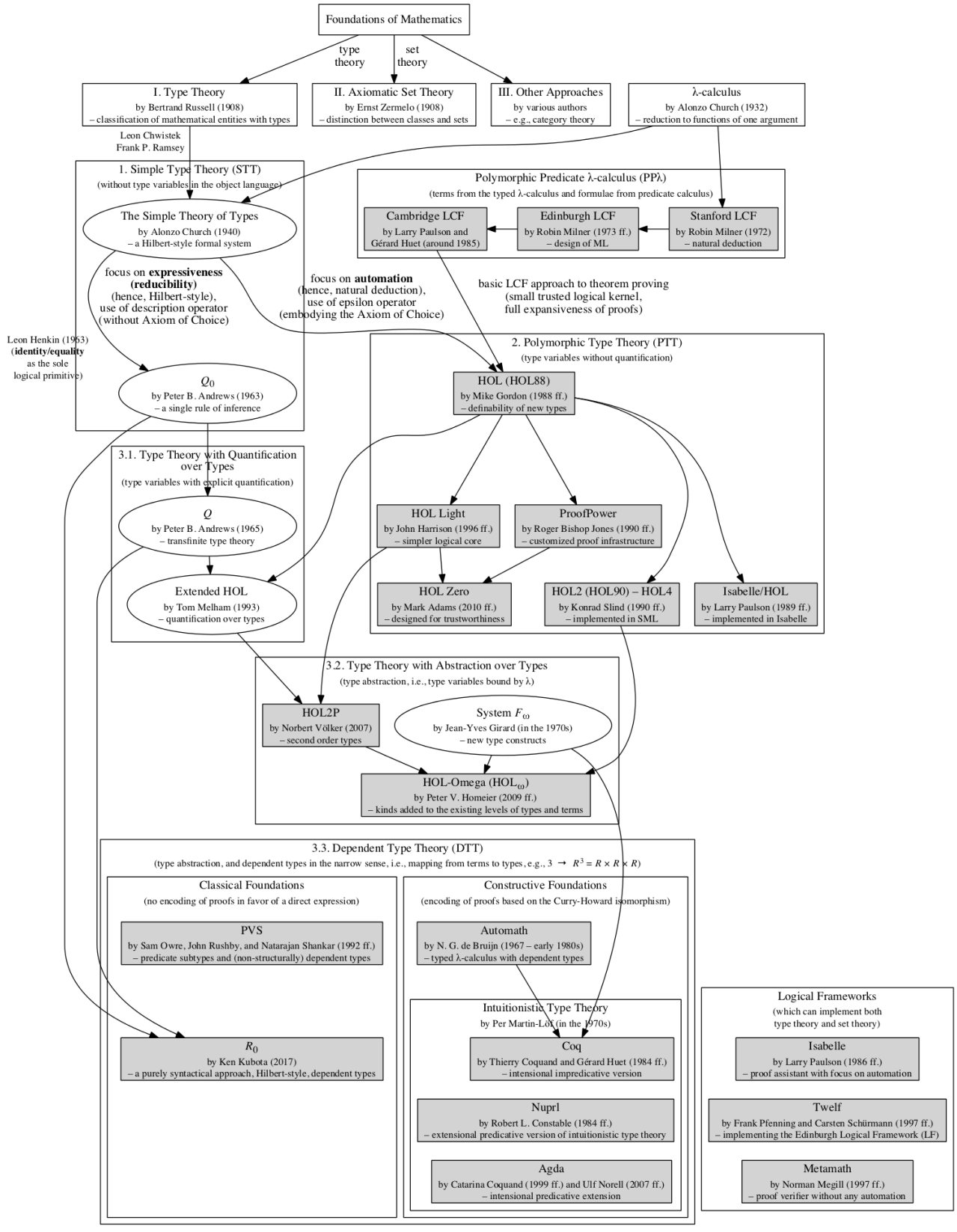
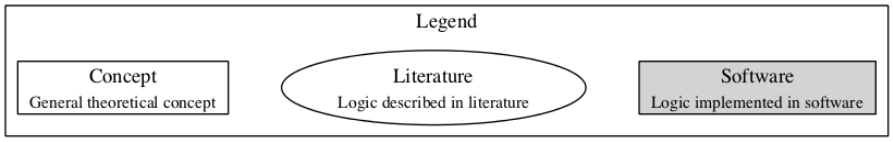

Foundations of Mathematics
type theory
set theory
I. Type Theory
by Bertrand Russell (1908)
– classification of mathematical entities with types
II. Axiomatic Set Theory
by Ernst Zermelo (1908)
– distinction between classes and sets
III. Other Approaches
by various authors
– e.g., category theory
λ-calculus
by Alonzo Church (1932)
– reduction to functions of one argument
Leon Chwistek
Frank P. Ramsey
1. Simple Type Theory (STT)
(without type variables in the object language)
The Simple Theory of Types
by Alonzo Church (1940)
– a Hilbert-style formal system
Polymorphic Predicate λ-calculus (PPλ)
(terms from the typed λ-calculus and formulae from predicate calculus)
Cambridge LCF
by Larry Paulson and Gérard Huet (around 1985)
Edinburgh LCF
by Robin Milner (1973 ff.)
– design of ML
Stanford LCF
by Robin Milner (1972)
– natural deduction
focus on expressiveness (reducibility) (hence, Hilbert-style), use of description operator (without Axiom of Choice)
focus on automation (hence, natural deduction), use of epsilon operator (embodying the Axiom of Choice)
basic LCF approach to theorem proving (small trusted logical kernel, full expansiveness of proofs)
Leon Henkin (1953) (identity/equality as the sole logical primitive)
Q_0
by Peter B. Andrews (1963)
– a single rule of inference
2. Polymorphic Type Theory (PTT)
(type variables without quantification)
HOL (HOL88)
by Mike Gordon (1988 ff.)
– definability of new types
3.1. Type Theory with Quantification over Types
(type variables with explicit quantification)
Q
by Peter B. Andrews (1965)
– transfinite type theory
Extended HOL
by Tom Melham (1993)
– quantification over types
HOL Light
by John Harrison (1996 ff.)
– simpler logical core
ProofPower
by Roger Bishop Jones (1990 ff.)
– customized proof infrastructure
HOL Zero
by Mark Adams (2010 ff.)
– designed for trustworthiness
HOL2 (HOL90) – HOL4
by Konrad Slind (1990 ff.)
– implemented in SML
Isabelle/HOL
by Larry Paulson (1989 ff.)
– implemented in Isabelle
3.2. Type Theory with Abstraction over Types
(type abstraction, i.e., type variables bound by λ)
HOL2P
by Norbert Völker (2007)
– second order types
System F_ω
by Jean-Yves Girard (in the 1970s)
– new type constructs
HOL-Omega (HOL_ω)
by Peter V. Homeier (2009 ff.)
– kinds added to the existing levels of types and terms
3.3. Dependent Type Theory (DTT)
(type abstraction, and dependent types in the narrow sense, i.e., mapping from terms to types, e.g., 3 → $R^3 = R \times R \times R$)
Classical Foundations
(no encoding of proofs in favor of a direct expression)
PVS
by Sam Owre, John Rushby, and Natarajan Shankar (1992 ff.)
– predicate subtypes and (non-structurally) dependent types
R_0
by Ken Kubota (2017)
– a purely syntactical approach, Hilbert-style, dependent types
Constructive Foundations
(encoding of proofs based on the Curry-Howard isomorphism)
Automath
by N. G. de Bruijn (1967 – early 1980s)
– typed λ-calculus with dependent types
Intuitionistic Type Theory
by Per Martin-Löf (in the 1970s)
Coq
by Thierry Coquand and Gérard Huet (1984 ff.)
– intensional impredicative version
NuprI
by Robert L. Constable (1984 ff.)
– extensional predicative version of intuitionistic type theory
Agda
by Catarina Coquand (1999 ff.) and Ulf Norell (2007 ff.)
– intensional predicative extension
Logical Frameworks
(which can implement both type theory and set theory)
Isabelle
by Larry Paulson (1986 ff.)
– proof assistant with focus on automation
Twelf
by Frank Pfenning and Carsten Schürmann (1997 ff.)
– implementing the Edinburgh Logical Framework (LF)
Metamath
by Norman Megill (1997 ff.)
– proof verifier without any automation



Contents

1 Literature and Links

0. Main Approaches

   I. Type Theory: [Russell, 1908] (reprint [Russell, 1967b])
   II. Set Theory: [Zermelo, 1908] (English translation [Zermelo, 1967])

1. Simple Type Theory

   - The Simple Theory of Types: [Church, 1940]
   - $\mathcal{Q}_0$: [Andrews, 2002, pp. 210–215; first edition Andrews, 1986, pp. 161–165; very first publication in Andrews, 1963, pp. 345 f., 350]. Elementary logic is developed without any appeal to the Axiom of Choice, which may be assumed later [Andrews, 2002, p. 236]. Adding an Axiom of Infinity (as Axiom 6) for number theory yields $\mathcal{Q}_0^\infty$ [Andrews, 2002, p. 259]. A short description [Andrews, 2014a] is available online [link].
     An implementation in Metamath (by Thierry Arnoux): [link to repository] (and [link to mailing list announcement])
     An implementation of a natural deduction variant (by Cris Perdue): http://prooftoys.org (and http://mathtoys.org)

2. Polymorphic Type Theory

   - HOL: [Gordon, 2001] (revised version of the 1985 paper), [Gordon and Melham, 1993].[1] The part on the logic written by Andrew Pitts[2] is available online [link] (with minor modifications).
     For a historical account on the HOL family, see [Gordon, 2000].
   - Isabelle/HOL: [Paulson, 1989b], http://www.cl.cam.ac.uk/research/hvg/Isabelle/
   - HOL4: [Slind and Norrish, 2008], http://hol-theorem-prover.org
   - ProofPower: [Arthan and Jones, 2005], http://www.lemma-one.com/ProofPower/index/
   - HOL Light: [Harrison, 2009],[3] http://www.cl.cam.ac.uk/~jrh13/hol-light/
   - HOL Zero: [Adams, 2010], http://www.proof-technologies.com/holzero/

[1] Further definitional principles are proposed and discussed in [Kumar et al., 2014], [Arthan, 2014], and [Arthan, 2016].
[2] Part III: The HOL Logic [Gordon and Melham, 1993, pp. 191–232]. "Part III presents the logic supported by HOL ('The HOL Logic'). It consists of two chapters: an informal introduction and a formal set-theoretic semantics (written by Andrew Pitts)." [Gordon and Melham, 1993, p. xv] In an email to the author, Andrew Pitts confirmed that originally he wrote the material in the file available online, which contains both (!) chapters.
[3] Further motivation for HOL Light is explained in [Harrison, 1995].

3. Type Theory with bound Type Variables and Dependent Type Theory
    1. Type Theory with Quantification over Types
        - $\mathcal{Q}$: [Andrews, 1965]
        - Extended HOL: [Melham, 1993b]
    2. Type Theory with Abstraction over Types
        - F_ω: [Girard, 1986][4]
        - HOL2P: [Völker, 2007], http://cswww.essex.ac.uk/staff/norbert/hol2p/
        - HOL-Omega (HOL_ω): [Homeier, 2009], http://www.trustworthytools.com/id17.html
    3. Dependent Type Theory
        1. Classical Foundations
            - PVS: [Owre, Rushby, and Shankar, 1992], http://pvs.csl.sri.com
            - $\mathcal{R}_0$: [Kubota, 2017], http://doi.org/10.4444/100.10
        2. Constructive Foundations
            - Automath: [Daalen, 1994],[5] http://www.win.tue.nl/automath/
              A modern reimplementation (by Freek Wiedijk): [Wiedijk, 2002], http://www.cs.ru.nl/~freek/aut/
            - Coq: [Coquand and Huet, 1985], [Coquand and Huet, 1988], [Coquand and Paulin, 1990], http://coq.inria.fr
            - Nuprl: [Constable, 1986], http://www.nuprl.org
            - Agda: [Norell, 2007],[6] http://wiki.portal.chalmers.se/agda/[7]

Logical frameworks can implement both type-theoretic and set-theoretic logics:[8]

- Isabelle: [Paulson, 1988][9], http://www.cl.cam.ac.uk/research/hvg/Isabelle/
- Twelf: [Pfenning and Schürmann, 1999], http://twelf.org
- Metamath: [Megill, 2007], http://metamath.org

Please note that logics and their implementation below the expressiveness of higher-order logic (e.g., first-order logic) are not considered.

For further overviews with an excellent discussion, including a comparison with set-theoretic approaches such as Mizar, see [Wiedijk, 2003] [online link] and [Wiedijk, 2007] [online link].

For a general overview about type theory and set theory, see the articles in the *Stanford Encyclopedia of Philosophy* on type theory [Coquand, 2015] [online link] and set theory [Bagaria, 2015] [online link].

A valuable resource containing many important historical mathematical papers is [Heijenoort, 1967a].

[4] For an introduction, see also chapter 30 of [Pierce, 2002, pp. 449 ff.].

[5] "The standard reference about Automath is [Nederpelt, Geuvers, and Vrijer, 1994]. This is a compilation of almost all important Automath publications. It contains as (A.3) a good introduction to Automath [Daalen, 1994]. The main important Automath paper that is missing from this collection is the paper [de Bruijn, 1991] about telescopes." [Wiedijk, 2002, p. 365]

[6] "The particular choice of type theory is not crucial and the theory we choose is roughly Luo's UTT [Luo, 1994] extended with Σ-types and η-laws." [Norell, 2007, p. 14]

[7] The current version is Agda 2. For Agda version 1, see: http://ocvs.cfv.jp/Agda/.

[8] The large field of logical frameworks cannot be unfolded here in detail. The reader is referred to further resources available online [link].

[9] Also available in [Paulson, 1989a].

Historically, "[t]he original LCF team at Stanford consisted of Robin Milner, assisted by Whitfield Diffie" [Gordon, 2000, p. 169] – now known for the Diffie-Hellman (DH) key exchange in cryptography. Both Rob Arthan and Roger Bishop Jones generously credit each other for having the main rôle in the creation of ProofPower.

Neither the graph nor the above list are comprehensive. Further logics are William M. Farmer's $\mathcal{Q}_0^u$ and $\mathcal{Q}_0^{uqe}$ (extensions of Andrews' $\mathcal{Q}_0$ with undefinedness). Further type-theoretic systems are TPS by Peter B. Andrews et al. and IMPS (with partial functions and subtypes) by William M. Farmer et al., and a well-known set-theoretic system is *Mizar* by Andrzej Trybulec.

- $\mathcal{Q}_0^u$: [Farmer, 2014]
- $\mathcal{Q}_0^{uqe}$: [Farmer, 2015]
- TPS: http://gtps.math.cmu.edu/tps.html
- IMPS: http://imps.mcmaster.ca
- Mizar: http://mizar.org

A more comprehensive list with descriptions is available at: http://www.nuprl.org/Intro/others.html.

For a survey on interactive theorem proving, see [Harrison, Urban, and Wiedijk, 2014], which includes a description of the recently completed formalization of the proof of the *Kepler conjecture*. Despite the currently limited expressiveness of the formal language (lacking explicit quantification over type variables and dependent types[10]), in 2014, theorem provers verified this proof at the front of mathematical research established by Thomas Hales together with Ferguson in 1998 and published in full in 2006.[11]

2 Design Decisions

1. Alonzo Church on the *reduction to functions of one argument*: "Adopting a device due to Schönfinkel, we treat a function of two variables as a function of one variable whose values are functions of one variable, and a function of three or more variables similarly." [Church, 1932, p. 352]

2. Alonzo Church *introducing λ-notation*: "If **M** is any formula containing the variable **x**, then $\lambda\mathbf{x}[\mathbf{M}]$ is a symbol for the function whose values are those given by the formula." [Church, 1932, p. 352]

3. Peter B. Andrews on the *description operator* (and Axiom 5 of $\mathcal{Q}_0$): "Henkin remarks at the end of [Henkin, 1963] that when one passes from the theory of propositional types to the full theory of finite types, it becomes necessary to add a constant $\iota_{1(01)}$ to denote a descriptor function, and an appropriate axiom involving this constant. We note that for this axiom it suffices to take the simple formula $\iota_{1(01)}(\lambda x_1(x_1 \equiv y_1)) \equiv y_1$ [...]." [Andrews, 1963, p. 350]

[10] "The theorem prover HOL Light [...] is based on a logic without dependent types, but we can still encode the index N as a type (roughly, an arbitrary indexing type of size N)." [Hales and Harrison, 2010, p. 6]

[11] The computer aided verification became a desire, as the traditional peer review of the proof of about 300 pages and 40,000 lines of additional computer code had to be given up after four years of team efforts. Initiated by Hales in 2006 under the project name *Flyspeck*, the verification was completed successfully in 2014, proving the *Kepler conjecture* from 1611 (forming a part of Hilbert's 18th problem) which "states that no arrangement of identical balls in ordinary 3-dimensional space has a higher packing density than the obvious cannonball arrangement" [Harrison, Urban, and Wiedijk, 2014, p. 192] with a density of $\pi/\sqrt{18}$ ($\approx$ 74 %). "The formal verifications have been carried out in the HOL Light and Isabelle proof assistants, with a second verification of the main statement in HOL Zero." [Hales, 2015, p. 2]

4. Peter B. Andrews on *finding as simple, natural, and expressive a formulation of type theory as possible*: "Chapter 5 introduces a system $\mathcal{Q}_0$ of typed λ-calculus which is essentially the system introduced by Alonzo Church in [Church, 1940], except that [...] equality relations rather than quantifiers and propositional connectives are primitive. [...] The discussion turns to finding as simple, natural, and expressive a formulation of type theory as possible. [...] This leads to an exposition of the basic ideas underlying $\mathcal{Q}_0$." [Andrews, 2002, p. xvi]

5. Peter B. Andrews on *expressiveness* as the main criterion for the design of $\mathcal{Q}_0$: "Therefore we shall turn our attention to finding a formulation of type theory which is as expressive as possible, allowing mathematical ideas to be expressed precisely with a minimum of circumlocutions, and which is as simple and economical as is possible without sacrificing expressiveness. The reader will observe that the formal language we find arises very naturally from a few fundamental design decisions." [Andrews, 2002, pp. 205 f.]

6. Peter B. Andrews on *equality as the basic primitive notion*: "A formulation of type theory based on these ideas was introduced by Church in [Church, 1940], and proved complete by Henkin in [Henkin, 1950]. In this system equality can be defined using connectives and quantifiers. However, it is also possible to define connectives and quantifiers in terms of equality. Equality is a very basic and simple notion, so instead of using Church's formulation of type theory, we shall use a slight variant of it (first introduced in [Henkin, 1963] and simplified in [Andrews, 1963]) in which equality is taken as the basic primitive notion." [Andrews, 2002, p. 208]

7. Peter B. Andrews introducing *ordered pairs* (*without extending the formal language*): "The expression $[\lambda g_{ooo} \,.\, g_{ooo} x_o y_o]$ can be used to represent the ordered pair $\langle x_{[o]}, y_o \rangle$". [Andrews, 2002, p. 208]

8. Peter B. Andrews introducing *induction* as part of the definition of natural numbers (*without extending the formal language*): "The Induction Principle simply limits the size of the set." [Andrews, 2002, p. 259] "We next define the natural numbers. These are equivalence classes of sets of individuals, and so have type $(o(o\iota))$.
DEFINITIONS. Let σ be the type symbol $(o(o\iota))$.
[...]
$\mathbb{N}_{o\sigma}$ stands for $[\lambda n_\sigma \forall p_{o\sigma} \,.\, [p_{o\sigma} 0_\sigma \;\wedge\; \forall x_\sigma \,.\, p_{o\sigma} x_\sigma \supset p_{o\sigma} \,.\, S_{\sigma\sigma} x_\sigma] \supset p_{o\sigma} n_\sigma]$
[...]
[...] The wff S represents the successor function. [...] $\mathbb{N}_{o\sigma}$ represents the set of natural numbers, i.e., the intersection of all sets which contain 0 and are closed under S." [Andrews, 2002, p. 260]

9. Peter B. Andrews introducing *recursion* by defining a recursion operator (*without extending the formal language*): "Indeed, since f is uniquely determined by h and g, we can define a recursion operator R such that $f = Rhg$." [Andrews, 2002, p. 281]
"DEFINITION. $R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}$ stands for $[\lambda h_{\sigma\sigma\sigma} \lambda g_\sigma \lambda n_\sigma \,\imath\, m_\sigma \forall w_{o\sigma\sigma} \,.\, [w_{o\sigma\sigma} 0_\sigma g_\sigma \;\wedge\; \forall x_\sigma \forall y_\sigma \,.\, w_{o\sigma\sigma} x_\sigma y_\sigma \supset w_{o\sigma\sigma}[S_{\sigma\sigma} x_\sigma] h_{\sigma\sigma\sigma} x_\sigma y_\sigma] \supset w_{o\sigma\sigma} n_\sigma m_\sigma]$.
6400. $\vdash [R_{\sigma\sigma\sigma(\sigma\sigma\sigma)} h_{\sigma\sigma\sigma} g_\sigma] 0_\sigma = g_\sigma \;\wedge\; \dot{\forall} n_\sigma \,.\, [R h_{\sigma\sigma\sigma} g_\sigma][S_{\sigma\sigma} n] = h_{\sigma\sigma\sigma} n_\sigma \,.\, [R h_{\sigma\sigma\sigma} g_\sigma] n_\sigma$"
[Andrews, 2002, p. 282, boldface as in the original] ("We [...] let $[\imath\, \mathbf{z}_\gamma \mathbf{A}]$ stand for $\iota_{\gamma(o\gamma)}[\lambda \mathbf{z}_\gamma \mathbf{A}]$." [Andrews, 2002, p. 234, boldface as in the original]. For the definition of the universal quantifier on numbers $\dot{\forall}$, see [Andrews, 2002, p. 260].)
"To illustrate the usefulness of R, define $+_{\sigma\sigma\sigma}$ as $R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}[\lambda x_\sigma S_{\sigma\sigma}]$, and $\mathbf{A}_\sigma + \mathbf{B}_\sigma$ as $+_{\sigma\sigma\sigma} \mathbf{A}_\sigma \mathbf{B}_\sigma$. [...] Similarly, $\times_{\sigma\sigma\sigma}$ can be defined as $[\lambda m_\sigma \,.\, R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}[\lambda x_\sigma \,.\, +_{\sigma\sigma\sigma} m_\sigma] 0_\sigma]$, and $\mathbf{A}_\sigma \times \mathbf{B}_\sigma$ as $\times_{\sigma\sigma\sigma} \mathbf{A}_\sigma \mathbf{B}_\sigma$." [Andrews, 2002, p. 284, boldface as in the original]

10. Andrew M. Pitts on the *natural deduction* system HOL in comparison to the *axiomatic (Hilbert-style) deductive system* $\mathcal{Q}_0$: "From a logical point of view, it would be better to have a simpler substitution primitive, such as 'Rule R' of Andrews' logic $\mathcal{Q}_0$, and then to derive more complex rules from it." [Gordon and Melham, 1993, p. 213]

11. Mike Gordon characterizing the *HOL logic* (without mentioning the definability of new types): "The HOL logic is a version of Church's simple theory of types [Church, 1940] modified by:
− allowing types to contain variables (i.e. be *polymorphic*) and
− simplifying the axioms and rules of inference." [Gordon, 2001, p. 4]

12. Mike Gordon on the *relation of Hilbert's epsilon operator to the description operator of Church's logic*: "The constant @ is a higher-order version of Hilbert's ε-operator; it is related to the constant ι in Church's formulation of higher-order logic. For more details, see Leisenring's book [12] and Church's original paper [Church, 1940]." [Gordon, 1988, p. 94]

13. Mike Gordon and Peter B. Andrews defining the conditional using *the epsilon operator vs. the description operator*.
Mike Gordon (HOL): "Many things that are normally primitive can be defined using the ε-operator. For example, the conditional term $\mathsf{Cond}\ t\ t_1\ t_2$ (meaning 'if t then t_1 else t_2') can be defined by:
$\mathsf{Cond}\ t\ t_1\ t_2 \quad = \quad \varepsilon x.((t = \mathsf{T}) \Rightarrow (x = t_1)) \ \wedge \ ((t = \mathsf{F}) \Rightarrow (x = t_2))$" [Gordon, 2001, p. 24]
Peter B. Andrews ($\mathcal{Q}_0$): "DEFINITION: Let $C_{\gamma o \gamma\gamma}$ be
$[\lambda x_\gamma \lambda y_\gamma \lambda p_o \, \imath \, q_\gamma \centerdot [p_o \wedge \centerdot \, x_\gamma = q_\gamma] \vee [\sim p_o \wedge \centerdot \, y_\gamma = q_\gamma]]$
$C_{\gamma o \gamma\gamma} x_\gamma y_\gamma p_o$ can be read 'if p_o then x_γ, else y_γ'.
5313. $\vdash [C_{\gamma o \gamma\gamma} x_\gamma y_\gamma T_o = x_\gamma] \wedge \centerdot \, C_{\gamma o \gamma\gamma} x_\gamma y_\gamma F_o = y_\gamma$" [Andrews, 2002, p. 235, boldface as in the original]
Formal verification of theorem **5313** in the $\mathcal{R}_0$ implementation (with type abstraction):
":= $COND$ ...
... $[\lambda t_\tau.[\lambda x_t.[\lambda y_t.[\lambda p_o.(\iota_{t(ot)}[\lambda q_t.(\vee_{ooo}(\wedge_{ooo} p_o (=_{ott} x_t q_t))(\wedge_{ooo}(\sim_{oo} p_o)(=_{ott} y_t q_t)))_o])_t]_{(to)}]_{(tot)}]_{(tott)}]$
wff 2080 : $[\lambda t.[\lambda x.[\lambda y.[\lambda p.(\iota\,[\lambda q.(\vee\,(\wedge\,p\,(=x\,q))\,(\wedge\,(\sim p)\,(=y\,q))))])]]]]_{\backslash 4o\backslash 3\backslash 2\tau}$ [...]
[...]
$\wedge\,(=(COND\,t\,x\,y\,T)\,x)\,(=(COND\,t\,x\,y\,F)\,y) \quad := \quad A5313$
$\wedge_{ooo}(=_{ott}(COND_{\backslash 4o\backslash 3\backslash 2\tau} t_\tau x_t y_t T_o) x_t)(=_{ott}(COND_{\backslash 4o\backslash 3\backslash 2\tau} t_\tau x_t y_t F_o) y_t)$
:= $A5313$" [Kubota, 2017, pp. 142, 151]

14. Mike Gordon on *explicit type variable quantification* in HOL: "In future versions of HOL it is expected that there will be explicit type variable quantification [Melham, 1993b], i.e., terms of the form $\forall\alpha.t$ (where α is a type variable). The right hand side of definitions will be required to be closed with respect to both term and type variables. Melham has shown that this will make defining mechanisms much cleaner and also permit an elegant treatment of type specifications." [Gordon, 2000, p. 175, fn. 7]

15. Tom Melham on *quantification over type variables at the level of terms (not types)*: "Note that we are not proposing an extension to the type language of HOL – the quantifications $\forall\alpha$.P and $\exists\alpha$.P are new term constructs, and not type constructs of the kind found (for example) in Girard's system F [...]. The extended logic proposed here resembles system $\mathcal{Q}$, a transfinite type theory due to Andrews [Andrews, 1965]. It is, however, still much weaker than Andrews' system." [Melham, 1993b, pp. 8 f., emphasis as in the original]

16. Tom Melham on *the relation of Melham's Extended HOL to Andrews' system* $\mathcal{Q}$: "I have found Andrews' book [Andrews, 1965] invaluable in working out many of the technical details of the

extension to HOL proposed here." [Melham, 1993b, p. 23]

17. Mike Gordon suggesting *type abstraction* for HOL: "The syntax and semantics of type variables are currently being studied by several logicians. A closely related area is the theory of 'second order' λ-terms like $\lambda\alpha.\, \lambda x{:}\alpha.\, x$, perhaps such terms should be included in the HOL logic." [Gordon, 2001, p. 22]

18. John Harrison on the *motivation for HOL Light*: "Part of my objective in HOL Light was to arrive at a simpler and more satisfying logical kernel that could nevertheless be the foundation of a practical tool. In particular HOL Light's foundations address a couple of the issues [mentioned in] some self-criticism [in] the HOL documentation: the complicated primitive substitution rule and [...] the way the epsilon operator is plumbed deep into the foundations. [...]
[...]
In HOL Light the basic logical notions including the quantifiers are all defined independent of the epsilon operator (in contrast to its use even to define 'there exists' in the original HOL). The choice operator is eventually introduced but it is quite late. [...]
[...]
Ultimately, all the quantifiers and connectives in HOL Light are defined in terms of equality alone. In this respect the foundations are reminiscent of one of Andrews'[...] systems, but with the distinction that the definitions are intuitionistically admissible. [...]
[...]
I think there is something quite satisfying about the derivation of higher-order logic from these simple foundations based just on equality in a simple world of functions. [...]" [Harrison, 2016]

19. Larry Paulson on the relevance of *natural deduction* for *automation*: "Church's axiom system is now antiquated, largely dating back to *Principia Mathematica*. There are improved formulations but most use the Hilbert style. Natural deduction is far superior for automated proof." [Paulson, 1989b, p. 3, emphasis as in the original]

20. Sam Owre and Natarajan Shankar on *(non-structurally) dependent types* in PVS: "One very important consequence of the above extension of the universe is that all type dependencies must be bounded [...]. This property is easily proved by induction on the structure of a PVS type [...]. In particular, there is no way to define a type constructor T^n in PVS [...]. If unbounded type dependencies were allowed in PVS, one can construct a dependent type such as $[n : nat \to T^n]$ whose representation is not in U as defined above." [Owre and Shankar, 1999, p. 35, emphasis as in the original] "PVS admits only a very restricted form of type dependency. In a dependent type $T(n)$, the parameter n can occur only within subtype predicates in $T(n)$. This means that the structure of $T(n)$ is invariant with respect to n. All possible ways of introducing type dependencies in PVS preserve this invariant. It follows that there is no way of defining a type $T(n)$, where $T(n)$ is A^n, i.e., the n-tuple over the type A." [Shankar and Owre, 2000, p. 45, emphasis as in the original]

21. Robert L. Constable et al. on choosing a *predicative type structure* for Nuprl: "The type structure hierarchy of Nuprl resembles that of *Principia Mathematica*, the ancestor of all type theories. The hierarchy manifests itself in an unbounded cumulative hierarchy of universes, U_1, U_2, ..., where by cumulative hierarchy we mean that U_i is in U_{i+1} and that every element of U_i is also an element of U_{i+1}. Universes are themselves types, and every type occurs in a universe. In fact, A is a type if and only if it belongs to a universe. Conversely all the elements of a universe are types." [Constable, 1986, p. 5, emphasis as in the original] "The concept of a universe in this role, to organize the hierarchy of types, is suggested in [Artin, Grothendieck, & Verdier 72] and was used by Martin-Löf [Martin-Löf 73]. This is a means of achieving a *predicative* type structure as

opposed to an *impredicative* one as in Girard [Girard 71] or Reynolds [Reynolds 83]." [Constable, 1986, p. 5, fn. 2, emphasis as in the original]

22. Gérard Huet on the *development of* Coq: "The logical language used by Coq is a variety of type theory, called the *Calculus of Inductive Constructions*. [...] The λ-calculus notation, originally used for expressing functionality, could also be used as an encoding of natural deduction proofs. This Curry-Howard isomorphism was used by N. de Bruijn in the *Automath* project, the first full-scale attempt to develop and mechanically verify mathematical proofs. [...] Exploiting this Curry-Howard isomorphism, notable achievements in proof theory saw the emergence of two type-theoretic frameworks; the first one, Martin-Löf's *Intuitionistic Theory of Types*, attempts a new foundation of mathematics on constructive principles. The second one, Girard's polymorphic λ-calculus F_ω, is a very strong functional system in which we may represent higher-order logic proof structures. Combining both systems in a higher-order extension of the Automath languages, T. Coquand presented in 1985 the first version of the *Calculus of Constructions*, CoC. [...] The formalism was extended in 1989 by T. Coquand and C. Paulin with primitive inductive definitions, leading to the current *Calculus of Inductive Constructions*. [...]
[...]
A first implementation of CoC was started in 1984 by G. Huet and T. Coquand. [...]" [Huet, 1995, emphasis as in the original]

23. Diederik van Daalen on *coding* logic using the *propositions-as-types* notion: "Now there are different ways of coding some logic into the objects-and-types framework. Here we only mention a so-called *functional interpretation* of logic, which gives rise to the *propositions-as-types* notion. This idea of interpreting logic was developed independently by de Bruijn and certain others, of whom we mention Howard [Howard, 1980], Prawitz [Prawitz 71], Girard [Girard 72] and Martin-Löf [Martin-Löf 75a]." [Daalen, 1994, p. 111, emphasis as in the original]

24. Ana Bove and Peter Dybjer on *variants of Martin-Löf's intuitionistic type theory*: "The Curry-Howard interpretation was the basis for Martin-Löf's intuitionistic type theory [19,20,21]. In this theory propositions and types are actually identified. Although Martin-Löf's theory was primarily intended to be a foundational system for constructive mathematics, it can also be used as a programming language [20]. From the early 1980's and onwards, a number of computer systems implementing variants of Martin-Löf type theory were built. The most well-known are the NuPRL [Constable, 1986] system from Cornell implementing an extensional version of the theory, and the Coq [32] system from INRIA in France implementing an intensional impredicative version. The Agda system implements an intensional predicative extension of Martin-Löf type theory. It is the latest in a sequence of systems developed in Göteborg." [Bove and Dybjer, 2009, p. 59]

25. Ulf Norell about using *encoding properties of values as types*: "Since dependent types allows types to talk about values, we can encode properties of values as types whose elements are proofs that the property is true. This means that a dependently typed programming language can be used as a logic." [Norell, 2009, p. 230]

26. Thierry Coquand on *direct notation of proofs using the intensional formulation of type theory*: "One advantage of the intensional formulation is that it allows for a direct notation of proofs based on λ-calculus (Martin-Löf 1971 and Coquand 1986)." [Coquand, 2015]

27. Freek Wiedijk introducing the notion of *Pollack-consistency*: "[W]e introduce the notion of *Pollack-consistency*. This property is related to a system being able to correctly parse formulas

that it printed itself. In current systems it happens regularly that this fails." [Wiedijk, 2012, p. 85, emphasis as in the original]

28. Mark Adams introducing the notion of *faithfulness*: "Also note that none of these notions fully address the issue of *faithfulness*, where internal representation and concrete syntax correctly correspond. A printer that printed `false` as `true` and `true` as `false` might be Pollack-consistent but would not be faithful." [Adams, 2016, p. 21, emphasis as in the original]

29. Mark Adams on the *Pollack-consistency* of HOL Zero (not taking into account unpublished $\mathcal{R}_0$): "We believe HOL Zero does not suffer from any incompleteness or ambiguity in its parsers or printers, and printed output can always be parsed back in to give the same internal representation. This would make HOL Zero's parsers and printers well-behaved and Pollack-consistent. As far as we know, this would be a first amongst not only HOL systems, but also various other theorem proving systems that support concrete syntax, such as Coq and Mizar." [Adams, 2016, p. 34]

30. Norman Megill on using the *minimum possible framework* with Metamath (not taking into account unpublished $\mathcal{R}_0$): "Unlike most other systems, Metamath attempts to use the minimum possible framework needed to express mathematics and its proofs. Other systems do not consider that aspect necessarily important, and their underlying computer programs can be large and complex in order to perform mathematical reasoning at a higher level. Metamath's proofs are often quite long compared to those of other systems, but they are completely transparent with nothing hidden from the user. All reasoning is done directly in the proof itself rather than by algorithms embedded in the verification program. Metamath is unique in this sense, offering an alternative approach for those attracted to its philosophy of simplicity." [Megill, 2017a]

31. Norman Megill on *reverse engineering* in mathematical history: "As humans, we observe interesting patterns in these 'meaningless' symbol strings as they evolve from the axioms, and we attach meaning to them. One result is the set of natural numbers, whose properties match those we observe when we count everyday objects, and their extensions to rational and real numbers. Of course, numbers were discovered centuries before set theory, and historically they were 'reversed engineered' back to the axioms of set theory. The proof of $2 + 2 = 4$ shows what was involved in that reverse engineering, representing the work of many mathematicians from Dedekind to von Neumann. At the other extreme of abstraction is the theory of infinite sets or transfinite cardinal numbers. Some of the world's most brilliant mathematicians have given us deep insight into this mysterious and wondrous universe, which is sometimes called 'Cantor's paradise.' " [Megill, 2017b]

32. Norman Megill on *eliminating the concepts of "free variable", "bound variable", and "proper substitution"* in Metamath: "In technical terms that logicians understand, we eliminate the cumbersome concepts of 'free variable,' 'bound variable,' and 'proper substitution' as primitive notions. These concepts are present in our system but are defined in terms of concepts expressed by the axioms and can be eliminated in principle. In standard systems, these concepts are really like additional, implicit axioms that are somewhat complex and cannot be eliminated." [Megill, 2007, p. 32]

3 Criticism

1. Mike Gordon on the use of the *epsilon operator* in HOL: "It must be admitted that the ε-operator looks rather suspicious." [Gordon, 2001, p. 24] "The inclusion of ε-terms into HOL 'builds in' the Axiom of Choice [...]." [Gordon, 2001, p. 24]

2. Freek Wiedijk on the *(current) HOL family*: "There is one important reason why the HOLs are not yet attractive enough to be taken to be the QED system:

   - The HOL type system is too poor. As we already argued in the previous section, it is too weak to properly do abstract algebra.
     But it is worse than that. In the HOL type system there are no *dependent types*, nor is there any form of *subtyping*. (Mizar and Coq both have dependent types and some form of subtyping. In Mizar the subtyping is built into the type system. In Coq a similar effect is accomplished through the use of automatic *coercions*.)
     For formal mathematics it is essential to both have dependent types and some form of subtyping." [Wiedijk, 2007, p. 130, emphasis as in the original]

3. Freek Wiedijk on the *(current) Coq system*: "There are two important reasons why Coq is not yet attractive enough to be taken to be the QED system:

   - The foundations of Coq are too complicated. They are baroque to say the least. Maybe they are even *beyond* baroque. They might even be called *rococo*.
     There is no paper in which the foundations of Coq are spelled out in full mathematical precision. [...]
     [...]
     Also, the foundations of Coq are sufficiently complicated that they are tinkered with, and therefore change between versions of the system.
   - As already noted in the previous section, Coq is not designed for classical mathematics, which means that doing classical & extensional mathematics in it is not as easy as one would like it to be." [Wiedijk, 2007, p. 130, emphasis as in the original]

4. John Harrison, Josef Urban, and Freek Wiedijk on the *(current) HOL family*: "Another limitation of the simple HOL type system is that there is no explicit quantifier over polymorphic type variables, which can make many standard results like completeness theorems and universal properties awkward to express, though there are extensions with varying degrees of generality that fix this issue [Melham, 1993a; Völker, 2007; Homeier, 2009]. Inflexibilities of these kinds certainly arise in simple type theories, and it is not even clear that more flexible *dependent* type theories (where types can be parametrized by terms) are immune. For example, in one of the most impressive formalization efforts to date [Gonthier *et al.*, 2013] the entire group theory framework is developed in terms of subsets of a single universe group, apparently to avoid the complications from groups with general and possibly heterogeneous types." [Harrison, Urban, and Wiedijk, 2014, pp. 170 f., emphasis as in the original]

5. John Harrison, Josef Urban, and Freek Wiedijk on the *(current) Coq system*: "Indeed, there does not seem to be any precise written specification of Coq's current foundations, other than the actual code." [Harrison, Urban, and Wiedijk, 2014, p. 188, fn. 29]

6. Peter B. Andrews on *a false soundness assertion in Henkin's article on completeness*: "As shown in [Andrews, 1972b], there is a nonstandard general model for $\mathscr{C}$ in which the Axiom of Extensionality $\forall x_\beta(f_{\alpha\beta}x_\beta = g_{\alpha\beta}x_\beta) \supset (f_{\alpha\beta} = g_{\alpha\beta})$ is not valid, since the sets in this model are so sparse that the denotation of the defined equality formula $\mathscr{Q}_{o\alpha\alpha}$ is not the actual equality relation. Thus, Theorem 2 of [Henkin, 1950] (which asserts the completeness and soundness of $\mathscr{C}$) is technically incorrect. The apparently trivial soundness assertion is false." [Andrews, 2014b, p. 70] (Note that [Andrews, 1972b] is based on the preceding [Andrews, 1972a].)

4 Historical Notes

1. Jean van Heijenoort on the *discovery of Russell's paradox* in 1901: "Bertrand Russell discovered what became known as the Russell paradox in June 1901 (see *1944*, p. 13). In the letter below, written more than a year later and hitherto unpublished, he communicates the paradox to Frege. The paradox shook the logicians' world, and the rumbles are still felt today.
The Burali-Forti paradox, discovered a few years earlier, involves the notion of ordinal number; it seemed to be intimately connected with Cantor's set theory, hence to be the mathematicians' concern rather than the logicians'. Russell's paradox, which makes use of the bare notions of set and element, falls squarely in the field of logic. The paradox was first published by Russell in *The principles of mathematics* ([Russell, 1903]) and is discussed there in great detail (see especially pp. 101–107). After various attempts, Russell considered the paradox solved by the theory of types ([Russell, 1908]). Zermelo [...] states that he had discovered the paradox independently of Russell and communicated it to Hilbert, among others, prior to its publication by Russell." [Heijenoort, 1967b, p. 124, emphasis as in the original]

2. Bertrand Russell in his letter to Frege on the *discovery of Russell's paradox*: "There is just one point where I have encountered a difficulty. You state [...] that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves." [Russell, 1967a, pp. 124 f., English translation of the letter written in German in 1902]

3. Bertrand Russell *introducing the notion of type*: "A class as one, we shall say, is an object of the same *type* as its terms [...]. [...] [T]he class as many is of a different type from the terms of the class [...]. [...] It is the distinction of logical types that is the key to the whole mystery." [Russell, 1903, pp. 104 f., emphasis as in the original]

4. Willard Van Orman Quine on the *development of type theory*: "It was in June 1901 that Russell discovered the paradox of the class of all classes that do not contain themselves as elements. He communicated it to Frege on 16 June 1902 [...]. Discussing 'the Contradiction', as he calls it, in *The principles of mathematics* ([Russell, 1903]) in a passage probably written in 1901, he mentions, without much elaboration, that 'the class as many is of a different type from the terms of the class' and that 'it is the distinction of logical types that is the key to the whole mystery'. The solution to the problem is presented in less than thirty lines (§ 104). [...] But before the volume came out (the preface is dated December 1902 and the volume 1903), Russell felt that the subject deserved more attention. He wrote Appendix B, of almost six pages, where the doctrine of types is put forward 'tentatively', since 'it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties'. At that time Russell knew, of course, of other paradoxes, for instance the Burali-Forti paradox and that of the greatest cardinal." [Quine, 1967, p. 150, emphasis as in the original]

5. Bertrand Russell describing *Russell's paradox*:
"I.
The Contradictions.
[...]
[...] Let w be the class of all those classes which are not members of themselves. Then, whatever class x may be, 'x is a w' is equivalent to 'x is not an x.' Hence, giving to x the value w, 'w is a

w' is equivalent to 'w is not a w.' " [Russell, 1908, p. 222, emphasis as in the original; see also Russell, 1967b, p. 153]

6. Bertrand Russell on *self-reference as common characteristic of paradoxes* (not mentioning negation/negativity): "In all the above contradictions (which are merely selections from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness." [Russell, 1908, p. 224; see also Russell, 1967b, p. 154]

7. Bertrand Russell on *making a restriction (condition, dependency) syntactically explicit (and verbally implicit)* as essential for the development of type theory: "The difficulty which besets attempts to restrict the variable is, that restrictions naturally express themselves as hypotheses that the variable is of such or such a kind, and that, when so expressed, the resulting hypothetical is free from the intended restriction. For example, let us attempt to restrict the variable to *men*, and assert that, subject to this restriction, 'x is mortal' is always true. Then what is always true is that if x is a man, x is mortal; and this hypothetical is true even when x is not a man. Thus a variable can never be restricted within a certain range if the propositional function in which the variable occurs remains significant when the variable is outside that range. But if the function ceases to be significant when the variable goes outside a certain range, then the variable is *ipso facto* confined to that range, without the need of any explicit statement to that effect. This principle is to be borne in mind in the development of logical types, to which we shall shortly proceed." [Russell, 1908, pp. 234 f., emphasis as in the original; see also Russell, 1967b, p. 162]

8. Bertrand Russell on the *vicious-circle principle* (not mentioning negation/negativity):
"IV.
The Hierarchy of Types.
[...] The division of objects into types is necessitated by the reflexive fallacies which otherwise arise. These fallacies, as we saw, are to be avoided by what may be called the 'vicious-circle principle;' *i. e.,* 'no totality can contain members defined in terms of itself.' " [Russell, 1908, pp. 236 f., emphasis as in the original; see also Russell, 1967b, p. 163]

9. Jean van Heijenoort on the *development of type theory and set theory*: "In spite of the great advances that set theory was making, the very notion of set remained vague. The situation became critical after the appearance of the Burali-Forti paradox and intolerable after that of the Russell paradox, the latter involving the bare notions of set and element. One response to the challenge was Russell's theory of types [...]. Another, coming at almost the same time, was Zermelo's axiomatization of set theory. The two responses are extremely different; the former is a far-reaching theory of great significance for logic and even ontology, while the latter is an immediate answer to the pressing needs of the working mathematician." [Heijenoort, 1967c, p. 199]

10. Ernst Zermelo on *establishing set theory*: "Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions 'number', 'order', and 'function', taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. At present, however, the very existence of this discipline seems to be threatened by certain contradictions, or 'antinomies', that can be derived from its principles—principles necessarily governing our thinking, it seems—and to which no entirely satisfactory solution has yet been found. In particular, in view of the 'Russell antinomy' ([Russell, 1903], pp. 101–107 and 366–368) of the set of all sets that do not contain themselves as elements, it no longer seems admissible today to assign to an arbitrary logically definable notion a set, or class, as its extension. Cantor's original definition of a set (*1895*) as 'a collection, gathered into a whole, of

certain well-distinguished objects of our perception or our thought' therefore certainly requires some restriction; it has not, however, been successfully replaced by one that is just as simple and does not give rise to such reservations. Under these circumstances there is at this point nothing left for us to do but to proceed in the opposite direction and, starting from set theory as it is historically given, to seek out the principles required for establishing the foundations of this mathematical discipline. In solving the problem we must, on the one hand, restrict these principles sufficiently to exclude all contradictions and, on the other, take them sufficiently wide to retain all that is valuable in this theory." [Zermelo, 1967, p. 200, emphasis as in the original, first published in German in 1908]

11. Alonzo Church *first proposing a third alternative to Russell's type theory and Zermelo's axiomatic set theory* in 1932: "Rather than adopt the method of Russell for avoiding the familiar paradoxes of mathematical logic, or that of Zermelo, both of which appear somewhat artificial, we introduce for this purpose, as we have said, a certain restriction on the law of excluded middle." [Church, 1932, p. 347]

12. Henk Barendregt on the *origin of the λ-notation*: "We end this introduction by telling what seems to be the story how the letter 'λ' was chosen to denote function abstraction. In [100] Principia Mathematica the notation for the function f with $f(x) = 2x + 1$ is $2\hat{x} + 1$. Church originally intended to use the notation $\hat{x}.2x + 1$. The typesetter could not position the hat on top of the x and placed it in front of it, resulting in
$\wedge x.2x + 1$.
Then another typesetter changed it into $\lambda x.2x + 1$." [Barendregt, 1997, p. 182]

13. Felice Cardone and J. Roger Hindley on the *invention of λ-calculus and the origin of the λ-notation*: "The λ-calculus was invented in about 1928 by Alonzo Church, and was first published in [Church, 1932]. Church was born in 1903 in Washington D.C. and studied at Princeton University. He made his career at Princeton until 1967, though in 1928–29 he visited Göttingen and Amsterdam.
Around 1928 he began to build a formal system with the aim of providing a foundation for logic which would be more natural than Russell's type theory or Zermelo's set theory, and would not contain free variables (for reasons he explained in [Church, 1932, pp. 346–347]). He chose to base it on the concept of function rather than set, and his primitives included abstraction $\lambda x[M]$ and application $\{F\}(X)$, which we shall call here '$\lambda x.M$' and '(FX)'.
(By the way, why did Church choose the notation 'λ'? In [Church, 1964, §2] he stated clearly that it came from the notation '$\hat{x}$' used for class-abstraction by Whitehead and Russell, by first modifying '$\hat{x}$' to '$\wedge x$' to distinguish function-abstraction from class-abstraction, and then changing '$\wedge$' to 'λ' for ease of printing. This origin was also reported in [Rosser, 1984, p.338]. On the other hand, in his later years Church told two enquirers that the choice was more accidental: a symbol was needed and 'λ' just happened to be chosen.)
As mentioned earlier, Church was not the first to introduce an explicit notation for function-abstraction. But he was the first to state explicit formal conversion rules for the notation, and to analyse their consequences in depth." [Cardone and Hindley, 2009, pp. 730 f.]

14. Felice Cardone and J. Roger Hindley on the *evolution of Church's simple type theory*: "Church's simple type theory was a function-based system, stemming from ideas of Frank Ramsey and Leon Chwistek in the 1920s, for simplifying the type theory of [Russell and Whitehead, 1913]. Church lectured on his system in Princeton in 1937–38 before publishing it in [Church, 1940], and his lectures were attended by Turing, who later made some technical contributions. [...]
Church's system was analysed and extended in a series of Princeton Ph.D. theses from the 1940s

onward, of which perhaps the best known are Leon Henkin's in 1947, published in [Henkin, 1950], and Peter Andrews', published in [Andrews, 1965]. Henkin gave two definitions of model of typed λ (*standard* and *general* models), and proved the completeness of simple type theory with respect to general models. Andrews extended Church's system to make a smooth theory of tran[s]finite types." [Cardone and Hindley, 2009, p. 737, emphasis as in the original]

15. Peter B. Andrews on *Henkin's rôle for the development of* $\mathcal{Q}_0$: "Quine described how to make these definitions in the short final section of [16], but Henkin developed this topic much further in [Henkin, 1963], introducing an axiomatic system and establishing its soundness and completeness. [...]
[...]
Henkin's work played a decisive role in my life. [...] Henkin's work developing a formulation of Church's type theory with equality (identity) as the sole logical primitive was particularly important for me. I used such a formulation of full type theory, called $\mathcal{Q}_0$, in my Ph.D. thesis [Andrews, 1965] and the textbook [Andrews, 2002].
[...]
It is important to realize the significance of Henkin's contribution in developing a formulation of type theory based on equality. It is a real improvement of the system $\mathscr{C}$ discussed in [Henkin, 1950], which has primitive constants for propositional connectives and quantifiers [...]" [Andrews, 2014b, pp. 68 f.]

16. Andrew M. Pitts on *type variables* in HOL: "In Church's original formulation of simple type theory, type variables are part of the meta-language and are used to range over object language types. Proofs that contain type variables were understood as proof schemes (i.e. families of proofs). To support such proof schemes *within* the HOL logic, type variables have been added to the object language type system." [Gordon and Melham, 1993, p. 195, emphasis as in the original]

17. Andrew M. Pitts on the *set of rules and axioms* for HOL: "The particular set of rules and axioms chosen to axiomatize the HOL logic is rather arbitrary. It is partly based on the rules that were used in the LCF logic PPλ, since HOL was implemented by modifying the LCF system. In particular, the substitution rule `SUBST` is exactly the same as the corresponding rule in LCF; the code implementing this was written by Robin Milner and is highly optimized. Because substitution is such a pervasive activity in proof, it was felt to be important that the system primitive be as fast as possible. From a logical point of view, it would be better to have a simpler substitution primitive, such as 'Rule R' of Andrews' logic $\mathcal{Q}_0$, and then to derive more complex rules from it." [Gordon and Melham, 1993, p. 213]

18. Mike Gordon on the *genesis of HOL*: "[...] [T]he terms [...] could be encoded [...] in such a way that the LSM expansion-law just becomes a derived rule [...]. This approach is both more elegant and rests on a firmer logical foundation, so I switched to it and HOL was born. [...]
The logic supported by Cambridge LCF has the usual formula structure of predicate calculus, and the term structure of the typed λ-calculus. The type system, due to Milner, is essentially Church's original one [Church, 1940], but with type variables moved from the meta-language to the object language (in Church's system, a term with type variables is actually a meta-notation – a term-schema – denoting a family of terms, whereas in LCF it is a single polymorphic term). [...]
[...] HOL employs the LCF substitution because I wanted to use the existing efficient code. As a result the HOL logic ended up with a rather *ad hoc* formal basis. Another inheritance from LCF is the use of a natural deduction logic (Church used a Hilbert-style formal system). [...]" [Gordon, 2000, p. 173, emphasis as in the original]

19. Mike Gordon on the *development of HOL*: "The design of HOL was largely taken 'off the shelf,' the theory being classical higher order logic and the implementation being LCF. The development of the system was, at first, primarily driven by hardware verification case studies." [Gordon, 2000, p. 174]

20. Mike Gordon on the *validation of HOL's definitional principles*: "[...] Dr Andrew Pitts was commissioned to validate HOL's definitional principles. He produced informal proofs that they could not introduce inconsistency [Gordon and Melham, 1993, Chapter 16]." [Gordon, 2000, p. 175]

21. Ondřej Kunčar and Andrei Popescu *classifying the HOL logic*: "Polymorphic HOL, more precisely, Classic[al] Higher-Order Logic with Infinity, Hilbert Choice and Rank-1 Polymorphism, endowed with a mechanism for constant and type definitions, was proposed in the [eigh]ties as a logic for interactive theorem provers by Mike Gordon, who also implemented the seminal HOL theorem prover [Gordon and Melham, 1993]." [Kunčar and Popescu, 2015, p. 234]

For a detailed treatment of epsilon terms, see [Slater, 2016].

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