C}D_iC. \quad (6.1)$$

Here we write $A^\mu = A_0^\mu$ (as in sect. 5) and omit colour indices, so that $AA = A^aA^a$, $AAA = f^{abc}A^aA^bA^c$, $AAAA = f^{abc}f^{ade}A^bA^cA^dA^e$. Also $F^{ij} = \partial^iA^j - \partial^jA^i$, and $D_i = \partial_i + gT^{1/2}A^i$.

The mode decomposition, giving rise to the static fields in eq. (6.1), is analogous to eq. (2.3). We now introduce renormalization constants relating bare (B) quantities to renormalized ones by

$$A_B^i = Z_S^{1/2}A^i, \qquad A_B^4 = Z_T^{1/2}A^4, \qquad C_B = Z_C^{1/2}C, \qquad g_B = Z_1Z_S^{-3/2}g. \quad (6.2)$$

Using the summation technique of sect. 3 and the appendix, as well as the algebraic manipulation program SCHOONSCHIP [38], we computed the nonstatic contributions to all one-loop diagrams with up to four external static fields (except the four-ghost diagram which is irrelevant). A high-temperature expansion was then obtained using eq. (A.7). This gives the following expression for the static effective action containing all one-loop nonstatic mode corrections. In terms of $G = g^2/24\pi^2$ we find

$$\begin{aligned}
&\Gamma^{3D}[A_0^\mu, C_0, \bar{C}_0] \\
&= \int d^3x \left\{ \mathscr{L}_{cl}^{3D} + \tfrac{1}{4}F^{ij}F^{ij}\left[G\left(\frac{N_f}{\hat{\varepsilon}_f} + \tfrac{1}{4}N(3\xi - 13)\frac{1}{\hat{\varepsilon}_b}\right) + Z_S - 1\right] \right. \\
&\quad + \tfrac{1}{2}(\partial_i A^4)^2\left[G\left(N_f\left(\frac{1}{\hat{\varepsilon}_f} - 1\right) + \tfrac{1}{4}N\left((3\xi - 13)\frac{1}{\hat{\varepsilon}_b} + 10 - 6\xi\right)\right) + Z_T - 1\right] \\
&\quad + \tfrac{1}{2}m^2(T)(A^4)^2 + gT^{1/2}(\partial_i A^j)A^iA^j\left[G\left(\frac{N_f}{\hat{\varepsilon}_f} + \tfrac{1}{8}N(9\xi - 17)\frac{1}{\hat{\varepsilon}_b}\right) + Z_1 - 1\right] \\
&\quad - gT^{1/2}(\partial_i A^4)A^4A^i\left[G\left\{N_f\left(\frac{1}{\hat{\varepsilon}_f} - 1\right) + \tfrac{1}{8}N\left[(9\xi - 17)\frac{1}{\hat{\varepsilon}_b} + 20 - 12\xi\right]\right\}\right. \\
&\quad \left. + Z_1Z_TZ_S^{-1} - 1\right] + \tfrac{1}{4}g^2TA^iA^jA^iA^j\left[G\left(\frac{N_f}{\hat{\varepsilon}_f} + N(\tfrac{3}{2}\xi - 1)\frac{1}{\hat{\varepsilon}_b}\right) + Z_1^2Z_S^{-1} - 1\right] \\
&\quad + \tfrac{1}{2}g^2TA^4A^iA^4A^iG\left[\left(N_f\left(\frac{1}{\hat{\varepsilon}_f} - 1\right) + N\left((\tfrac{3}{2}\xi - 1)\frac{1}{\hat{\varepsilon}_b} + \tfrac{5}{2} - \tfrac{3}{2}\xi\right)\right) + Z_1^2Z_TZ_S^{-2} - 1\right] \\
&\quad + \frac{g^4T}{12\pi^2}V(A^4) + \frac{1}{2\xi}(\partial_i A^i)^2(Z_S - 1) - \partial_i\bar{C}\partial_i C\left[\tfrac{3}{8}GN(\xi - 3)\frac{1}{\hat{\varepsilon}_b} + Z_C - 1\right] \\
&\quad \left. - gT^{1/2}\partial_i\bar{C}A^iC\left(\frac{3GN\xi}{4\hat{\varepsilon}_b} + Z_CZ_1Z_S^{-1} - 1\right) + \dots \right\}. \qquad (6.3)
\end{aligned}$$

Here the induced "electric mass" is to leading order

$$m^2(T) = g^2\left\{\tfrac{1}{3}NT^2 + \tfrac{1}{6}N_fT^2 - (1/4\pi^2)\sum_f M_f^2\right\}, \qquad (6.4)$$

with N_f the number of flavours, and M_f the mass of the quark of flavour f, while N is the number of colours. The induced quartic potential of the temporal gauge/scalar field A^4 is

$$V(A^4) = \left(1 - \frac{N_f}{4N}\right)(A^a A^a)^2 + \tfrac{1}{8}(N - N_f)\, d_{abe} d_{cde} A^a A^b A^c A^d, \tag{6.5}$$

where $A = A^4$. We have explicitly written the colour indices; the d-symbol for SU(N) is defined e.g. in ref. [37].

6.2. DIMENSIONAL REDUCTION

To obtain a form of dimensional reduction the counterterms in eq. (6.3) must be chosen in such a way that the nonstatic terms are cancelled as far as possible. The following choice is seen to be optimal

$$Z_S = 1 - G\left[\frac{N_f}{\hat{\varepsilon}_f} + \tfrac{1}{4}N(3\xi - 13)\frac{1}{\hat{\varepsilon}_b}\right], \tag{6.6}$$

$$Z_T = Z_S + G\left[N_f + \tfrac{1}{4}N(6\xi - 10)\right], \tag{6.7}$$

$$Z_C = 1 + \tfrac{36}{8}N(3 - \xi)\frac{1}{\hat{\varepsilon}_b}, \tag{6.8}$$

$$Z_1 = 1 - G\left(\frac{N_f}{\hat{\varepsilon}_f} + \tfrac{1}{8}N(9\xi - 17)\frac{1}{\hat{\varepsilon}_b}\right). \tag{6.9}$$

This choice corresponds to normalization conditions of type (2.13). The singular part of these counterterms (cf. eqs. (A.8) and (A.9)) coincide with those found in the MS-scheme in the vacuum theory [37], even though eq. (6.3) is the result of a high-T expansion. Eqs. (6.6)–(6.9) extend previous calculations by Nadkarni [8], who used an entirely different computation technique.

We now arrive at an extremely simple form of the renormalized static effective action

$$\Gamma^{3D} = \int d^3x \left\{ \mathscr{L}_{cl}^{3D} + \tfrac{1}{2}m^2(T)(A^4)^2 + \frac{g^4 T}{24\pi^2}\left[\left(1 - \frac{N_f}{N}\right)\mathrm{tr}(A^4)^4 + \frac{3}{2}\frac{N_f}{N^3}\left(\mathrm{tr}(A^4)^2\right)^2\right]\right\} \tag{6.10}$$

To exhibit the diagrammatic origin of (6.5), we rewrote this expression in terms of $A = A^a T^a_{ad}$, so that the trace is in the adjoint representation of SU(N) [37]. The power of BRS invariance shows up here, as the only induced terms in eq. (6.10) are the ones absent in the classical action (6.1). A result equivalent to eq. (6.10) has earlier been found by Nadkarni*, who calculated in the static temporal gauge, and used altogether different computation methods. The agreement is remarkable, because the induced quartic term is not a priori gauge independent (it is, of course, invariant under static gauge transformations!).

We note that the quartic term in eq. (6.10) becomes negative for N_f large enough. However, in ordinary QCD ($N = 3$, $N_f = 6$, i.e. for temperatures much higher than the topquark mass) its coefficient is now positive in contrast to QED. The quartic term still comes entirely from the regulator diagram (cf. subsect. 5.3), but now the gluons contribute with a negative sign themselves, thus overpowering the fermionic efforts to give the term its QED-like form. This mechanism is, of course, familiar from the β-function.

Indeed, the β-function defined by the finite-temperature renormalization group à la eq. (2.16) is easily found from eqs. (6.2), (6.6) and (6.9). The running coupling $\alpha(T) = g^2(T)/4\pi$ in the MD-renormalization scheme then assumes the familiar form

$$\alpha(T) = \left[c \log T^2/\Lambda_T^2\right]^{-1} \tag{6.11}$$

in terms of an RG-invariant scale $\Lambda_T^2 = T_0^2 \exp(-1/\alpha(T_0)c)$, with $c = (11N - 2N_f)/12\pi$. Note that the closing remark of sect. 3.3 could be repeated almost wordly here. The behaviour of the running coupling in the context of sect. 2.3 has been investigated in detail in the real-time formalism [40,41].

The attractive appearance of the reduced effective action (6.10) should not obscure the fact that its validity is strictly limited to the one-loop approximation, in sharp contrast to its QED analogue (5.5). In the next order a "magnetic mass" (cf. the end of sect. 5.1), is expected to appear [14], and even if it does not appear in naive perturbation theory the magnetic mass gap will be generated nonperturbatively [13,35]. In any case, a consistent perturbation scheme in the static 3D-theory should dress its bare propagators with a magnetic mass as well as with its (chromo)-electric analogue already present in eq. (6.10). This means, however, that the corrections to DR in eq. (4.9) enter, i.e. on the indicator Δ of subsect. 4.3 does not vanish, and DR holds only up to a given order in perturbation theory. This order can easily be determined, for each Green function separately, by the technique of subsect. 4.5. Notice that neither the electric nor the magnetic "mass" has a direct physical meaning in QCD; both are parameters in the 3D-effective action whose T-dependence is governed by the finite-temperature renormalization group based on normalization conditions à la eq. (2.13) which have no gauge-invariant meaning. The "masses" in the dressed gluon propagator are thus gauge-dependent, and have

* His published result [11,39] is correct only for $N = 2, 3$, but unpublished calculations agree with eq. (6.5). In fact, a private communication with Nadkarni allowed us to correct an error due to a bug in the Asymm command of SCHOONSCHIP in its M68000 implementation.

no other function but parametrizing a particular perturbation method. In the full four-dimensional thermal theory they are not even well-defined [42].

To close this section we wish to comment on a recent paper in which DR in QCD has been studied in great detail [43]. The author concludes that quarks do not decouple at infinite temperature, in agreement with our general results, but that nonstatic gluons do. The latter conclusion is a consequence of a different treatment of the induced static gluon mass coming from nonstatic gluon loops compared to the one deriving from quark loops. The former is correctly resummed into the gluon propagator, whereas the latter is treated as a counterterm. This procedure leads to infrared divergences in higher orders, which are "cured" by the insertion of a T-independent infrared-cutoff which comes out of the blue. The infinite-temperature limit is then taken in the cutoff theory, after which the nonstatic gluons turn out to decouple. Treating gluon loops in the gluon self-energy on the same footing as quark loops, that is, resumming the mass coming from gluon loops into the gluon propagator as well, would conform with the role the induced mass plays in the context of the finite-T renormalization group (cf. sect. 2), and would obviate the need for a totally arbitrary infrared-cutoff. Since the general techniques used in ref. [43] are quite correct, following the last-mentioned procedure would have led to the conclusion that neither quarks nor nonstatic gluons decouple to all orders.

7. Conclusions

The principal message of this paper is that the problem of dimensional reduction at high temperature ought to be studied in the context of the finite-temperature renormalization group. This is nothing spectacular, and just extrapolates the correct setting of the ordinary (Appelquist–Carazzone) decoupling theorem. The precise meaning of T-dependent masses and coupling constants may be determined in this context, after which a simple criterion for complete dimensional reduction can be formulated.

Applying this criterion to specific models leads to the conclusion that dimensional reduction to all orders only takes place under exceptional circumstances, which are realized in QED in $D \leqslant 4$, and also in supersymmetric theories. What is obstructive to general dimensional reduction is essentially a relic of the hierarchy problem, which is enhanced at finite temperature because any field then contains scalar modes. In vacuum field theory small masses can be protected against growing heavy due to quantum corrections by fine-tuning the bare parameters of the theory; this, admittedly unattractive, possibility makes the Appelquist–Carazzone theorem work for scalar fields. On the other hand, large T-dependent masses, as generated by the finite-temperature renormalization group, are simply there, and cannot be removed by fine-tuning parameters which by definition are T-independent. The presence of these large masses blocks dimensional reduction in non-exceptional circumstances.

In the literature dimensional reduction at high T has mainly been used to simplify the study of QCD [1,8,9,11,39,43,44]. The implicit assumption here is that

QCD undergoes dimensional reduction to all orders in naive perturbation theory, after which the effective three-dimensional theory should be studied nonperturbatively. Such a study would then, of course, be of great relevance for the full theory $(\mathrm{QCD})_4$ at high but finite temperatures. Nonperturbative phenomena in $(\mathrm{QCD})_3$, like mass generation or gauge symmetry breaking [39] would necessarily occur in (QCD_4) at high temperature as well.

In our opinion, the results of this paper imply that the above programme is mortally flawed. Dimensional reduction in QCD does *not* take place to all orders in perturbation theory, not even at strictly infinite temperature. It holds to a given, low order in perturbation theory, so that an extrapolation of nonperturbative features of the reduced to the full theory cannot be motivated by calling upon the alleged dimensional reduction process. This is not to say that such extrapolations are meaningless; they simply have to be justified by other means. Indeed, a careful study [13] shows that $(\mathrm{QCD})_3$ is actually more similar to *vacuum* $(\mathrm{QCD})_4$ than to its infinite temperature limit!

Most of the common wisdom on the infrared behaviour of high-temperature QCD has been based on studies of $(\mathrm{QCD})_3$ in the above vein; although this wisdom should not be dismissed lightly, we believe that a full-fledged four-dimensional real-time approach may teach us a great deal more [42]. (For one thing, the decisive role of dissipation is completely obscured in the three-dimensional theory.)

We wish to add here that dimensional reduction, to the extent that it *does* occur, is of little help in the evaluation of essential thermal observables like the energy density, transport coefficients, and correlation functions of certain composite operators.

It would be interesting to find out what the results of this paper mean in the rather different context of dimensional reduction in the Kaluza–Klein sense. One-loop quantum corrections have been studied [45, 46], but to our knowledge no general treatment exists. As we have seen, one-loop results are rather atypical here. The important issue is clearly whether the bare parameters of the higher-dimensional theory are allowed to depend on the final compactification length(s). If not, mass generation due to nonstatic loops should be avoided at all cost, and this may lead to the conclusion that only supersymmetric theories can compactify in a satisfactory manner (this is well-known [47] but usually comes from the requirement that the "low-energy" theory avoids the hierarchy problem, rather than from the demand that the compactification itself takes place at all).

In this context it may also be worthwhile to remark that eq. (6.10) obviously demonstrates that quantum corrections may induce a nonabelian Higgs sector even if only *one* dimension is compactified. At the classical level a compactification of at least *two* dimensions is needed to achieve this [48].

The Master's thesis of E. Koopman was of significant help in the preparation of this paper. I wish to thank S. Nadkarni and R. Alvarez-Estrada for correspondence and discussions. I am indebted to Ch.G. van Weert for a critical reading of the manuscript. Finally, I acknowledge critical remarks by B. de Wit which led to the present version of subsect. 4.1.

Appendix

In this appendix certain details of our summation technique are uncovered. We have to evaluate dimensionless expressions of the type

$$I(y^2;\alpha,\beta,\rho)=2\tilde{T}^{-2\varepsilon}\sum_{n=0}^{\infty}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(k^2)^{\alpha}}{\left[k^2+(n+\rho)^2+y^2\right]^{\beta}},\tag{A.1}$$

where α and β are positive integers, $\tilde{T}=2\pi T/v$, $\rho=1$ for bosons and $\rho=\frac{1}{2}$ for fermions, and $D=3-2\varepsilon$, $\varepsilon>0$. Adapting a well-known formula in dimensional regularization [37] to the three-dimensional case, it follows that

$$I(y^2;\alpha,\beta,\rho)=\pi^{-1/2}(4\pi)^{\varepsilon-1}\frac{\Gamma(\alpha+\frac{3}{2}-\varepsilon)\Gamma(\beta-\alpha-\frac{3}{2}+\varepsilon)}{\Gamma(\beta)\Gamma(\frac{3}{2}-\varepsilon)}$$

$$\cdot S(y^2;\beta-\alpha-\tfrac{3}{2}+\varepsilon,\rho),\tag{A.2}$$

with

$$S(y^2;\lambda,\rho)=\sum_{n=0}^{\infty}\left[y^2+(n+\rho)^2\right]^{-\lambda}.\tag{A.3}$$

This sum is defined as it stands for $\lambda>\frac{1}{2}$, i.e. $\varepsilon>\alpha-\beta+2$. If $\beta-\alpha>2$ then one can immediately set $\varepsilon=0$.

To analyze the analytic structure of S as a function of ε, as well as to obtain large- and small-y expansions, we write the sum as a Mellin–Barnes integral

$$S(y^2;\lambda,\rho)=\frac{\pi^{1/2}y^{-2\lambda}}{\Gamma(\lambda)}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}z}{2\pi i}\left(\frac{y}{2}\right)^{z}\zeta(z,\rho)\frac{\Gamma(z)\Gamma(\lambda-z/2)}{\Gamma(z/2+\frac{1}{2})},\tag{A.4}$$

where $\zeta(z,\rho)$ is the Hurwitz zeta-function [49], and $1<c<2\lambda$. To derive eq. (A.4), use consecutively item (B), p. 6/7, and eq. (2.58) of ref. [50], and eqs. (8.6.1), (6.1.18) and (6.1.17) of ref. [51]. It follows from the representation (A.4) that $S(\lambda)$ has poles in $\lambda=\frac{1}{2}-k$, $k=0,1,2,\dots$ (due to the pinch singularity encountered by moving Re λ to the left, so that the contour is pinched between the poles of $\zeta(z,\rho)$ in $z=1$, and $\Gamma(\lambda-\frac{1}{2}z)$). For large y (A.4) generates an asymptotic expansion by shifting the contour to the left and picking up the poles of the integrand (a convergent series in $1/y$ cannot be obtained from (A.4) because the contour cannot be closed to the left [52]). One finds

$$S(y^2; \lambda, \rho) = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda)} y^{-2\lambda+1} + \zeta(0, \rho) y^{-2\lambda} + \dots . \tag{A.5}$$

For general ρ, the terms represented by the dots are $\mathcal{O}(y^{-2\lambda-1})$. For $\rho = 1$, however, we have $\zeta(z, 1) = \zeta(z)$, and a glance at (A.4) combined with $\zeta(-2k) = 0$, $k = 1, 2, \dots$, shows that for the omitted terms vanish faster than any positive power of $1/y$ for $y \to \infty$. For $\rho = \frac{1}{2}$ one has $\zeta(z, \frac{1}{2}) = (2^z - 1)\zeta(z)$, and the same conclusion follows; in fact even the second term in eq. (A.5) vanishes for fermions.

A convergent expansion in y may be obtained by closing the contour to the right. This yields ($y < 1$)

$$S(y^2; \lambda, \rho) = (1/\Gamma(\lambda)) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(2k + 2\lambda, \rho)\Gamma(\lambda + k) y^{2k}. \tag{A.6}$$

Combining (A.6) with (A.1) and taking the limit $\varepsilon \to 0$ then gives a series expansion for I. We first state the result for the singular case $\beta - \alpha \leqslant 2$:

$$I(y^2; \alpha, \beta, \rho) = \frac{\Gamma(\alpha + \frac{3}{2})}{2\pi^2(\beta - 1)!}$$

$$\times \left\{ (-1)^{\alpha-\beta} \frac{\pi^{1/2} y^{4+2\alpha-2\beta}}{2(\alpha - \beta + 2)!} \left(\frac{1}{\varepsilon} - 2\psi(\rho) - \gamma - \log \frac{4\pi T^2}{\nu^2} \right) \right.$$

$$\left. + \sum_{k=0}^{\infty}{}' \frac{(-1)^k}{k!} \Gamma(k + \beta - \alpha - \tfrac{3}{2}) \zeta(2k + 2\beta - 2\alpha - 3, \rho) y^{2k} \right\}. \tag{A.7}$$

The prime on the summation sign denotes that the term $k = 2 + \alpha - \beta$ must be omitted. For bosons $\psi(1) = -\gamma$, while for fermions $\psi(\frac{1}{2}) = -\gamma - \log 4$, so that the pole terms become

$$\frac{1}{\hat{\varepsilon}_b} = \frac{1}{\varepsilon} + \gamma - \log \frac{4\pi T^2}{\nu^2}, \qquad \frac{1}{\hat{\varepsilon}_f} = \frac{1}{\varepsilon} + \gamma - \log \frac{\pi T^2}{4\nu^2} \qquad \text{(A.8), (A.9)}$$

for bosons and fermions, respectively. For $\beta - \alpha > 2$ we simply obtain

$$I(y^2; \alpha, \beta, \rho) = \frac{\Gamma(\alpha + \frac{3}{2})}{2\pi^2(\beta - 1)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma(k + \beta - \alpha - \tfrac{3}{2})$$

$$\times \zeta(2k + 2\beta - 2\alpha - 3, \rho) y^{2k}. \tag{A.10}$$

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