

IMPROVING MATHEMATICS IN KEY STAGES 2 AND 3

Guidance Report



Education
Endowment
Foundation

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The guidance was based on a review of the research evidence conducted by Prof Jeremy Hodgen (University College London Institute of Education), Dr Colin Foster (University of Nottingham), Dr Rachel Marks (University of Brighton), and Prof Margaret Brown (King's College London). The EEF would like to thank the many other researchers and practitioners who provided support and feedback on drafts of this guidance.

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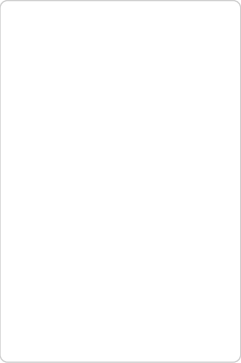
About the Education Endowment Foundation

The Education Endowment Foundation (EEF) is an independent charity supporting teachers and school leaders to use evidence of what works—and what doesn't—to improve educational outcomes, especially for disadvantaged children and young people.

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FOREWORD



Leaving school with a good GCSE in maths is a prerequisite for progressing into quality jobs, apprenticeships, and further education. The skills we learn at school help us with everyday life too. Yet too many of our young people do not make the grade and, as a result, risk social and economic exclusion. These pupils are disproportionately drawn from disadvantaged homes. Last year, over half of those eligible for free school meals had not achieved the expected level in English and maths by age 16. That's one in two young people from low-

income households who are automatically denied access to secure and well-paid careers, as well as to further study. This is not just a personal tragedy for the individual; it's a waste of talent on a national scale and a huge barrier to improving social mobility.

To truly break this link between family income and educational attainment, we have to start early and make sure that all young people—regardless of background—have access to great maths teaching in primary and secondary school.

At the Education Endowment Foundation, we believe the best way to do this is through better use of evidence: looking at what has—and has not—worked in the past can put us in a much better place to judge what is likely to work in the future. But it can be difficult to know where to start. There are thousands of studies of maths teaching out there, most of which are presented in academic papers and journals. Teachers are inundated with information about programmes and training courses too, all of which make claims about impact.

How can anyone know which findings are the most secure, reliable, and relevant to their school and pupils?

This is why we've produced this guidance report. It offers eight practical, evidence-based recommendations that are relevant to all pupils—but particularly to those struggling with their mathematics. To develop the recommendations, we reviewed the best available international research and consulted experts, teachers, and academics to arrive at key principles for effective teaching.

I hope this report will help to support consistently excellent, evidence-informed maths teaching in England that creates great opportunities for all children, regardless of their family background. It is a starting point for a more evidence-informed approach to teaching maths and the EEF and its partners, particularly our network of Research Schools, will be producing a range of supporting resources, tools, and training to help you implement the recommendations in your classrooms.



Sir Kevan Collins

Chief Executive
Education Endowment Foundation

INTRODUCTION

What does this guidance cover?

This guidance report focuses on the teaching of mathematics to pupils in Key Stages 2 and 3. The decision to focus on these Key Stages was made after an initial consultation period with teachers, academics, and other stakeholders. The consultation suggested that these were areas where guidance could make a big impact as not only are schools seeking advice on adjusting to a new curriculum, there is also concern about pupils making a transition between the stages.

This report is not intended to provide a comprehensive guide to mathematics teaching. We have made recommendations where there are research findings that schools can use to make a significant difference to pupils' learning, and have focused on the questions that appear to be most salient to practitioners. There are aspects of mathematics teaching not covered by this guidance. In these situations, teachers must draw on their knowledge of mathematics, professional experience and judgement, and assessment of their pupils' knowledge and understanding.

The focus is on improving the quality of teaching. Excellent maths teaching requires good content knowledge, but this is not sufficient. Excellent teachers also know the ways in which pupils learn mathematics and the difficulties they are likely to encounter, and how mathematics can be most effectively taught.¹ The guidance draws on a review of the evidence on effective maths teaching conducted by Prof Jeremy Hodgen, Dr Colin Foster, and Dr Rachel Marks. As such, it is not a new study in itself, but rather is intended as an accessible overview of existing research with clear, actionable guidance. More information about how this guidance was created is available at the end of the report.

Who is this guidance for?

This guidance is aimed primarily at subject leaders, headteachers, and other staff with responsibility for leading improvements in mathematics teaching in primary and secondary schools. Classroom teachers and teaching assistants will also find this guidance useful as a resource to aid their day-to-day teaching.

It may also be used by:

- governors and parents to support and challenge school staff;
- programme developers to inform their development of both professional development for teachers and interventions for pupils; and
- educational researchers to conduct further testing of the recommendations in this guidance, and fill in gaps in the evidence.

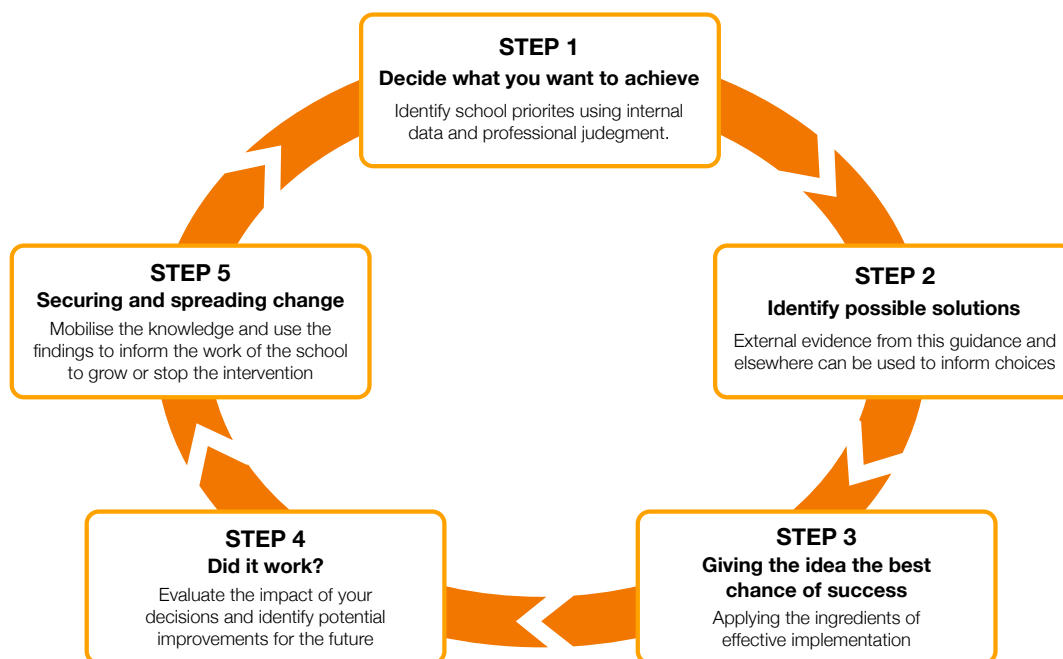
INTRODUCTION

Acting on the guidance

We recognise that the effective implementation of these recommendations—such that they make a real impact on children—is both critical and challenging. There are several key principles to consider when acting on this guidance.

1. Continuing Professional Development (CPD) will be an important component of implementation and is key to raising the quality of teaching and teacher knowledge. A summary of the best available evidence regarding CPD is available from the Teacher Development Trust.*
2. These recommendations do not provide a ‘one size fits all’ solution. It is important to consider the delicate balance between implementing the recommendations faithfully and applying them appropriately in a school’s particular context. Implementing the recommendations effectively will therefore require careful consideration of context as well as sound professional judgement.
3. It is important to consider the precise detail provided beneath the headline recommendations. For example, schools should not use **Recommendation 7** to justify the purchase of lots of interventions. Rather, it should provoke thought about the most appropriate interventions to buy.
4. Inevitably, change takes time, and we recommend taking at least two terms to plan, develop, and pilot strategies on a small scale before rolling out new practices across the school. Gather support for change across the school and set aside regular time throughout the year to focus on this project and review progress.

Figure 1: An evidence-informed school improvement cycle



* <https://tdtrust.org/about/dgt>

INTRODUCTION

What support is available for using this guidance?

- The EEF has partnered with the Institute for Effective Education to launch a national network of Research Schools. Research Schools will become a focal point for evidence-based practice in their region, building affiliations with large numbers of schools, and supporting the use of evidence at scale. More information about the Research Schools Network, and how it supports the use of EEF guidance reports, can be found at:
<https://researchschool.org.uk>
 - A number of resources will be produced by the EEF and its partners to support the implementation of the guidance report, such as audit and observation tools and further examples.
 - An upcoming guidance report will make recommendations regarding effective implementation of evidence-based approaches.
- It will draw on the best available evidence regarding implementation and examine the role of leadership, CPD, and evaluation. This report will be made available at: eef.li/guidance

SUMMARY OF RECOMMENDATIONS

1



Use assessment to build upon pupils' existing knowledge and understanding

- Assessment should be used not only to track pupils' learning but also to provide teachers with information about what pupils do and do not know.
- This should inform the planning of future lessons and the focus of targeted support.
- Effective feedback will be an important element of teachers' response to assessment.
- Feedback should be specific and clear, encourage and support further effort, and be given sparingly.
- Teachers not only have to address misconceptions but also understand why pupils may persist with errors.
- Knowledge of common misconceptions can be invaluable in planning lessons to address errors before they arise.

2



Use manipulatives and representations

- Manipulatives (physical objects used to teach maths) and representations (such as number lines and graphs) can help pupils engage with mathematical ideas.
- However, manipulatives and representations are just tools: how they are used is essential.
- They need to be used purposefully and appropriately to have an impact.
- There must be a clear rationale for using a particular manipulative or representation to teach a specific mathematical concept.
- Manipulatives should be temporary; they should act as a 'scaffold' that can be removed once independence is achieved.

3



Teach strategies for solving problems

- If pupils lack a well-rehearsed and readily available method to solve a problem they need to draw on problem solving strategies to make sense of the unfamiliar situation.
- Select problem solving tasks for which pupils do not have ready-made solutions.
- Teach them to use and compare different approaches.
- Show them how to interrogate and use their existing knowledge to solve problems.
- Use worked examples to enable them to analyse the use of different strategies.
- Require pupils to monitor, reflect on, and communicate their problem solving.

4



Enable pupils to develop a rich network of mathematical knowledge

- Emphasise the many connections between mathematical facts, procedures, and concepts.
- Ensure that pupils develop fluent recall of facts.
- Teach pupils to understand procedures.
- Teach pupils to consciously choose between mathematical strategies.
- Build on pupils' informal understanding of sharing and proportionality to introduce procedures.
- Teach pupils that fractions and decimals extend the number system beyond whole numbers.
- Teach pupils to recognise and use mathematical structure.

5



Develop pupils' independence and motivation

- Encourage pupils to take responsibility for, and play an active role in, their own learning
- This requires pupils to develop metacognition—the ability to independently plan, monitor and evaluate their thinking and learning
- Initially, teachers may have to model metacognition by describing their own thinking.
- Provide regular opportunities for pupils to develop metacognition by encouraging them to explain their thinking to themselves and others.
- Avoid doing too much too early.
- Positive attitudes are important, but there is scant evidence on the most effective ways to foster them.
- School leaders should ensure that all staff, including non-teaching staff, encourage enjoyment in maths for all children.

6



Use tasks and resources to challenge and support pupils' mathematics

- Tasks and resources are just tools—they will not be effective if they are used inappropriately by the teacher.
- Use assessment of pupils' strengths and weaknesses to inform your choice of task.
- Use tasks to address pupil misconceptions.
- Provide examples and non-examples of concepts.
- Use stories and problems to help pupils understand mathematics.
- Use tasks to build conceptual knowledge in tandem with procedural knowledge.
- Technology is not a silver bullet—it has to be used judiciously and less costly resources may be just as effective.

7



Use structured interventions to provide additional support

- Selection should be guided by pupil assessment.
- Interventions should start early, be evidence-based and be carefully planned.
- Interventions should include explicit and systematic instruction.
- Even the best designed intervention will not work if implementation is poor.
- Support pupils to understand how interventions are connected to whole-class instruction.
- Interventions should motivate pupils—not bore them or cause them to be anxious.
- If interventions cause pupils to miss activities they enjoy, or content they need to learn, teachers should ask if the interventions are really necessary.
- Avoid 'intervention fatigue'. Interventions do not always need to be time-consuming or intensive to be effective.

8



Support pupils to make a successful transition between primary and secondary school

- There is a large dip in mathematical attainment and attitudes towards maths as children move from primary to secondary school.
- Primary and secondary schools should develop shared understandings of curriculum, teaching and learning.
- When pupils arrive in Year 7, quickly attain a good understanding of their strengths and weaknesses.
- Structured intervention support may be required for Year 7 pupils who are struggling to make progress.
- Carefully consider how pupils are allocated to maths classes.
- Setting is likely to lead to a widening of the attainment gap between disadvantaged pupils and their peers, because the former are more likely to be assigned to lower group.

1 Use assessment to build upon pupils' existing knowledge and understanding



Mathematical knowledge and understanding can be thought of as consisting of several components and it is quite possible for pupils to have strengths in one component and weaknesses in another.² It is therefore important that assessment is not just used to track pupils' learning but also provides teachers with up-to-date and accurate information about the specifics of what pupils do and do not know. This information allows teachers to adapt their teaching so it builds on pupils' existing knowledge, addresses their weaknesses, and focuses on the next steps that they need in order to make progress. Formal tests can be useful here, although assessment can also be based on evidence from low-stakes class assessments, informal observation of pupils, or discussions with them about mathematics. More guidance on how to conduct useful and accurate assessment is available in the EEF's guidance on Assessing and Monitoring Pupil Progress, available online.

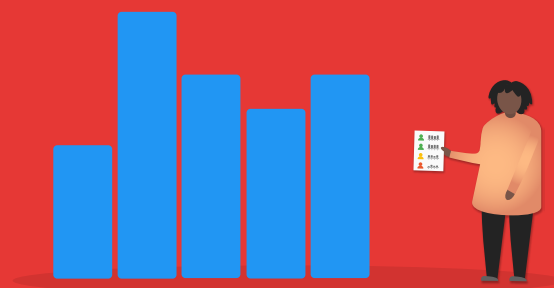
Responding to Assessment

Teachers' knowledge of pupils' strengths and weaknesses should inform the planning of future lessons and the focus of targeted support (see **Recommendation 7**). Teachers may also need to try a different approach if it appears that what they tried the first time did not work. Effective feedback will be an important element of teachers' responses to assessment information. Consider the following characteristics of effective feedback:³

- **be specific, accurate, and clear** (for example, 'You are now factorising numbers efficiently, by taking out larger factors earlier on', rather than, 'Your factorising is getting better');
- **give feedback sparingly so that it is meaningful** (for example, 'One of the angles you calculated in this problem is incorrect—can you find which one and correct it?');
- **compare what a pupil is doing right now with what they have done wrong before** (for example, 'Your rounding of your final answers is much more accurate than it used to be');

- **encourage and support further effort** by helping pupils identify things that are hard and require extra attention (for example, 'You need to put extra effort into checking that your final answer makes sense and is a reasonable size');
- **provide guidance to pupils on how to respond to teachers' comments**, and give them time to do so; and
- **provide specific guidance on how to improve** rather than just telling pupils when they are incorrect (for example, 'When you are unsure about adding and subtracting numbers, try placing them on a number line', rather than 'Your answer should be -3 not 3').

Feedback needs to be efficient. Schools should be careful that their desire to provide effective feedback does not lead to onerous marking policies and a heavy teacher workload.⁴ Effective feedback can be given orally; it doesn't have to be in the form of written marking. A summary of the evidence regarding different marking approaches and their impact on workload ('A Marked Improvement') is available on the [EEF website](#).



Addressing misconceptions

A misconception is an understanding that leads to a 'systematic pattern of errors'.⁵ Often misconceptions are formed when knowledge has been applied outside of the context in which it is useful. For example, the 'multiplication makes bigger, division makes smaller' conception applies to positive, whole numbers greater than 1. However, when subsequent mathematical concepts appear (for example, numbers less than or equal to 1), this conception, extended beyond its useful context, becomes a *mis*-conception.*

It is important that misconceptions are uncovered and addressed rather than side-stepped or ignored. Pupils will often defend their misconceptions, especially if they are based on sound, albeit limited, ideas. In this situation, teachers could think about how a misconception might have arisen and explore with pupils the 'partial truth' that it is built on and the circumstances where it no longer applies.⁶ Counterexamples can be effective in challenging pupils' belief in a misconception. However, pupils may need time and teacher support to develop richer and more robust conceptions.

Knowledge of the common errors and misconceptions in mathematics can be invaluable when designing and responding to assessment, as well as for predicting

the difficulties learners are likely to encounter in advance.⁷ Teachers with knowledge of the common misconceptions can plan lessons to address potential misconceptions before they arise, for example, by comparing examples to non-examples when teaching new concepts. A non-example is something that is not an example of the concept.

Evidence summary

- Feedback is generally found to have large effects on learning. The review identified two meta-analyses indicating that the effects of feedback in mathematics are similar to other subjects. There is considerable variability in reported effects, and, feedback can have powerful negative as well as positive impacts on learning.
- There is a detailed literature on the misconceptions that pupils often develop when learning maths.

* More information about the common misconceptions and misunderstandings which students develop in different mathematical topics can be found in the following texts.

— Hansen, A. (ed.) (2017). *Children's Errors in Mathematics* (4th edn). London: Sage.

— Ryan, J., and Williams, J. (2007). *Children's mathematics 4-15: Learning from errors and misconceptions*. McGraw-Hill Education.

— Hart, K. M., Brown, M. L., Kuchemann, D. E., Kerslake, D., Ruddock, G., and McCartney, M. (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.

2 Use manipulatives and representations



Manipulatives and representations can be powerful tools for supporting pupils to engage with mathematical ideas. However, manipulatives and representations are just tools: how they are used is important. They need to be used purposefully and appropriately in order to have an impact.⁸ Teachers should ensure that there is a clear rationale for using a particular manipulative or representation to teach a specific mathematical concept. The aim is to use manipulatives and representations to reveal mathematical structures and enable pupils to understand and use mathematics independently.

Box A: What are manipulatives and representations?

A manipulative is a physical object that pupils or teachers can touch and move which is used to support the teaching and learning of mathematics. Common manipulatives include Cuisenaire rods and Dienes blocks.

The term 'representation' refers to a particular form in which mathematics is presented.⁹ Examples of different representations include:

- two fractions represented on a number line;
- a quadratic function expressed algebraically or presented visually as a graph; and
- a probability distribution presented in a table or represented as a histogram.

What does effective use of manipulatives look like?

Manipulatives can be used across both Key Stages. The evidence suggests some key considerations:

- Ensure that there is a clear rationale for using a particular manipulative or representation to teach a specific mathematical concept. Manipulatives should be used to provide insights into increasingly sophisticated mathematics.
- Enable pupils to understand the links between the manipulatives and the mathematical ideas they represent.¹⁰ This requires teachers to encourage pupils to link the materials (and the actions performed on or with them) to the mathematics of the situation, to appreciate the limitations of concrete materials, and to develop related mathematical images, representations and symbols.¹¹
- Try to avoid pupils becoming reliant on manipulatives to do a type of task or question. A manipulative should enable a pupil to understand mathematics by illuminating the underlying general relationships, not just 'getting them to the right answer' to a specific problem.¹²
- Manipulatives should act as a 'scaffold', which can be removed once independence is achieved. Before using a manipulative, it is important to consider how it can enable pupils to eventually do the maths without it. When moving away from manipulatives, pupils may find it helpful to draw diagrams or imagine using the manipulatives.
- Manipulatives can be used to support pupils of all ages. The decision to remove a manipulative should be made in response to the pupils' improved knowledge and understanding, not their age.

Box B: Using manipulatives—an example

A teacher said, 'Give me a two-digit number ending in 0'. A pupil said, 'Forty'.

The teacher said, 'I'm going to subtract the tens digit from the number: $40 - 4$ gives me 36.'

The pupils tried this with other two-digit numbers ending in 0 and discovered that the result was always a multiple of 9.

The teacher said, 'I'm going to use multilink cubes to see whether this will help us see why we always get a multiple of 9.'

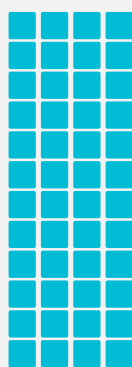
The teacher made four sticks of 10 cubes each.

'So here is 40. What does it look like if we remove 4 cubes?' (a)

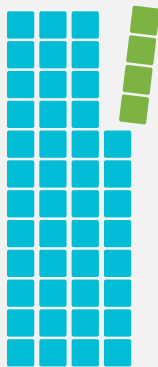
A pupil came to the front of the classroom and removed 4 cubes. (b)

'How else could you do it?' asked the teacher.

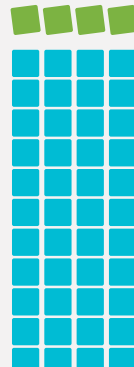
Another pupil removed 4 cubes in a different way. (c)



(a)



(b)



(c)

A pupil said, 'Ah yes! If we take away one from each 10 then we are left with four 9s.'

Another pupil said, 'And if we started with 70 we'd have 10 sevens take away 1 seven is 9 sevens.'

The teacher wrote:

$$40 - 4 = 10 \times 4 - 1 \times 4 = 9 \times 4$$

$$70 - 7 = 10 \times 7 - 1 \times 7 = 9 \times 7$$

and the pupils discussed what is going on here, before the teacher concluded with the generalisation:

$$10t - t = (10 - 1)t = 9t$$

What about other types of representation?

The evidence indicates that number lines are a particularly effective representation for teaching across both Key Stages 2 and 3, and that there is strong evidence to support the use of diagrams as a problem-solving strategy. The specific evidence

regarding the use of representations more generally is weaker. However, it is likely that the points above regarding effective use of manipulatives apply to all other representations.

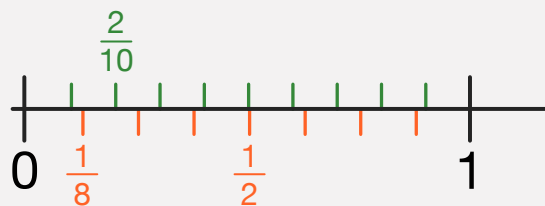
Box C: Using a number line

The teacher noticed that some pupils were incorrectly adding fractions by adding the numerators and the denominators. She gave the class this task:

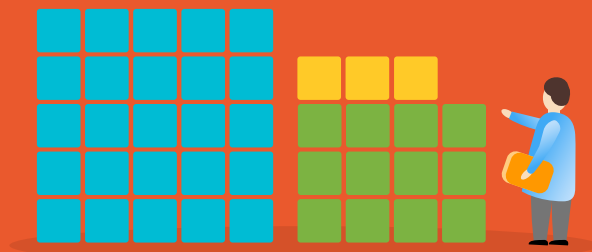
'Is this true?'

$$\frac{1}{2} + \frac{1}{8} = \frac{2}{10}$$

Some pupils noticed that $\frac{2}{10}$ is less than $\frac{1}{2}$. With the teacher's help, the pupils represented the three fractions using a number line.



This helped pupils to see that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$, and then to work out that the answer is $\frac{5}{8}$. The pupils then invented their own examples of incorrect and correct fraction additions using number lines to make sense of it.



While in general the use of multiple representations appears to have a positive impact on attainment, more research is needed to inform teachers' choices about which, and how many, representations to use when.¹³ There is promising evidence that comparison and discussion of different representations can help pupils develop conceptual understanding. Teachers should purposefully select different representations of key mathematical ideas to discuss and compare, with the aim of supporting pupils to develop more abstract, diagrammatic representations. However, while using multiple representations can aid understanding, teachers should be aware that using too many representations at one time may cause confusion and hinder learning.¹⁴

Evidence summary

- The review identified five relevant meta-analyses concerned with the use of concrete manipulatives and representations. The evidence was stronger in support of concrete manipulatives.
- Two systematic reviews by the US What Works Clearinghouse provide evidence to support the use of visual representations, particularly in problem solving and to support pupils who are struggling with mathematics.

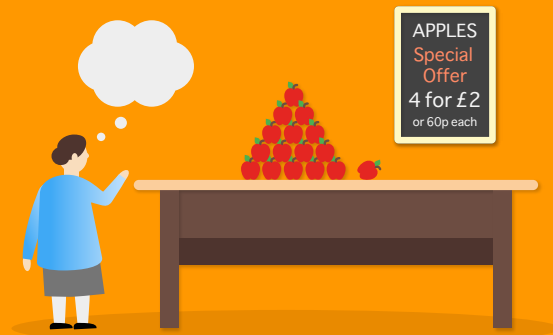
3 Teach strategies for solving problems



Problem solving generally refers to situations in which pupils do not have a readily-available method that they can use. Instead, they have to approach the problem flexibly and work out a solution for themselves. To succeed in this, pupils need to draw on a variety of problem-solving strategies (see **Box D**) which enable them to make sense of unfamiliar situations and tackle them intelligently.

The evidence suggests that teachers should consider the following when developing these skills.¹⁵

- Select genuine problem-solving tasks that pupils do not have well-rehearsed, ready-made methods to solve. Sometimes problem-solving is taken to mean routine questions set in context, or ‘word problems’, designed to illustrate the use of a specific method. But if students are only required to carry out a given procedure or algorithm to arrive at the solution, it is not really problem solving; rather, it is just practising the procedure.
- Consider organising teaching so that problems with similar structures and different contexts are presented together, and, likewise, that problems with the same context but different structures are presented together. Pupils need to experience identifying similar mathematics that underlies different situations, and also to identify and interrogate multiple relationships between variables in one situation.
- Teach pupils to use and compare different approaches. There are often multiple ways to approach a problem. Much can be learned by examining different solutions to the same problem and looking for similarities in solution approaches to different problems. Pupils will need to distinguish between superficial similarity (for example, two problems both about carrots) and deeper similarities, relating to mathematical structure, which make similar strategies effective (such as two problems in different contexts that are both about enlargement). Teach pupils to interrogate and use their existing mathematical knowledge to solve problems. Pupils should be encouraged to search their knowledge of similar problems they have encountered for strategies that were successful, and for facts and concepts that might be relevant.
- Encourage pupils to use visual representations. Help students to make use of appropriate diagrams and representations that provide insight into the structure of a problem and into its mathematical formulation.
- Use worked examples to enable pupils to analyse the use of different strategies. Worked examples, or ‘solved problems’, present the problem and a correct solution together, they remove the need to carry out the procedures required to reach the solution and enable pupils to focus on the reasoning and strategies involved. Worked examples may be complete, incomplete, or incorrect, deliberately containing common errors and misconceptions for learners to uncover. Analysing and discussing worked examples helps students develop a deeper understanding of the logical processes used to solve problems.
- Require pupils to monitor, reflect on, and communicate their reasoning and choice of strategy. While working on a problem, encourage pupils to ask questions like, ‘What am I trying to work out?’, ‘How am I going about it?’, ‘Is the approach that I’m taking working?’, and ‘What other approaches could I try?’ When the problem is completed, encourage pupils to ask questions like, ‘What worked well when solving this problem?’, ‘What didn’t work well?’, ‘What other problems could be solved by a similar approach?’, ‘What similar problems to this one have I solved in the past?’ Pupils should communicate their thinking verbally and in writing—using representations, expressions, and equations—to both teachers and other pupils.



Box D: What is a problem-solving strategy?

A problem-solving strategy is a general approach to solving a problem. The same general strategy can be applied to solving a variety of different problems. For example, a useful problem-solving strategy is to identify a simpler but related problem. Discussing the solution to the simpler problem can give insight into how the original, harder problem may be tackled and the underlying mathematical structure.

A strategy is different from an algorithm, which is a well-established sequence of predetermined steps that are executed in a particular order to carry out a commonly-required procedure.

Evidence summary

- The review identified three relevant meta-analyses concerned with the teaching of problem-solving. These provide some evidence for the use of problem-solving, although the effects were varied.
- A systematic review by the US What Works Clearinghouse provides evidence for specific teaching approaches. The evidence strength is judged to be strongest in support of the use of visual representations and worked examples, and encouraging pupils to monitor and reflect on the problem-solving process.

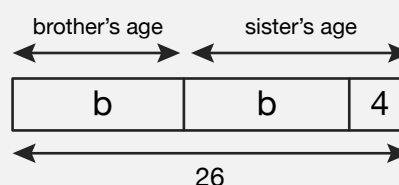
Box E: Using the bar model to compare strategies

A class was working on this problem:

*'A sister is 4 years older than her brother.
The total of their ages is 26.
How old are they?'*

One pupil used the bar model to solve the problem:

So the brother is 11 years old and the sister is 15 years old.



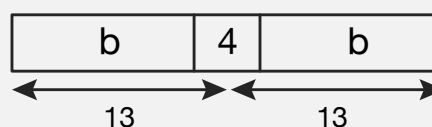
$$b + b + 4 = 26$$

$$\text{so } b + 4 = 22$$

Another pupil said, 'Couldn't we just halve 26 and then add and take away 4? So their ages are $13 + 4$ and $13 - 4$, which is 17 and 9.'

Another pupil said, 'No, you don't add and take away 4, you add and take away 2'. The teacher drew this revised bar model diagram:

'How does this help us to see whether it's 4 or 2 that you add and subtract?'



A pupil said, 'There are two lots of b in the 26, there is only one 4, so it was correct to add and subtract 2, not 4.'

4 Enable pupils to develop a rich network of mathematical knowledge



This recommendation presents the evidence regarding teaching specific topics in mathematics. Although this recommendation concerns particular topics, teaching should emphasize the many connections between different mathematical facts, procedures, and concepts to create a rich network.

Currently, the evidence about effective teaching approaches is stronger regarding number (including fractions, ratio and proportion) and algebra than for other areas such as geometry. However, it is likely that some of the approaches below (particularly choosing between strategies, paying attention to mathematical structure, and building on pupils' informal knowledge) apply across mathematical topics. Teachers should adopt such approaches while drawing on their knowledge of maths, their own professional experience and the other recommendations in this guidance.

Ensure that pupils develop fluent recall of number facts

Quick retrieval of number facts is important for success in mathematics.¹⁶ It is likely that pupils who have problems retrieving addition, subtraction, multiplication, and division facts, including number bonds and multiples, will have difficulty understanding and using mathematical concepts they encounter later on in their studies.

Teach pupils to understand procedures

Pupils are able to apply procedures most effectively when they understand how the procedures work and in what circumstances they are useful.¹⁷ Fluent recall of a procedure is important, but teachers should ensure that appropriate time is spent on developing understanding.

One reason for encouraging understanding is to enable pupils to reconstruct steps in a procedure that they may have forgotten.¹⁸ The recommendations in this guidance on visual representations, misconceptions, and setting problems in real-world contexts are useful here.

Teach pupils to choose between mathematical strategies

Teachers should help pupils to compare and choose between different methods and strategies for solving

problems in algebra, number, and elsewhere. Pupils should be taught a range of mental, calculator, and pencil-and-paper methods, and encouraged to consider when different methods are appropriate and efficient.

The evidence suggests that using a calculator does not generally harm students' mental or pencil-and-paper calculation skills. In fact, studies have shown using a calculator can have positive impacts, not only on mental calculation skills, but also on problem-solving and attitudes towards maths.¹⁹ Calculators should be integrated into the teaching of mental and other calculation approaches, and pupils should be taught to make considered decisions about when, where, and why to use particular methods. The aim is to enable pupils to self-regulate their use of calculators, consequently making less (but better) use of them.²⁰

Build on pupils' informal understanding of multiplicative reasoning to introduce procedures

Multiplicative reasoning is the ability to understand and think about multiplication and division. It is an important skill which is required for tasks that involve ratios, rates, and proportions, and is often required in real-life contexts such as "best buy" problems. For example, a common misconception—when asked to work out the quantities required for 10 people from a recipe for 4 people—involves many pupils adding 6 (because the difference in the number of people is 6) rather than multiplying by 2., or doubling and adding half as much again (because the ratio is 4:10, 2:5, or 1:2.).



There is some evidence to suggest that delaying the teaching of formal methods in order to focus on developing pupils' multiplicative reasoning is beneficial.²¹ Teachers should consider using a series of contexts and tasks which progressively build on pupils' informal understanding. After introducing formal procedures and algorithms, teachers should return to pupils' initial informal strategies and show when they lead to the same answers, as well as when and why they may be less effective.

Teach pupils that fractions and decimals extend the number system beyond whole numbers

Fractions are often introduced to pupils with the idea that they represent parts of a whole—for example, one half is one part of a whole that has two equal parts. This is an important concept, but does not extend easily to mixed fractions that are greater than 1. Another important concept is often overlooked: fractions are numbers which can be represented on the number line. They have magnitudes or values, and they can be used to refer to numbers in-between whole numbers.²²

Understanding that fractions are numbers, and being able to estimate where they would occur on a number line, can help pupils to estimate the result of adding two fractions and so recognise, and address, misconceptions such as the common error of adding fractions by adding the numerators and then adding the denominators.

Number lines are a useful tool for teaching these concepts. They can be used to:

- represent the magnitude, or value, of fractions, decimals, and rational numbers generally; and
- compare the magnitudes of fractions, decimals, and whole numbers.

Teach pupils to recognise and use mathematical structure

Paying attention to underlying mathematical structure helps pupils make connections between problems, solution strategies, and representations that may, on the surface, appear different, but are actually mathematically equivalent. Teachers should support pupils to use language that reflects mathematical structure, for example by rephrasing pupils' responses that use vague, non-mathematical language with appropriate mathematical language. Some examples of teachers supporting pupils to recognise mathematical structure are:

- encouraging pupils to read numerical and algebraic expressions as descriptions of relationships, rather than simply as instructions to calculate²³ (for example, pupils often regard the equals sign as an instruction to calculate rather than an indication of an equivalence,²⁴ understanding a relationship as an equivalence would mean thinking of ' $17 \times 25 = 10 \times 25 + 7 \times 25$ ' as ' 17×25 is the same as $10 \times 25 + 7 \times 25$ '); and
- enabling pupils to understand the inverse relations between addition and subtraction, and between multiplication and division.²⁵

Evidence summary

- The review identified two relevant meta-analyses concerning the teaching of algebra. However, these meta-analyses are at a general level and do not provide evidence about specific teaching approaches.
- Four systematic reviews by the US What Works Clearinghouse provide evidence for specific teaching approaches in number and algebra. The evidence strength is judged to be stronger in support of focusing on fluent recall, encouraging the deliberate choice of strategies and using number lines to represent fractions and decimals.

5 Develop pupils' independence and motivation



Teachers should encourage pupils to take responsibility for, and play an active role in, their own learning.²⁶ This will require pupils to develop metacognition (the ability to independently plan, monitor, and evaluate their thinking and learning) and motivation towards learning maths.

Develop pupils' metacognition through structured reflection on their learning

Developing pupils' metacognition can help them to become more effective and independent mathematicians. It is often thought of as pupils' ability to think about their own thinking and learning.

Examples of pupils demonstrating this ability include:

- examining existing knowledge to inform the selection of a particular approach to solving a mathematical task;
- monitoring whether the chosen approach has been successful; and then
- deliberately changing or continuing the approach based on that evidence.

Ultimately the aim is for pupils to be able to do this automatically and independently, without needing support from the teacher or their peers, however, these are complex skills which will initially require explicit teaching and support. Teachers should model metacognition (see example in **Box F**) by simultaneously describing their own thinking or asking questions of their pupils as they complete a task.²⁷ Worked examples could be usefully employed by the teacher to make their thinking explicit.²⁸ Teachers should carefully increase their expectations regarding pupils' independence as the pupils gain competence and fluency. Teachers can provide regular opportunities for pupils to develop independent metacognition through:

- encouraging self-explanation—pupils explaining to themselves how they planned, monitored, and evaluated their completion of a task; and
- encouraging pupils to explain their metacognitive thinking to the teacher and other pupils.²⁹

Box F: Modelling metacognition during problem solving

While demonstrating the solving of a problem, a teacher could model how to plan, monitor, and evaluate their thinking by reflecting aloud on a series of questions. These could include:

- *What is this problem asking?*
- *Have I ever seen a mathematical problem like this before? What approaches to solving it did I try and were they successful?*
- *Could I represent the problem with a diagram or graph?*
- *Does my answer make sense when I re-read the problem?*
- *Do I need help or more information to solve this problem? Where could I find this?*

Evidence summary

- The review identified three relevant meta-analyses concerned with the teaching of problem-solving. These provide some evidence for the use of problem-solving, although the effects were varied.
- A systematic review by the US What Works Clearinghouse provides evidence for specific teaching approaches. The evidence strength is judged to be strongest in support of the use of visual representations and worked examples, and encouraging pupils to monitor and reflect on the problem-solving process.



Developing metacognition is not straightforward and there are some important challenges to consider.

- Teachers need to ensure that pupils' metacognition does not detract from concentration on the mathematical task itself.³⁰ This might happen if pupils are expected to do too much, too early, without effective scaffolding from their teacher.
- Regardless of the strategy being taught, pupils need significant time to imitate, internalise, and independently apply strategies, with strategies

used repeatedly across many maths lessons.³¹ It is likely that the time required to develop metacognition is much greater than for other skills and knowledge.

- Discussion and dialogue can be useful tools for developing metacognition, but pupils may need to be taught how to engage in discussion.³² Teachers should model effective discussion and 'what to do as a listener'.³³ Orchestrating productive discussions requires considerable skill and so may require targeted professional development.

Build pupils' long-term motivation towards learning and doing mathematics

The development of positive attitudes and motivation is, of course, itself an important goal for teaching. It can also support the development of self-regulation and metacognition, as these capabilities require deliberate and sustained effort, which can require motivation over a long period of time. Motivation is complex and may be influenced by a like or dislike of maths, beliefs about whether one is good or bad at maths, and beliefs about whether mathematics is useful or not.³⁴

Unfortunately, many pupils hold negative attitudes about mathematics, and pupils' attitudes tend to worsen as they get older. In a recent survey, the proportion of pupils who reported that they do not like learning maths was 17% at Year 5 and 48% at Year 9.³⁵

Although positive attitudes are important, there is a lack of evidence regarding effective approaches to developing them. It is likely to be important to model positive attitudes towards mathematics throughout the whole school. School leaders should ensure that all staff, including non-teaching staff, encourage and model motivation, confidence, and enjoyment in maths for all children.

Teachers should engage parents to encourage their children to value, and develop confidence in, mathematics.³⁶ However, teachers should exercise caution when engaging parents directly in pupils' mathematics learning, for example by helping with homework, as interventions designed to do this have often not been linked to increased attainment.³⁷

Box G: Modelling metacognition during problem solving

Maths anxiety is a type of anxiety that specifically interferes with mathematics, and is not the same as general anxiety. It can have a large detrimental impact on pupils' learning by overloading their working memory or causing them to avoid mathematics. Mathematics anxiety tends to increase with age, but there are signs of it appearing even in children in Key Stage 1.³⁸ Unfortunately, while there is some promising research, there is a limited understanding of how to reduce it.³⁹ Gaining an awareness of, and ability to recognise, the problem is the first step.

Teachers should look out for pupils avoiding maths or displaying signs of anxiety ('freezing', sweating, fidgeting) when using maths, and use their knowledge of their pupils, and professional judgement, to support them to overcome their anxiety.

6 Use tasks and resources to challenge and support pupils' mathematics



Tasks are critical to the learning of mathematics because the tasks used in the classroom largely define what happens there. However, the evidence suggests that the choice of one particular task or resource over another is less important than the way that teachers set about using them in the classroom.⁴⁰ Tasks and resources are tools which need to be deployed effectively to have a positive impact on learning.

Effective use of tasks and resources requires a considerable level of skill: many teachers will require focused support to achieve this. School leaders should make this a priority for CPD.

Evidence summary

- The review was not able to identify meta-analyses on the use of tasks, although there is a great deal of literature on the use and design of tasks. One large-scale and robust study indicated that teacher knowledge is key to realising the learning potential of a task.
- The review identified two relevant meta-analyses concerned with the effects of different textbooks. These provide moderate evidence indicating that the effects of one textbook scheme over another is at best small.
- The review identified 11 meta-analyses addressing aspects of technology.
- Despite the large number of reviews, the evidence regarding technology is limited.

Use of assessment of pupils' strengths and weaknesses to inform and use of tasks

A teacher asked pupils which of the following were equations of straight lines passing through the point (1, 2):

$$x = 1 \quad 5x = 3 + y \quad x = y - 1 \quad y = 2x^2$$

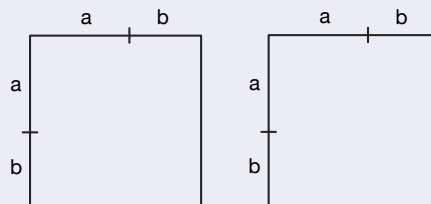
Some pupils checked to see whether (1, 2) satisfied these equations, but did not check that the equations were straight lines. Others dismissed all of the equations as they were not written in the form $y = mx + c$. This allowed the teacher to identify what students knew and did not know, about the equations of straight-line graphs.

After discussing the answers and using graphical software to show the graphs, the teacher asked the pupils to create some examples and non-examples of their own for straight lines passing through the point (2, -3). They now produced correct examples written in various forms, such as $3x + 2y = 0$, and their non-examples included both lines that did not pass through (2, -3) as well as curves.

Use tasks to address pupils' misconceptions

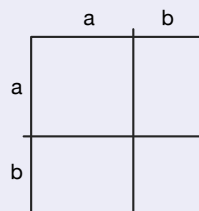
The teacher noticed that some pupils seemed to assume that $a^2 + b^2 = (a + b)^2$.

She drew two identical diagrams on the board.



She asked the pupils to represent $a^2 + b^2$ on one diagram and $(a + b)^2$ on the other.

By drawing two additional lines, as shown here, pupils could see that in general $a^2 + b^2 + 2ab = (a + b)^2$.

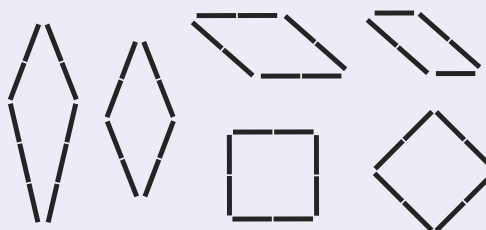


Provide examples and non-examples of concepts

A teacher asked for some definitions of 'rhombus', which she noted on the board.

She then revealed these shapes on the board, one at a time, each time asking, "Put your hand up if you think it's a rhombus".

She wrote the number of votes underneath each shape. She then told the class that the first shape was not a rhombus but the second was. She asked for new votes for the remaining shapes, which she again recorded.



She then gave the answer for the third and fourth shapes and asked for new votes for the remaining two shapes, which she again recorded.

She then gave the answer for the last two shapes and asked once more for some definitions of a rhombus, which the class then discussed.

Here, the examples and non-examples were carefully chosen, in terms of both shape and orientation, in order to highlight common misunderstandings, such as that a square is not a rhombus.

Using tasks effectively

What does the effective use of a task look like? The boxes that follow provide some key considerations with examples of what they mean in practice. There are many potential sources for useful tasks. Tasks will

often need preparation and adaptation to serve these purposes. It is likely that many tasks, even seemingly routine ones, can be used by a skilful teacher to support pupils to learn.

Discuss and compare different solution approaches

A teacher asked a class to come up with different ways of calculating 5×18 .

Here are some of their approaches:

- 'I can multiply 5 by 20, then take two 5s away':
 $5 \times 18 = 5 \times 20 - 5 \times 2 = 100 - 10 = 90$
- 'To multiply by 5, it's easy. I can multiply by 10 then halve the answer.'
 $10 \times 18 = 180, 180 \div 2 = 90$
- '18 is 9 times 2, so I can multiply 5 by 9, then multiply the answer by 2.'
 $5 \times 9 = 45, 45 \times 2 = 90$

The class discussed what similarities there were, how easy each method was to understand, and how efficient it was to execute mentally. The teacher asked the class to try to find similar ways to calculate 12×15 .

Use tasks to build conceptual knowledge in tandem with procedural knowledge

A teacher asked a class to perform this calculation using long multiplication:

$$\begin{array}{r} 34 \\ \times 52 \\ \hline \end{array}$$

The teacher then asked the pupils to move the digits around to produce a new 2-digit by 2-digit multiplication, for example 53×24 . She asked them to find the arrangement that gives the largest answer.

This task provided opportunities for pupils to rehearse the long multiplication algorithm (procedural knowledge) while at the same time developing conceptual understanding of place value.

Use stories and problems to help pupils understand mathematics

A teacher presented a class with this task:

'1,127 divided by 23 is equal to 49.'

'What number divided by 24 is equal to 49?'

Some pupils noticed that 24 is 1 more than 23, and so gave the answer 1,128.

Some used trial and improvement.

Some calculated 24×49 , which gave the correct answer, but not everyone understood why.

To help provide some insight into the structure of the task, the teacher read out the story below and then asked, *'How could we use this to help us solve the task?'*

A class of 23 children earns £1,127 for clearing litter from a beach.

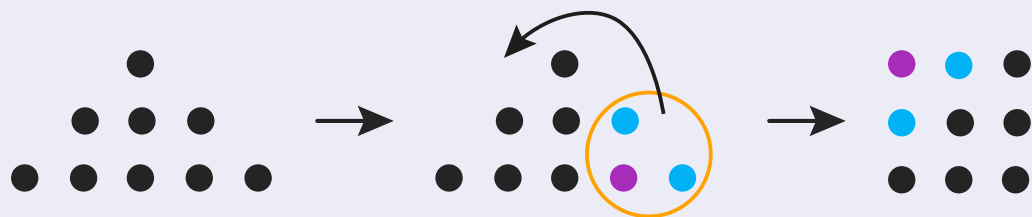
They share the money equally and find that they get £49 each.

Pupils realised that they needed to pose a question like: *'How much money would a class of 24 children need to earn, so that when they shared it out equally, they also each got £49?'*

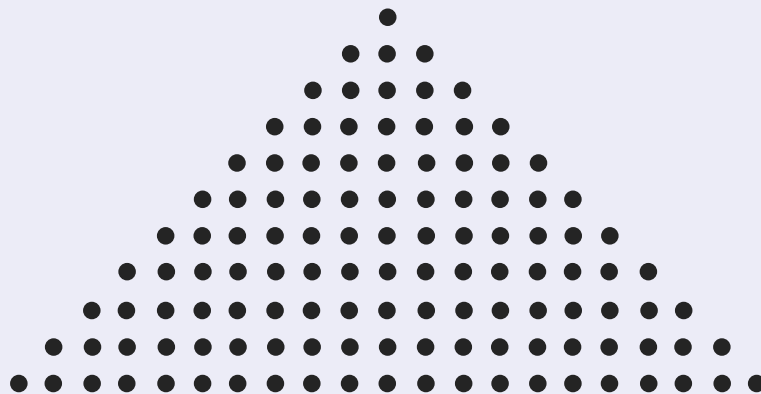
In this context, the pupils found it easier to see that an additional £49 would be needed for the 24th child, meaning that the answer would be $1,127 + 49 = 1,176$. Here the context illuminated the mathematical structure.

Provide opportunities for pupils to investigate mathematical structure and make generalisations

A teacher used the diagram below to show how a 'dot-triangle' with a base of 5 dots could be changed into a square array of 3×3 dots, just by moving 3 of the dots.



She then asked pupils how they could calculate (not count) the number of dots in a 'dot-triangle' with a base of 21 dots.



Some pupils just counted all of the dots.

Some thought that the triangle could be changed into a 19×19 array (as $3 = 5 - 2$ and $19 = 21 - 2$).

Others noticed that the dots could be rearranged into an 11×11 array (or, more generally, an $(n + 1) \times (n + 1)$ array for a base of $2n + 1$ dots).

These 'dot triangles' were used by the teacher to help pupils see that the sum of the odd numbers is a square number.



Using resources effectively

It is unlikely that introducing a resource on its own, whether it is a textbook or a new technology, will (on its own) have a positive impact on teaching or learning. Resources must support, or at least be accompanied by, an improvement in the quality of teaching to make a real difference.

Technology is a resource that appears to hold great promise for maths teaching, but the reality of its impact in the classroom has not always matched expectations. A great range of technological hardware and software is used in mathematics classrooms, including mobile devices, dynamic geometry software, exploratory computer environments, and educational games. Whatever the type of technology used, the evidence suggests a set of core principles for using it effectively. Detailed guidance can be found in the EEF's Digital Technology Literature Review, available online. Here are three key considerations:⁴¹

1. Identify a clear role for the technology in your pupils' learning. Ask questions such as: What teaching strategies and tasks will help pupils explore the relationship between equations and graphs using dynamic geometry software? Will spreadsheets allow pupils to view and transform data more effectively? How can tasks be designed to harness the power of the technology itself to provide feedback to pupils?
2. Training for teachers should not just focus on the technological skills involved in using new equipment. Ongoing professional development on how the technology can be used to improve teaching is likely to be needed if it is going to make a difference.
3. Before adopting a technology, consider the potential costs, including the impact on teachers' workload. Added up, these costs can be greater than for similarly effective approaches which do not involve technology.

7 Use structured interventions to provide additional support



Schools should focus on improvements to core classroom teaching that support all children in the class. With this in place, the need for catch up intervention should decrease. Nevertheless, some high-quality, structured intervention may still be required for some pupils to make progress. Selection of the intervention should be guided by effective assessment of pupils' individual strengths and weaknesses.

Evidence summary

- The guidance in this section draws on an EEF-funded review of maths interventions conducted by Ann Dowker.

The easiest way to identify high-quality interventions is to look for those that have been rigorously evaluated and have had a positive impact on pupil outcomes.*

However, few evaluations of maths catch-up interventions have been conducted, and an intervention with a rigorous and positive evaluation might not be available. The EEF is prioritising evaluation of these programmes. In the meantime, schools may want to adopt and implement an intervention with the features common to successful interventions:⁴²

- Interventions **should happen early**, both because mathematical difficulties can affect performance in other areas of the curriculum, and in order to reduce the risk of children developing negative attitudes and anxiety about mathematics.
- The intervention should be informed by the **evidence base regarding effective teaching and the typical development of mathematical capabilities**. How does the intervention relate to the recommendations in this guidance report?
- Interventions should include **explicit and systematic teaching**. This should include providing models of proficient problem solving, verbalization of thought processes, guided practice, corrective feedback, and frequent cumulative review.⁴³
- Effective implementation is essential to success. **Interventions require careful planning and use of school resources, including staff**. The best designed programme will not work if teaching staff are unavailable, excessively overburdened, or not adequately trained to deliver the programme. The EEF's '[Making Best Use of Teaching Assistants](#)' provides guidance on the effective deployment of Teaching Assistants.
- Ensure that **connections are made between intervention and whole-class instruction**. Interventions are often quite separate from classroom activities. The key is to ensure that learning in interventions is consistent with, and extends, work inside the classroom and that pupils understand the links between them. It should not be assumed that pupils can consistently identify and make sense of these links on their own.
- **The intervention should motivate pupils** and prevent or counteract the association of mathematics with boredom or anxiety. The use of games, for example, is a recurring feature of promising programmes, especially with primary school children.

*Evaluations of interventions can be found in the upcoming EEF review of mathematics programmes, on the projects section of the EEF website, or in the Evidence 4 Impact database maintained by the Institute for Effective Education.



- **Pay careful attention to what a pupil might miss if they take part in an intervention.** Will they be missing activities they enjoy? Will they miss out on curriculum content that they need to learn? Would class teaching be more effective? It is essential that the intervention is more effective than the instruction pupils would otherwise receive; if this is not the case, intervention pupils may fall further behind their peers.
- It is important to **avoid ‘intervention fatigue’** from the perspective of both teachers and pupils. Interventions do not always need to be very time consuming or intensive to be effective.

Box H: Specific learning difficulties in mathematics

There are major debates about whether there is a specific disorder that can be termed ‘dyscalculia’, whether it should be regarded as the lower end of a continuum of mathematical capabilities, or whether it occurs when pupils struggle with maths despite otherwise typical academic performance. We still do not yet know whether children with more severe or more specific mathematical difficulties require fundamentally different types of interventions from others.⁴⁴ If pupils are really struggling with maths, the most effective response is likely to be to attain a good understanding of their strengths and weaknesses, and target support accordingly (see **Recommendation 1**).

8

Support pupils to make a successful transition between primary and secondary school



There is a large dip in mathematical attainment and attitudes towards maths as children move from primary to secondary school in England. For example, one large national study of primary attainment in England found that, at the end of Year 7—a full year after the transition to secondary school—pupils' performance on a test of primary numeracy was below their performance at the end of Year 6.⁴⁵

It is clear that schools should be concerned about supporting pupils to make an effective transition. Unfortunately, there is very little evidence concerning the effectiveness of particular interventions that specifically address this dip. The broader research evidence, however, does suggest some key considerations:

- **Are primary and secondary schools developing a shared understandings of curriculum, teaching, and learning?** Both primary and secondary teachers are likely to be more effective if they are familiar with the mathematics curriculum and teaching methods outside of their age-phase.
- **How are primary schools ensuring that pupils leave with secure mathematical knowledge and understanding?** Primary schools could provide pupils with an effective defence against the common problems of transitioning to a new school.
- **When pupils arrive in Year 7, are secondary teachers attaining a good understanding of their strengths and weaknesses?** (See Recommendation 1.) Do they use this information

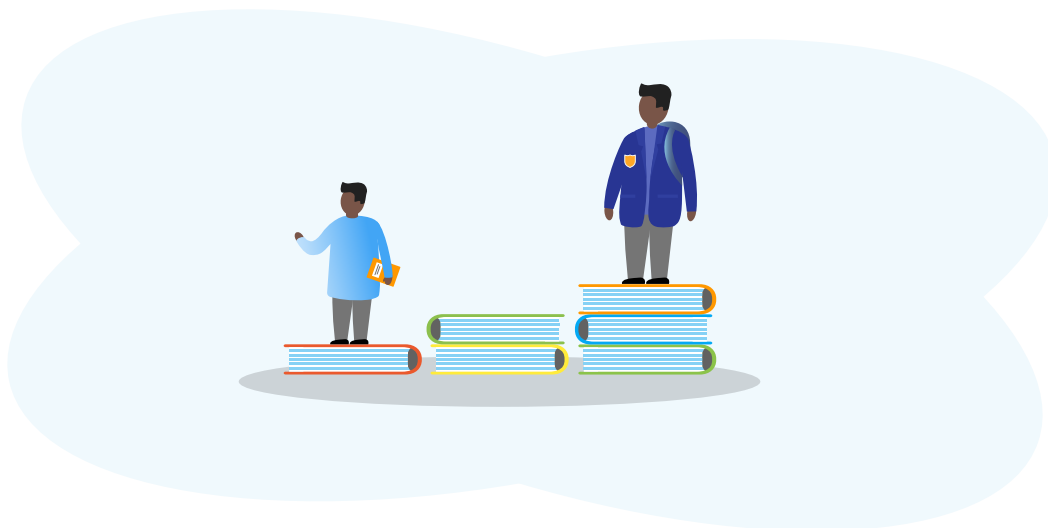
to build on key aspects of the primary mathematics curriculum in ways that are engaging, relevant and not simply repetitive?

- **How are secondary schools providing structured intervention support for Year 7 pupils who are struggling to make progress** (see Recommendation 7)?
- **How are pupils allocated to maths classes when they enter Year 7?** The research evidence suggests that allocating pupils to maths classes based on their prior attainment (often called 'setting' or 'ability grouping') does not, on average, lead to an increase in attainment overall and may widen attainment gaps. It has a slightly negative impact on pupils allocated to lower sets, although pupils allocated to higher sets may benefit slightly. Disadvantaged pupils are more likely to be assigned to lower sets, so setting is likely to lead to a widening of the attainment gap between disadvantaged pupils and their peers.⁴⁶



Evidence summary

- There is little evidence concerning the effectiveness of particular interventions that specifically address the transition. This recommendation considers the broader evidence regarding effective teaching, and applies it to the specific problem posed by the transition. There are a large number of meta-analyses on setting, which consistently suggest it widens the gap between lower and higher sets.



GLOSSARY

Manipulatives	A physical object that pupils or teachers can touch and move, used to support the teaching and learning of mathematics. Popular manipulatives include Cuisenaire rods and Dienes blocks.
Meta-analysis	A particular type of systematic research review which focuses on the quantitative evidence from different studies and combines these statistically to seek a more reliable or more robust conclusion than can be drawn from separate studies.
Multiple	For any integers a and b , a is a multiple of b if a third integer c exists so that $a = bc$. For example, 14, 49 and 70 are all multiples of 7 because $14 = 7 \times 2$, $49 = 7 \times 7$ and $70 = 7 \times 10$; -21 is also a multiple of 7 since $-21 = 7 \times -3$. ⁴⁷
Mixed fraction	A whole number and a fractional part expressed as a common fraction. Example: $1\frac{1}{2}$ is a mixed fraction. ⁴⁸
Non-example	Something that is not an example of a concept.
Number bonds	A pair of numbers with a particular total, for example number bonds for ten are all pairs of whole numbers with the total 10. ⁴⁹
Proportion	A part to whole comparison. Example: where £20 is shared between two people in the ratio 3:5, the first receives £7.50 which is $\frac{3}{8}$ of the whole £20. This is his proportion of the whole. If two variables x and y are related by an equation of the form $y = kx$, then y is directly proportional to x ; it may also be said that y varies directly as x . When y is plotted against x this produces a straight-line graph through the origin. If two variables x and y are related by an equation of the form $xy = k$, or equivalently $y = k/x$, where k is a constant and $x \neq 0$, $y \neq 0$, they vary in inverse proportion to each other. ⁵⁰
Rate	A measure of how quickly one quantity changes in comparison to another quantity. For example, speed is a measure of how distance travelled changes with time. ⁵¹
Ratio	A part-to-part comparison. The ratio of a to b is usually written $a:b$. For example, in a recipe for pastry fat and flour are mixed in the ratio 1:2 which means that the fat used has half the mass of the flour, i.e. amount of fat/amount of flour = $\frac{1}{2}$. Thus ratios are equivalent to particular fractional parts. ⁵²
Representations	'Representation' refers to a particular form in which mathematics is presented. ⁵³ Examples of different representations include: <ul style="list-style-type: none"> • two fractions could be represented on a number line; • a quadratic function could be expressed algebraically or presented visually as a graph; and • a probability distribution could be presented in a table or represented as a histogram.
Systematic review	A synthesis of the research evidence on a particular topic, that uses strict criteria to exclude studies that do not fit certain methodological requirements. Systematic reviews that provide a quantitative estimate of an effect size are called meta-analyses.

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HOW WAS THIS GUIDANCE COMPILED?

This guidance report draws on the best available evidence regarding the teaching of maths at Key Stages 2 and 3. The primary source of evidence for the recommendations is an evidence review conducted by Prof. Jeremy Hodgen, Dr Colin Foster, and Dr Rachel Marks.

The guidance report was created over three stages:

1. Scoping.

The process began with a consultation with teachers, academics, and other experts. The EEF team selected the area of interest (mathematics at Key Stages 2 and 3), appointed an Advisory Panel and evidence review team, and agreed research questions for the evidence review. The Advisory Panel consisted of both expert teachers and academics.

2. Evidence review.

The evidence review team conducted searches for the best available international evidence. Where possible, the review focused on meta-analyses and systematic reviews.

3. Writing recommendations.

The EEF worked with the support of the Advisory Panel to draft the recommendations. Academic and teaching experts were consulted on drafts of the report.

We would like to thank the many researchers and practitioners who provided support and feedback on drafts of this guidance.


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