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Lecture 1: Cardinal B-splines and convolution operators

1. Cardinal B-splines

A **cardinal spline of order** k is a piecewise polynomial of degree $< k$, in C^{k-2}, and with breakpoints the *integers*. The space S_k of all cardinal splines of order k is (obviously) a linear space of infinite dimension, but nonetheless, can be considered 'small' in the following way:

S_k is invariant under shifts. Here, a **shift** is translation by an integer, hence the **shift-invariance** means that

$$f \in S_k \quad \Longrightarrow \quad E^\alpha f \in S_k, \quad \text{all } \alpha \in \mathbb{Z},$$

with

$$E^\alpha : f \mapsto f(\cdot - \alpha).$$

This means that, for whatever $f \in S_k$ we choose, the sequence

$$E(f) := (E^\alpha f : \ \alpha \in \mathbb{Z})$$

is also in S_k. The spline space S_k is then 'small' in the sense that it is a **principal** shift-invariant space: there exists a function $f \in S_k$ whose $E(f)$ already spans the entire space (that's a vague statement, but will be made precise later, without a need to topologize S_k. If you insist on a rigorous statement *now*, equip S_k with the topology of uniform convergence on compact sets (which makes it a Fréchet space). Then the finite span of $E(f)$ is dense in S_k).

A **generator** f in the above sense is not unique. The general theory of shift-invariant spaces will be invoked later on to show that 'almost' any $f \in S_k$ generates that space. In particular, the space S_k is **local** in the sense that the compactly supported functions in it form a dense subspace, hence one may look for a compactly supported generator. Among those, the **B-spline** is the one with *minimal support*.

None of the properties listed above is proved now. All will be obtained as simple consequences of the general theory of shift-invariant (**SI**) spaces.

Definition 1: B-splines. The B-spline B_k of order k is the k-fold convolution of the B-spline of order 1, i.e.,

$$B_k = B_1 * B_{k-1}.$$

The B-spline of order 1,

$$B_1 := \chi_{[0..1)},$$

is the support function of $[0\,..\,1)$. □

It follows that

$$(2) \qquad B_k(x) = \int_{\mathbb{R}} B_1(x-t) B_{k-1}(t)\,\mathrm{d}t = \int_{x-1}^{x} B_{k-1}(t)\,\mathrm{d}t.$$

Thus (i) $\operatorname{supp} B_k = [0\,..\,k]$, and (ii) $B_k > 0$ on $(0\,..\,k)$ (as follows by induction). Also, differentiating (2), we obtain that

$$(3) \qquad DB_k = B_{k-1} - E^1 B_{k-1} =: -\Delta B_{k-1},$$

with Δ the **forward difference**. Since B_2 is the continuous piecewise linear hat function (check directly!), it turns out (by induction) that $B_k \in C^{k-2}(\mathbb{R})$. (This implies that, in terms of the B-splines $B(\cdot|t_0, \ldots, t_k)$ discussed in the first part of the course, $B_k = B(\cdot|0, \ldots, k)$.)

Fourier transform. The basic connections between convolution and Fourier transformation imply that

$$\widehat{B_k} = \widehat{B_1}^{\,k}.$$

The computation of $\widehat{B_1}$ can be done directly (do it!): $\widehat{B_1}(\omega) = \frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}$, hence

$$\widehat{B_k}(\omega) = \frac{(1-\mathrm{e}^{-\mathrm{i}\omega})^k}{(\mathrm{i}\omega)^k}.$$

Zeros of the Fourier transform. It is elementary to prove that $\widehat{B_1}(\omega) = 0$ iff $\omega \in 2\pi\mathbb{Z}\backslash 0 =: \mathcal{L}$, and that all these zeros are *simple*. It follows then that $\widehat{B_k}$ has a zero of exact order k at each point of $\mathcal{L}$, and vanishes nowhere else. (Note: the Fourier transform extends to an entire function on $\mathbb{C}$, but still all the zeros of $\widehat{B_k}$ lie on the real line, so it does not matter here whether we talk about the *real* zeros or *general* zeros, i.e., zeros on the whole of $\mathbb{C}$.) Note that, thus, $\widehat{B_k}$ does not have a 2π-*periodic zero*.

Definition 4: the SF conditions. We say that the compactly supported ϕ satisfies the **Strang-Fix (SF) conditions of order** k if its Fourier transform $\widehat{\phi}$ vanishes to order k at each $\omega \in \mathcal{L}$, while $\widehat{\phi}(0) \neq 0$. □

Note that the Fourier transform of ϕ is entire, thanks to the compact support assumption, hence the differentiability of $\widehat{\phi}$ at the points of $\mathcal{L}$ is granted.

Thus, the B-spline B_k satisfies the SF-conditions of order k, and does not have any 2π-periodic zeros. But who cares? In order to motivate the above discussion I state, without proof, a few theorems (all of them will be proved during the course; some will be proved in the very near future).

Since the entire discussion is univariate (and will stay so), and since many of our theorems have suitable multivariate analogs, I'd like to distinguish those theorems that make the difference between the univariate theory and the multivariate one. The $*$ denotes those theorems that are truly univariate.

First, let us introduce the notion of **semi-discrete convolution**:

Definition 5. Let ϕ be a compactly supported function/distribution. The semi-discrete convolution operator $\phi*'$ is the linear map

$$\phi*' : \mathbb{C}^{\mathbb{Z}} \to S_\star(\phi) : c \mapsto \sum_{j\in\mathbb{Z}} E^j\phi c(j) = \sum_{j\in\mathbb{Z}} \phi(\cdot - j)c(j),$$

with the infinite sum taken, offhand, pointwise. The space

$$S_\star(\phi)$$

is *defined* to be the range of this map, i.e., it is the infinite linear span of the shifts of ϕ.

The space $S_\star(\phi)$ is one of several interpretations of the notion 'the shift-invariant space generated by ϕ'. Note that it is well-defined only for a compactly supported ϕ.

Example. For the B-spline B_k, the space $S_\star(B_k)$ can be characterized as the collection of all functions in $C^{k-2}(\mathbb{R})$ that, on each interval $[j \,..\, j+1)$, $j \in \mathbb{Z}$, coincide with some polynomial in $\Pi_{<k}$. The claim we have just made is not obvious. It follows, however, from the characterization of spline spaces that appeared in the first part of this course. □

Definition 6: linear independence. We say that the sequence $E(\phi) = (E^j\phi : \; j \in \mathbb{Z})$ of shifts of ϕ is linearly independent if the semi-discrete convolution $\phi*'$ is 1-1, i.e., its kernel,

$$\ker(\phi*') := \{c \in \mathbb{C}^{\mathbb{Z}} : \phi*'c = 0\},$$

is trivial, i.e., contains only the zero sequence. □

Remark: The notation $\phi*'$ and the terminology 'semi-discrete' were introduced because there are occasions when we will apply $\phi*'$ to functions defined on all of $\mathbb{R}$ rather than just on $\mathbb{Z}$. The convention then is that, still,

$$\phi*' f := \sum_{j\in\mathbb{Z}} \phi(\cdot - j) f(j).$$

Theorem: the curse of periodic zeros. *Let ϕ be a compactly supported function. Let $\theta \in \mathbb{C}$. Then $\widehat{\phi}$ vanishes on $\theta + 2\pi\mathbb{Z}$ if and only if $\phi*' e_{i\theta} = 0$. This says that $E(\phi)$ is linearly dependent if $\widehat{\phi}$ has a 2π-periodic zero in $\mathbb{C}$.*

Here, to recall,

$$e_\xi : x \mapsto \mathrm{e}^{\xi x}$$

is the **exponential with frequency** ξ.

Theorem*: the factorization theorem. *Let ϕ be a compactly supported distribution* (if you do not know what a distribution is, assume $\phi \in L_1(\mathbb{R})$) *that satisfies the SF conditions of order k. Then there exists a compactly supported distribution η* (sorry, one cannot avoid distributions here) *such that*

$$\phi = B_k * \eta.$$

Theorem: the polynomial reproduction theorem. *Let ϕ be a compactly supported distribution (or a function in $L_1(\mathbb{R})$). Then ϕ satisfies the SF conditions of order k if and only if the map*

$$\phi*' : \Pi_{<k} \to \Pi_{<k} : p \mapsto \sum_{\alpha\in\mathbb{Z}} E^\alpha \phi p(\alpha)$$

is a well-defined automorphism (i.e., 1-1 onto its domain). In particular, if ϕ satisfies the SF conditions of order k, then $\Pi_{<k} \subset S_\star(\phi)$.

2. Convolution operators

A convolution operator is a map of the form $f \mapsto v*f$, with v some function (or, more generally, a distribution) known as the **kernel** (of the convolution, much different from the kernel of a linear map). Many convolution operators act as **approximate identities**, i.e., rather than a single map, a sequence $(v_n*)_n$ of such operators is given, and $\|f - v_n*f\| \to 0$ (as $n \to \infty$), for each f in the underlying nls.

Previously in class two such operators were introduced and discussed: the *Dirichlet kernel* and the *Fejér kernel*. These kernels, and a few others, are connected to the cardinal B-splines discussed above and the goal here is to outline these connections (and to make brief remarks concerning the approximation power of these kernels).

Definition. Let X be a nls, and let $(A_n)_n$ be a sequence of maps on X. We say that $(A_n)_n$ **provides approximation order** k if

$$X_0 := \{f \in X : \|f - A_n f\| = O(n^{-k})\}$$

is dense in X. □

Pertinent examples for X are the spaces $C(\mathbb{T})$, $L_2(\mathbb{T})$, $C(\mathbb{R})$, $L_2(\mathbb{R})$. In these examples, the dense subspace X_0 comprises functions that are *sufficiently smooth*: one of the most fundamental principles of Approximation Theory is the close relation between the smoothness of a function and its approximability.

Examples of approximate identities that are related to B-splines.

In all these examples, the B-splines are related to the Fourier transform (or Fourier coefficients) of the kernels. (But, to fully appreciate these examples, it pays to look first at the next lecture, specifically at the periodization of $f \in L_1(\mathbb{R})$.)

Example 1. We take $\widehat{v_n} := c_n \chi_{[-n..n]}$. A direct calculation shows that

$$v_n(x) = c_n \frac{\sin(nx)}{\pi x}.$$

(Exercise: find the values of $(c_n)_n$ (that makes this an approximate identity?).)

The *Dirichlet kernel* is the periodic analog of the above example: we only need to interpret χ_Ω as the sequence that assumes the value 1 at each of the integers in Ω, and the value 0 at all other integers. Thus, in the periodic case we get here

$$D_n(x) = \frac{1}{2\pi} \sum_{j=-n}^{n} \mathrm{e}^{\mathrm{i}jx}.$$

□

Example 2. With $\chi_n := \chi_{[-n..n]}$, we take $\widehat{v_n} := c_n \chi_{(n/2)} * \chi_{(n/2)}$. Thus, $\widehat{v_n}$ is a hat function supported on $[-n \,..\, n]$.

The Fejér kernel F_n is the periodic analog of the above example, i.e., its Fourier coefficients are obtained by evaluating $\widehat{v_n}$ at the integers:

$$F_n(x) = \sum_{j=-n}^{n} (1 - \frac{|j|}{n})\, \mathrm{e}^{\mathrm{i}jx}.$$

Since F_n was obtained by convolving the D-kernel on the Fourier domain, it must be the square of the D-kernel, hence it is *positive*. □

In Fourier analysis, the Fejér kernel is favored over the Dirichlet kernel (primarily because of the fact that $F_n * f$ converges in $C(\mathbb{T})$ pointwise). However, from the point of view of Approximation Theory, the Dirichlet kernel is much better. To understand why, solve the following exercises:

Exercise. Let $f \in L_2(\mathbb{T})$ be a smooth function (say, in $W_2^k(\mathbb{T})$, i.e., having k derivatives in $L_2(\mathbb{T})$). Compare the decay of

$$n \mapsto \|f - D_n * f\|_{L_2}$$

to the decay of

$$n \mapsto \|f - F_n * f\|_{L_2}.$$

□

Another kernel of interest is the **de La Vallée Poussin kernel**. Its continuous version is obtained by summing, on the Fourier domain, shifts of (the Fourier transform) of the (continuous analog of the) Fejér kernel, i.e., shifts of the hat function, thus getting

$$\widehat{v_n} := \text{ the broken line that connects } (-\infty, 0),\ (-2n, 0),\ (-n, 1),\ (n, 1),\ (2n, 0),\ (\infty, 0).$$

Thus the Fourier coefficients of V_n, de La Vallée Poussin's kernel, are obtained by evaluating the above $\widehat{v_n}$ at the integers.

Exercise. Prove that (a) $V_n * f$ converges in $C(\mathbb{T})$ to f for every f, and (b) that $\|f - V_n * f\|_{L_2} = O(n^{-k})$, for every $f \in W_2^k(\mathbb{T})$. □

Next: Poisson's summation formula, and its applications