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DIMENSIONAL REGULARIZATION AND THE RENORMALIZATION GROUP

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Abstract: The behaviour of a renormalized field theory under scale transformations $x \to \lambda x$; $p \to p/\lambda$ can be found in a simple way when the theory is regularized by the continuous dimension method. The techniques proposed here have several applications in dimensionally regularized theories: short distance behaviour is expressed in terms of the single poles at $n = 4$, and all coefficients in front of the higher poles $1/(n-4)^k$ are expressed in terms of those of the single poles $1/(n-4)$, without calculating any diagram. In some cases the Taylor series in $1/(n-4)$ can be summed, leading to some exact results for the infinities of the full theory.

1. Introduction

The continuous dimension method has recently been advocated by many authors as a useful device for obtaining a finite perturbation expansion of a renormalized field theory [1, 2]. As its main advantages one usually mentions the facts that no additional regulator diagrams are needed that would make the algebra more complicated, and that complicated symmetries like local gauge invariance are left intact [1]

In this article we would like to point out still another useful aspect: it becomes rather transparent how the theory behaves under spacetime scale transformations. This behaviour is described by a simple differential equation, closely related to the Callan-Symanzik equation.

Our technique cuts both ways: we also find an important set of equations between the residues of different poles at $n \to 4$.

2. The role of the unit of mass in the subtraction procedure

Let $\mathcal{L}(\varphi_i, \lambda)$ be a Lagrangian for a renormalizable field theory with fields φ_i, in which λ is an expansion parameter, for instance a coupling constant.

The continuous dimension method consists in considering the theory in a "space-

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time" with dimension $n \neq 4$. For all integrals that can occur in the perturbation expansion, one can define in a unique way its finite part as an analytic function of n.

If we now consider the limit $n \to 4$ we encounter poles of the type $(n-4)^{-k}$, which can only be removed if we let the coefficients in $\mathfrak{L}$ also vary as a function of n. Thus the Lagrangian $\mathfrak{L}(\varphi_i, \lambda)$ is replaced by

$$\mathfrak{L} = \mathfrak{L}(\varphi_i, \lambda) + \Delta\mathfrak{L}(\varphi_i, \lambda, n) ,$$

$$\Delta\mathfrak{L} = \lambda \frac{A(\varphi_i)}{n-4} + \lambda^2 \left[\frac{B(\varphi_i)}{n-4} + \frac{C(\varphi_i)}{(n-4)^2} \right] + \cdots . \tag{2.1}$$

The Lagrangian (2.1) is constructed such that all Green functions remain finite as $n \to 4$, order by order in the perturbation expansion in λ.

Now, if the Lagrangian $\mathfrak{L} + \Delta\mathfrak{L}$ can be obtained from $\mathfrak{L}$ itself just by renormalizing the parameters and rescaling the fields (and, if necessary, performing gauge transformations), then the theory is called renormalizable. We shall only consider that case.

The S-matrix will depend only on those parameters which are invariant under field transformations or gauge transformations. Let us first consider these parameters and let us furthermore assume, for the time being, that there are only two of them: a mass parameter m, and a coupling constant λ, occuring in the form of terms like

$$-\frac{1}{2} m^2 \varphi^2 \qquad \text{and} \qquad -\frac{1}{24} \lambda \varphi^4 \tag{2.2}$$

in the Lagrangian. The more general case of an arbitrary number of parameters will be discussed in sect. 5. In this section we shall not use the particular form of the terms (2.2) except that they define the dimension of these parameters: if we assign to a derivative ∂_μ dimension 1, the Lagrangian has dimension n. A boson field φ has therefore dimension $(n-2)/2$. A mass m has dimension 1 and λ in (2.2) has dimension

$$n - 4\frac{n-2}{2} = 4 - n .$$

Now in the theory at non-rational n, all divergent integrals in the pertubation expansion can be redefined in terms of convergent integrals in a unique way [1]. This procedure conserves all possible symmetry aspects of the original Lagrangian, including its scale transformation properties. So, at non-rational n the parameters m and λ in (2.2) have a well defined and unique meaning, and their dimension is not influenced by the interactions*. We therefore call these parameters the bare

* Note that our subtraction procedure at $n \neq 4$ differs in a crucial way from that of Wilson, ref. [3], who uses a cut-off Λ, whereas we first redefine divergent integrals. This is why Wilson gets anomalous dimensions also at $n \neq 4$ where we do not.

parameters which we shall denote with a suffix B: λ_B and m_B. However, as stated above, we have to let these bare parameters go to infinity while $n \to 4$. This we do by expressing them in terms of two finite (but rather arbitrary) parameters λ_R and m_R, and n (R standing for "renormalized"). We shall choose λ_R and m_R to be dimensionless and independent of n, although we may substitute, if we wish, a function that depends in a given way, but always smoothly, on n.

To express λ_B and m_B in terms of λ_R, m_R and n we need a unit of mass, μ, that is kept fixed. So, in practice, we shall have an expansion in terms of λ_R of the following type:

$$\lambda_B = \mu^{4-n} \left[\lambda_R + \frac{a_{12}(m_R)\lambda_R^2}{n-4} + \frac{a_{13}(m_R)\lambda_R^3}{n-4} + \ldots + \frac{a_{23}(m_R)\lambda_R^3}{(n-4)^2} + \frac{a_{24}(m_R)\lambda_R^4}{(n-4)^2} + \ldots + \frac{a_{34}(m_R)\lambda_R^4}{(n-4)^3} + \ldots + \ldots\right],$$

$$m_B = \mu \left[m_R + \frac{b_{11}(m_R)\lambda_R}{n-4} + \frac{b_{12}(m_R)\lambda_R^2}{n-4} + \ldots + \frac{b_{22}(m_R)\lambda_R^2}{(n-4)^2} + \ldots + \ldots\right]. \tag{2.3a}$$

The coefficients $a_{\nu i}$, $b_{\nu i}$ can be calculated [1, 2, 4] from the Feynman diagrams* of order i in λ. If we want not only the S-matrix, but also Green functions to remain finite at $n \to 4$, we also must renormalize fields and gauge parameters. Our methods will apply to these renormalizations also (see sect. 5).

It will be more convenient not to expand in terms of λ_R, and to write eqs. (2.3a) as,

$$\lambda_B = \mu^{4-n} \left[\lambda_R + \sum_{\nu=1}^{\infty} \frac{a_\nu(m_R, \lambda_R)}{(n-4)^\nu}\right],$$

$$m_B = \mu \left[m_R + \sum_{\nu=1}^{\infty} \frac{b_\nu(m_R, \lambda_R)}{(n-4)^\nu}\right]. \tag{2.3b}$$

Now, of course, eq. (2.3) is not the only expression for λ_B and m_B which will lead to a finite S-matrix as $n \to 4$, because one can always substitute

$$\lambda_R \to \lambda_R + e_1(n-4) + e_2(n-4)^2 \ldots,$$

$$m_R \to m_R + f_1(n-4) + f_2(n-4)^2 \ldots, \tag{2.4}$$

so that we get a different series of the kind

* Actually, the coefficients a are independent of m_R and b are proportional to m_R (see sect. 4), but for sake of clarity we do not want to use that information here.

$$\lambda_{\mathrm{B}} = \mu^{4-n}\left[\sum_{k=1}^{\infty} c'_k(m'_{\mathrm{R}},\lambda'_{\mathrm{R}})(n-4)^k + \lambda'_{\mathrm{R}} + \sum_{\nu=1}^{\infty} \frac{a'_\nu(m'_{\mathrm{R}},\lambda'_{\mathrm{R}})}{(n-4)^\nu}\right] ,$$

$$m_{\mathrm{B}} = \left[\sum_{k=1}^{\infty} d'_k(m'_{\mathrm{R}},\lambda'_{\mathrm{R}})(n-4)^k + m'_{\mathrm{R}} + \sum_{\nu=1}^{\infty} \frac{b'_\nu(m'_{\mathrm{R}},\lambda'_{\mathrm{R}})}{(n-4)^\nu}\right] . \tag{2.5}$$

This corresponds to the usual ambiguity in any renormalization formalism, but it is easy to convince oneself that the subsidiary condition that all coefficients c_i and d_i in (2.5) be zero makes the expansion (2.3) unique, (once we have chosen a value for μ), because two different ones cannot possibly both yield an S-matrix that is finite, order by order in the perturbation expansion. We shall from now on assume that all necessary transformations of the type (2.4) have been performed such that this subsidiary condition holds, so that we have series like (2.3) and not (2.5)*. On the one hand this requirement also leads to a definition of λ_{R} and m_{R} in a given theory (in general they do not correspond to the more usual definition like on-mass-shell coupling constant or physical mass), but on the other hand it will be clear that this definition will depend on our "unit of mass" μ.

3. The scaling properties of λ_{R} and m_{R}

Suppose we calculate a diagram with loops that is ultra-violet divergent but does not become infra-red divergent as $m \to 0$. After application of our regulation technique and insertion of the series (2.3) the result will contain powers of

$$\log\frac{k^2}{\mu^2} , \tag{3.1}$$

where k is a typical external momentum and μ is our "unit of mass".

It will be clear that the perturbation expansion will only be applicable if (3.1) is resonably small, that is, we must choose μ to be of the order of k. If we want to study limits like $k^2 \to \infty$ or $k^2 \to 0$, it is of importance to vary μ also. So it is of importance to compare the same theory at different choices of μ. Our result will be the discovery of a well known fact [5] in a new way: the anomalous scaling behaviour of renormalizable theories.

Consider a new unit of mass, μ', somewhat larger than μ:

$$\mu' = \mu(1+\epsilon), \qquad |\epsilon| \ll 1 , \tag{3.2}$$

and express λ_{B} and m_{B} in terms of the new unit:

* For practical calculations this condition for the counter terms is probably not the most convenient one, due to the appearance of numbers like the Euler constant γ in the higher coefficients.

$$\lambda_B = (\mu')^{4-n}\,[1+\epsilon(n-4)]\left[\lambda_R + \sum_{\nu=1}^{\infty}\frac{a_\nu}{(n-4)^\nu}\right]$$

$$= (\mu')^{4-n}\left[\epsilon(n-4)\lambda_R + \lambda_R + \epsilon a_1 + \sum_{\nu=1}^{\infty}\frac{a_\nu + \epsilon a_{\nu+1}}{(n-4)^\nu}\right], \tag{3.3}$$

and similarly

$$m_B = \mu'[1-\epsilon]\left[m_R + \sum_{\nu=1}^{\infty}\frac{b_\nu}{(n-4)^\nu}\right]. \tag{3.4}$$

Note that the series (3.3) is not of the desired type because of the occurence of a term proportional to $n-4$. We have a series of the type (2.5) instead of (2.3). This can be cured by substituting for λ_R a slightly n-dependent quantity, that is, by performing a transformation of the type (2.4):

$$\lambda_R = \widetilde{\lambda}_R - \epsilon(n-4)\widetilde{\lambda}_R\,. \tag{3.5}$$

The series (3.3) and (3.4) then become (note that also a and b depend on λ):

$$\lambda_B = (\mu')^{4-n}\left[\widetilde{\lambda}_R + \epsilon a_1 - \epsilon\widetilde{\lambda}_R a_{1,\lambda} + \sum_{\nu=1}^{\infty}\frac{a_\nu(m_R,\widetilde{\lambda}_R) + \epsilon a_{\nu+1} - \epsilon\widetilde{\lambda}_R a_{\nu+1,\lambda}}{(n-4)^\nu}\right], \tag{3.6}$$

and

$$m_B = \mu'\left[m_R - \epsilon m_R - \epsilon\widetilde{\lambda}_R b_{1,\lambda} + \sum_{\nu=1}^{\infty}\frac{b_\nu(m_R,\widetilde{\lambda}_R) - \epsilon b_\nu - \epsilon b_{\nu+1,\lambda}}{(n-4)^\nu}\right], \tag{3.7}$$

where $a_{\nu,\lambda}$ stands for $\partial a_\nu(m,\widetilde{\lambda}_R)/\partial\widetilde{\lambda}_R$, and similarly $b_{\nu,\lambda}, a_{\nu,m}, b_{\nu,m}$. Now, eqs. (3.6) and 3.7) have the desired form of eq. (2.3).

Indeed, the values of λ_R and m_R have changed in a non-trivial way. We have to define

$$\lambda_R' = \widetilde{\lambda}_R + \epsilon(a_1 - \widetilde{\lambda}_R a_{1,\lambda})\,,$$

$$m_R' = m_R - \epsilon(m_R + \widetilde{\lambda}_R b_{1,\lambda})\,. \tag{3.8}$$

In sect. 2 we admitted that λ_R, m_R may depend explicitly on n, but all physically relevant quantities are smooth functions of λ_R, m_R and n, and, therefore, only the values of λ_R and m_R at $n=4$ are relevant. At $n=4$ we have $\widetilde{\lambda}_R = \lambda_R$, so we obtained the desired transformation properties of λ and m:

If we change our unit of mass μ into $\mu' = \mu(1+\epsilon)$, then we have to change our renormalized parameters λ_R and m_R respectively into

$$\lambda'_R = \lambda_R + \epsilon\left[1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right] a_1(m_R, \lambda_R)\,,$$

$$m'_R = m_R - \epsilon\left[m_R + \lambda_R \frac{\partial}{\partial\lambda_R} b_1(m_R, \lambda_R)\right], \tag{3.9}$$

in order to obtain the same renormalized S-matrix following the same prescription of sect. 2.

Note that only the residues of the single poles of λ_B and m_B contribute to these scaling properties of λ_R and m_R.

Eqs. (3.9) are in fact differential equations for λ_R and m_R as a function of μ. Examples of such equations will be discussed in sect. 6.

4. Identities for the coefficients a_ν and b_ν

We have not yet written down the complete series that replaces (2.3) after a scale transformation $\mu \to \mu'$. The substitution that has to be made in (3.3) is the product of (3.5) and (3.8):

$$\lambda_R = \lambda'_R - \epsilon(n-4)\lambda'_R + \epsilon(-a_1 + \lambda'_R a_{1,\lambda})\,;$$

$$m_R = m'_R + \epsilon(m'_R + \lambda'_R b_{1,\lambda}). \tag{4.1}$$

We get from (3.3):

$$\lambda_B = (\mu')^{4-n}\left[\lambda'_R + \sum_{\nu=1}^{\infty} \frac{1}{(n-4)^\nu}\{a_\nu(m'_R, \lambda'_R) + \epsilon a_{\nu+1} - \epsilon\lambda'_R a_{\nu+1,\lambda}\right.$$
$$\left. + \epsilon a_{\nu,\lambda}(-a_1 + \lambda'_R a_{1,\lambda}) + \epsilon a_{\nu,m}(m'_R + \lambda'_R b_{1,\lambda})\}\right], \tag{4.2}$$

and

$$m_B = \mu'\left[m'_R + \sum_{\nu=1}^{\infty} \frac{1}{(n-4)^\nu}\{b_\nu(m'_R, \lambda'_R) - \epsilon b_\nu - \epsilon\lambda'_R b_{\nu+1,\lambda}\right.$$
$$\left. + \epsilon b_{\nu,\lambda}(-a_1 + \lambda'_R a_{1,\lambda}) + \epsilon b_{\nu,m}(m'_R + \lambda'_R b_{1,\lambda})\}\right]. \tag{4.3}$$

Note that a_ν, b_ν, m_R and λ_R are all dimensionless. Obviously, the functions a_ν and b_ν are independent of μ.

But, as has been argued in sect. 2, the series (2.3) is the only series that leads to a finite S-matrix at $n \to 4$. Hence the series (4.2) and (4.3) must be the same as (2.3), so the "correction terms" proportional to ϵ must all cancel. Those terms proportional to ϵ that are of zeroth order in $1/(n-4)$, have been made to vanish by construction, but for the higher order ones this observation leads to important identities between the coefficients a_ν, b_ν:

$$a_{\nu,\lambda}(-a_1+\lambda_R a_{1,\lambda})+a_{\nu,m}(m_R+\lambda_R b_{1,\lambda})=\lambda_R a_{\nu+1,\lambda}-a_{\nu+1}\,; \tag{4.4a}$$

$$b_{\nu,\lambda}(-a_1+\lambda_R a_{1,\lambda})+b_{\nu,m}(m_R+\lambda_R b_{1,\lambda})-b_\nu=\lambda_R b_{\nu+1,\lambda}\,. \tag{4.4b}$$

This is the set of equations we alluded to in the beginning. For instance, it follows that in (2.3a)

$$\frac{\partial a_{12}}{\partial m_R}=0\,; \tag{4.5a}$$

$$2a_{23}=2a_{12}^2+m_R\frac{\partial a_{13}}{\partial m_R}\,; \tag{4.5b}$$

$$m_R\frac{\partial b_{11}}{\partial m_R}=b_{11}\,; \tag{4.5c}$$

$$2b_{22}=a_{12}b_{11}+\frac{b_{11}^2}{m_R}+m_R\frac{\partial b_{12}}{\partial m_R}-b_{12}\,. \tag{4.5d}$$

and so on. We see that a_{12} is independent of m_R and b_{11} is proportional to m_R. Now, one can show [1] that all counter terms in the dimensional regularization procedure are not only polynomials in terms of external momenta but also in terms of masses m_R. In particular, there is no singularity at $m_R \to 0$. Given this fact, it is not difficult to derive from eqs. (4.4) that all coefficients a_{ij} are independent of m_R and all coefficients b_{ij} are linear in m_R, so the last term in (4.5b) and the last two terms in (4.5d) may be dropped; likewise one can then substitute $a_{\nu,m}=0$; $b_{\nu,m}=b_\nu/m_R$ in (4.4). A qualitative interpretation of (4.4) can then perhaps be made: the subdivergences of a diagram are expressed in terms of over-all divergences and subdivergences of subgraphs. We have not checked the possibility of this interpretation in detail. The validity of (4.4) has been checked in some examples.

Eqs. (4.4) may be reformulated in terms of a differential equation for the functions

$$a(\lambda_R, m_R, n)=\sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda_R, m_R)}{(n-4)^\nu}\,,\quad \text{etc.}$$

This equation, for the $\lambda\varphi^4$ theory, will be discussed in sect. 7.

5. Generalization towards an arbitrary number of parameters

The preceding section was meant to illustrate our arguments in detail for the case of only two different kinds of parameters. Let us now take a renormalizable theory with an arbitrary but fixed number of parameters λ^k, which in a bare Lagrangian have dimension D^k,

$$D^k = \rho_{(k)}(4-n) + \sigma_{(k)} \,. \tag{5.1}$$

Again we write

$$\lambda_{\mathrm{B}}^k = \mu^{D^k}\left[\lambda_{\mathrm{R}}^k + \sum_{\nu=1}^{\infty} \frac{a_\nu^k(\lambda_{\mathrm{R}})}{(n-4)^\nu}\right]. \tag{5.2}$$

In gauge field theories also renormalization of gauge parameters α^k may be needed:

$$\alpha_{\mathrm{B}}^k = \mu^{D_\alpha^k}\left[\alpha_{\mathrm{R}}^k + \sum_{\nu=1}^{\infty} \frac{b_\nu^k(\lambda_{\mathrm{R}},\alpha_{\mathrm{R}})}{(n-4)^\nu}\right]. \tag{5.3}$$

Renormalization of the fields can also be considered:

$$\varphi_{\mathrm{B}}^k = \mu^{D_\varphi^k}\left[\varphi_{\mathrm{R}} + \sum_{\nu=1}^{\infty} \frac{c_\nu^k(\lambda_{\mathrm{R}},\alpha_{\mathrm{R}},\varphi_{\mathrm{R}})}{(n-4)^\nu}\right]. \tag{5.4}$$

Both α_{R} and φ_{R} do not enter into (5.2) so the physical parameters λ_{R} will satisfy equations among themselves. In order to find the scaling behaviour, or the pole equations, for α_{R} and φ_{R} one simply absorbs (5.3) and (5.4) into (5.2). From now on we assume that that may have been done.

Consider

$$\mu' = \mu(1+\epsilon)\,; \tag{5.5}$$

the equivalent of eq. (3.3) is

$$\begin{aligned}\lambda_{\mathrm{B}}^k = (\mu')^{D^k}\Big[&\epsilon\rho_{(k)}(n-4)\lambda_{\mathrm{R}}^k + \lambda_{\mathrm{R}}^k + \epsilon\rho_{(k)}a_1^k - \epsilon\sigma_{(k)}\lambda_{\mathrm{R}}^k \\ &+ \sum_{\nu=1}^{\infty}\frac{a_\nu^k - \epsilon\sigma_{(k)}a_\nu^k + \epsilon\rho_{(k)}a_{\nu+1}^k}{(n-4)^\nu}\Big]\end{aligned} \tag{5.6}$$

Let us write

$$\lambda_{\mathrm{R}}^k = \lambda'^k_{\mathrm{R}} - \epsilon\rho_{(k)}(n-4)\lambda'^k_{\mathrm{R}} + \delta\lambda_{\mathrm{R}}^k\,, \tag{5.7}$$

and substitute that into eq. (5.6).

The term $\delta\lambda_{\mathrm{R}}^k$ is required to be such that the first term of the series in $1/(n-4)$ then obtained, will be just λ'_{R}; hence

$$\delta\lambda^k_{\mathrm{R}} = \epsilon\sigma_{(k)}\lambda'^k_{\mathrm{R}} - \epsilon\rho_{(k)}a^k_1 + \sum_l \epsilon a^k_{1,l}\rho_{(l)}\lambda'^l_{\mathrm{R}}\,; \tag{5.8}$$

where $a^k_{\nu,l}$ stands for $(\partial/\partial\lambda^l_{\mathrm{R}})a^k_\nu(\lambda_{\mathrm{R}})$. Now

$$\lambda^k_{\mathrm{B}} = (\mu')^{D^k}\Big[\lambda'^k_{\mathrm{R}} + \sum_{\nu=1}^{\infty}\frac{1}{(n-4)^\nu}\{a^k_\nu(\lambda') - \epsilon\sigma_{(k)}a^k_\nu + \epsilon\rho_{(k)}a^k_{\nu+1}$$

$$-\sum_l \epsilon\rho_{(l)}a^k_{\nu+1,l}\lambda'^l_{\mathrm{R}} + \epsilon\sum_l a^k_{\nu,l}\Big(\sigma_{(l)}\lambda^l_{\mathrm{R}} - \rho_{(l)}a^l_1$$

$$+\sum_m a^l_{1,m}\rho_{(m)}\lambda'^m_{\mathrm{R}}\Big)\}\Big]\,. \tag{5.9}$$

Putting $n = 4$ in eq. (5.7) we find the scaling properties of the parameters λ_{R}:

$$\mu' = \mu(1+\epsilon)\,,$$

$$\lambda'^k_{\mathrm{R}} = \lambda^k_{\mathrm{R}} - \epsilon\sigma_{(k)}\lambda^k_{\mathrm{R}} + \epsilon\rho_{(k)}a^k_1 - \epsilon\sum_l a^k_{1,l}\rho_{(l)}\lambda^l_{\mathrm{R}}\,, \tag{5.10}$$

and the generalization of eqs. (4.4) becomes

$$\sum_l a^k_{\nu,l}(\sigma_{(l)}\lambda^l_{\mathrm{R}} - \rho_{(l)}a^l_1 + \sum_m a^l_{1,m}\rho_{(m)}\lambda^m_{\mathrm{R}}) - \sigma_{(k)}a^k_\nu$$

$$= \sum_l a^k_{\nu+1,l}\rho_{(l)}\lambda^l_{\mathrm{R}} - \rho_{(k)}a^k_{\nu+1}\,. \tag{5.11}$$

In most theories the parameters with $\sigma(k) = 0$ will only get counter terms that are independent of the other parameters. This observation leads to a simplification similar to the one in the end of sect. 4.

6. Scaling behaviour of some theories

Let us define a scaling parameters s by

$$\mu = \mu_0 e^s\,, \qquad \lambda^k_{\mathrm{R}} = \lambda^k_{\mathrm{R}}(s)\,. \tag{6.1}$$

Then we can rewrite eqs. (5.10) in terms of a differential equation:

$$\frac{\mathrm{d}\lambda_{\mathrm{R}}^{k}}{\mathrm{d}s} = -\sigma_{(k)}\lambda_{\mathrm{R}}^{k} + \rho_{(k)} a_1^k(\lambda_{\mathrm{R}}) - \sum_l a_{1,l}^k(\lambda_{\mathrm{R}})\rho_{(l)}\lambda_{\mathrm{R}}^l \,. \tag{6.2}$$

This equation is nearly, but not quite, identical to the Callan-Symanzik equation [5]. Here the coefficients are expressed in terms of the dimensions and the poles of the parameters.

Let us again consider the case that there is only one parameter λ with $\sigma = 0$, as in $\lambda\varphi^4$ theory. Then a_1 only depends on this parameter (see sect. 4) and we have a (non-linear) first order differential equation with one variable. Using the notation of (2.3a) and dropping the suffix R, we get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s} = a_1(\lambda) - \lambda\frac{\mathrm{d}a_1}{\mathrm{d}\lambda} = -a_{12}\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{6.3}$$

It will be clear that the behaviour of the theory under scale transformations depends crucially on the relative sign of λ and a_{12}.

Two examples of one-parameter theories are well known: (i) the scalar theory with coupling $-\lambda\varphi^4$. The sign of λ is usually taken to be positive, in order for the Hamiltonian to be positive definite*; (ii) quantum electrodynamics. Here λ is to be replaced by e^2, which is always positive.

Both of these theories have $a_{12} < 0$ which implies that $\lambda_{\mathrm{R}}(s)$ diverges as $s \to +\infty$. This implies that the small distance behaviour of these theories is not described by the usual perturbation expansion. The long distance behaviour on the other hand, can be found quite accurately because for $s \to -\infty$, $\lambda \to 0$ and the first terms of the perturbation expansion converge rapidly there. In this region we can give the solution of eq. (6.3):

$$\lambda_{\mathrm{R}}(s) = \frac{1}{C + a_{12}s}\,. \tag{6.4}$$

eq. (6.4) may be used to find infra-red behaviour for massless theories, but in general has no meaning if massive particles occur, as a consequence of "ultra-violet" divergencies at $m^2/\mu^2 \to \infty$.

In a pure Yang-Mills theory, also, λ is to be replaced by g^2, where g is the usual coupling constant. So $\lambda > 0$. However, in that theory $a_{12} > 0$ [4, 7] and the solution (6.4) is valid in the ultra-violet region $s \to +\infty$. The same behaviour is found in $\lambda\varphi^4$ theory if λ is taken the unusual sign [6].

A Yang-Mills theory with fermions can be written down [4, 7] with the property $a_{12} = 0$. In this theory the two-loop diagrams are decisive for its scale behaviour.

Suppose $a_{13} \neq 0$. A small change in the theory can make a_{12} very small but not zero. In that case the function $a_1 - \lambda(\partial a_1/\partial\lambda)$ in (6.3) must have a zero for a small

* The correctness of this argument is dubious. See ref. [6] and the last paragraph of this section.

value λ_0 of λ. We then have a conformally invariant theory with a small coupling constant, so that both the perturbation expansion and the conformal invariance can be studied.

The situation becomes quite complicated if we have two or more parameters λ^i with $\sigma_{(i)} = 0$ In general, eq. (6.3) will be replaced by

$$\frac{\mathrm{d}\lambda_{\mathrm{R}}^i}{\mathrm{d}s} = -a_{jk}^i \lambda_{\mathrm{R}}^j \lambda_{\mathrm{R}}^k + \mathcal{O}(\lambda^3) . \tag{6.5}$$

where a_{jk}^i are determined by the poles of the one-loop graphs. In many cases the solutions diverge at both ends of the s-scale [7]. Neither the ultra-violet, nor the infra-red behaviour can then be calculated. In a certain example we also found that a parameter λ_{R} can easily change sign, which is one more reason to doubt the argument that the corresponding classical Hamiltonian must be positive definite.

7. Renormalization and the perturbation expansion

In this section we show an application of eq. (5.11). Let us again consider eqs. (2.3) and now write

$$\lambda_{\mathrm{R}} + \sum_{\nu=1}^{\infty} a_\nu(\lambda_{\mathrm{R}}) \frac{1}{(n-4)^\nu} = \lambda_{\mathrm{B}}(\lambda_{\mathrm{R}}, n) , \tag{7.1}$$

where we confine to those theories for which a_ν are independent of m_{R} (like in $\lambda \varphi^4$ theory, see sect. 4).

Putting also

$$a_0 = \lambda_{\mathrm{R}} , \tag{7.2}$$

eq. (4.4a) becomes

$$\frac{\partial a_\nu}{\partial \lambda_{\mathrm{R}}} (-a_1 + \lambda_{\mathrm{R}} a_{1,\lambda}) = \lambda_{\mathrm{R}} \frac{\partial a_{\nu+1}}{\partial \lambda_{\mathrm{R}}} - a_{\nu+1} , \tag{7.3}$$

for $\nu = 0, 1, 2, \ldots$, or

$$\frac{\partial \lambda_{\mathrm{B}}}{\partial \lambda_{\mathrm{R}}} A(\lambda_{\mathrm{R}}) + (n-4) \left(\lambda_{\mathrm{R}} \frac{\partial \lambda_{\mathrm{B}}}{\partial \lambda_{\mathrm{R}}} - \lambda_{\mathrm{B}} \right) = 0 . \tag{7.4}$$

where $A(\lambda_{\mathrm{R}})$ stands for $a_1 - \lambda_{\mathrm{R}} a_{1,\lambda}$.

The general solution of eq. (7.4) is

$$\lambda_{\mathrm{B}}(\lambda_{\mathrm{R}}, n) = \exp \left\{ \int_{C_0}^{\lambda_{\mathrm{R}}} \mathrm{d}\lambda \frac{1}{\lambda + (n-4)^{-1} A(\lambda)} + C_1(n) \right\} . \tag{7.5}$$

The integration path in λ-space may be arbitrary. We must take in mind that $A(\lambda)$ is for small λ of order λ^2, so we write:

$$A(\lambda) = \lambda^2 \widetilde{A}(\lambda)\,, \tag{7.6}$$

and

$$\lambda_B = \lambda_R \exp\left\{ -\int_0^{\lambda_R} d\lambda \frac{1}{(n-4)\widetilde{A}^{-1}(\lambda) + \lambda} + C_2(n) \right\}. \tag{7.7}$$

According to (7.2) we require that $\lambda_B \to \lambda_R + O(\lambda_R^2)$ if $\lambda_R \to 0$. It follows that in eq. (7.7)

$$C_2(n) = 0. \tag{7.8}$$

Strictly speaking eq. (7.7) with (7.8) should be interpreted as follows: expanding it with respect to λ_R one obtains a series of which each term has a certain singularity at $n = 4$. Substituting that series in the expansion of the S-matrix in terms of λ_B (which also has singularities at $n = 4$) one obtains as a result a series for the S-matrix, in which each term is finite at $n = 4$.

But it is appealing to assume that if a perturbation expansion contains only finite and well-defined terms, then the full theory will also be finite. In that case one merely needs to substitute eq. (7.7) itself into the "full theory" at $n \neq 4$, to obtain a finite theory at $n = 4$. Of course, we have no means to check such an assumption, but it is natural and let us see its consequences.

At $n \to 4$, eq. (7.7) behaves like (if $\widetilde{A}(0) \neq 0$):

$$\lambda_B = \frac{1}{\widetilde{A}(0)}(n-4) + \frac{\widetilde{A}'(0)}{(\widetilde{A}(0))^3}(n-4)^2 \log(n-4) + R_1(n,\lambda_R)(n-4)^2\,, \tag{7.9}$$

where $R_1(n,\lambda_R)$ is a function that behaves smoothly at $n \to 4$, and $R_1(4,\lambda_R)$ depends on λ_R:

$$R_1(4,\lambda_R) \simeq -\frac{1}{(\widetilde{A}(0))^2 \lambda_R} + O(\log \lambda_R), \tag{7.10}$$

for small λ_R.

So here is another remarkable result for renormalization theory: If the bare coupling constant λ_B in φ^4 theory (or g_B^2 in pure Yang-Mills theory) is given an n dependence as in eq. (7.9), then the theory is finite at $n \to 4$.

Note that we may give λ_R a smooth n dependence, so $R_1(n)$ is an arbitrary function of n, finite at $n = 4$. The numbers

$$\widetilde{A}(0) = -a_{12} \qquad \text{and} \qquad \widetilde{A}'(0) = -2a_{13}\,, \tag{7.11}$$

(see eq. (2.3a)), correspond to the one-loop and two-loop poles respectively, and can be calculated exactly. For instance

$$\widetilde{A}(0) = \frac{3}{16\,\pi^2} \tag{7.12}$$

in φ^4 theory (with the usual sign for λ) and

$$\widetilde{A}(0) = -\,\frac{11}{12\,\pi^2}\,, \tag{7.13}$$

for the coupling constant g^2 in pure SU(2) Yang-Mills theory.

Although $\lambda_B \to 0$ we have not a free field theory (the interaction strength is roughly equal to λ_R in eq. (7.10)).

Note also that our result, formula (7.9), only holds for the "full" theory, not the perturbation series, as can be deduced from the singular behaviour of $R_1(4, \lambda_R)$ at $\lambda_R \to 0$ (see eq. (7.10)).

In the same way the mass-renormalization can be calculated. Taking $b_\nu(\lambda)$ proportional to m_R, or

$$m_B = \rho(\lambda_R, n)\, m_R\,,$$

we find from (4.4b):

$$\rho(\lambda_R, n) = \exp \int_0^{\lambda_R} \mathrm{d}\lambda\, \frac{\widetilde{B}(\lambda)}{\lambda \widetilde{A}(\lambda) + n - 4}\,, \tag{7.14}$$

where

$$\lambda b_{1,\lambda} = m_R\, \lambda \widetilde{B}(\lambda)\,.$$

Here also one could consider first taking $n \to 4$. We find that finiteness of the theory then requires the mass renormalization (if $\widetilde{A}(0) \neq 0$):

$$m_B = (n-4)^{-\widetilde{B}(0)/\widetilde{A}(0)}\, R_2(n)\,, \tag{7.15}$$

with $R_2(n)$ finite at $n \to 4$, and proportional to m_R.

Note that $\widetilde{B}(0) = b_{11}/m_R$ (see (2.3a)), corresponds to the one-loop mass renormalization and can be calculated exactly also. In the $\lambda\varphi^4$ theory

$$\widetilde{B}(0) = -\,\frac{1}{32\,\pi^2}\,, \tag{7.16}$$

and eq. (7. 15) becomes

$$m_B = (n-4)^{\frac{1}{6}}\; R_2(n)\,. \tag{7.17}$$

It is remarkable that only one-loop infinities contribute to this mass renormalization, while only the one and two-loop infinities determine the coupling constant renormalization. It is left as an exercise for the reader to see what happens in a theory with $\tilde{A}(0) = 0$.

8. Conclusions

Making use of the observation that at $n \neq 4$ all integrals in perturbation field theory can be made finite, and that these finite expressions have the "naive" scaling properties, we found that the scaling behaviour at $n = 4$ of dimensionally regularized field theories can be formulated in terms of simple equations (3.9), (5.10), (6.2). These equations are closely related to the Callan-Symanzik equations [5], but they have the advantage that the coefficients are completely determined by the residues of the poles at $n = 4$, which one adds to the parameters in the Lagrangian, in order to make the theory finite at $n \to 4$. In particular the single-loop contribution to these poles (which are the most important ones) can be obtained rapidly for complicated theories by means of a "pole-algebra", to be derived in a future publication [4].

Further we derived equations between residues of lower and higher poles at $n = 4$, which are so stringent that in certain cases the complete singular behaviour at $n \to 4$ of the bare parameters in the Lagrangian can be determined exactly (eqs. (7.9), (7.15)). The results in sect. 7 should, however, be interpreted with care, since they hold only for the summed theory (if such a thing exists), not for the individual terms in the perturbation expansion.

References

[1] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.
[2] C.G. Bollini and J.J. Giambiagi, Phys. Letters 40B (1972) 566;
J.F. Ashmore, Nuovo Cimento Letters 4 (1972) 289.
[3] K.G. Wilson, Cornell preprint CLNS-198 (1972), Quantum field theory models in less than 4-dimensions.
[4] G. 't Hooft, subsequent publication.
[5] C.G. Callan, Phys. Rev. D2 (1970) 1541;
K. Symanzik, Comm. Math. Phys. 18 (1970) 48.
[6] K. Symanzik, DESY preprints 72/68 (1972) and 72/73 (1972).
[7] D.J. Gross, F. Wilczek, Princeton preprint (1973); G. 't Hooft, Marseille conference, 19–23 June 1972, unpublished.