Inflationary observables in loop quantum cosmology

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Based on

- M. Bojowald and G.C., `arXiv:1011.2779`.
- M. Bojowald, G.C., and S. Tsujikawa, in preparation ($\times 2$).

Outline

1. LQC

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Status

- Big bang singularity problem addressed (a) at kinematical level, (b) in exactly solvable quantum model, (c) in effective dynamics.
- Effective equations known for pure FRW and linear perturbations in all sectors (S, T, V).

But:

- Role of different parametrization schemes?
- Conservation law for the curvature perturbation?
- Mukhanov equation of scalar perturbations?
- Scalar spectrum and other cosmological observables?
- Observational constraints?

Canonical variables and quantization

Flat FRW background: $\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}x^i\mathrm{d}x_i)$.

$$A_a^i = c\,{}^0e_a^i\,, \qquad E_i^a = p\,{}^0e_i^a$$

$$\{c, p\} = \frac{8\pi G\gamma}{3}$$

$$p \to \hat{p}\,, \qquad c \to \hat{h} = \widehat{\mathrm{e}^{\mathrm{i}\mu(p)c}}$$

$$\hat{H}(\hat{E}, \hat{h})|\Psi\rangle = 0 \qquad \text{“WDW” equation}$$

$$\langle\Psi_{\rm sc}|\hat{H}(\hat{E}, \hat{h})|\Psi_{\rm sc}\rangle \approx 0 \qquad \text{effective dynamics}$$

Background equations

Two corrections: inverse-volume and holonomy. We consider only the former.

$$\mathcal{H}^2 = \frac{8\pi G}{3}\,\alpha\left[\frac{\varphi'^2}{2\nu} + pV(\varphi)\right]$$

$$\varphi'' + 2\mathcal{H}\left(1 - \frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p}\right)\varphi' + \nu p V_{,\varphi} = 0$$

where

$$\alpha \approx 1 + \alpha_0\delta_{\rm Pl}, \qquad \nu \approx 1 + \nu_0\delta_{\rm Pl}$$

$$\delta_{\rm Pl} \equiv \left(\frac{a_{\rm Pl}}{a}\right)^\sigma$$

Strategy

Perturbation theory in classical constraints.

$$E_i^\alpha = p\delta_i^\alpha + \delta E_i^\alpha \,, \qquad A_\alpha^i = c\delta_\alpha^i + \left(\delta\Gamma_\alpha^i + \gamma\delta K_\alpha^i\right)$$

$$\{\delta K_\alpha^i(\mathbf{x}), \delta E_j^\gamma(\mathbf{y})\} = 8\pi G\delta_\alpha^\gamma\delta_j^i\delta(\mathbf{x},\mathbf{y})$$

Write effective constraints with inverse-volume correction functions. E.g.,

$$H[N] \sim \int \mathrm{d}^3x N[\alpha(E)\mathcal{H}_g + \nu(E)\mathcal{H}_\pi + \rho(E)\mathcal{H}_\nabla + \mathcal{H}_V]$$

Anomalies

- Closure of the effective constraint algebra imposed, $\{C_a, C_b\} = f_{ab}{}^c(A, E)C_c$.
- Perturbed equations contain counterterms f, f_1, g_1, h, f_3 which guarantee anomaly cancellation in the constraint algebra [Bojowald & Hossain 2007,2008; Bojowald et al. 2008,2009]
- Anomaly cancellation shown only in the quasi-classical regime with inverse-volume corrections (small counterterms). Case with holonomy corrections unknown.

Counterterms. I

$$
\begin{aligned}
f &= -\frac{\alpha_0}{2}\delta_{\rm Pl}\,, \\
f_1 &= \frac{1}{2}\left(\frac{\sigma\nu_0}{3}-\alpha_0\right)\delta_{\rm Pl}, \\
h &= \alpha_0\left(\frac{1}{2}-\sigma\right)\delta_{\rm Pl}\,, \\
g_1 &= \left[\nu_0\left(\frac{\sigma}{3}+1\right)-\alpha_0\right]\delta_{\rm Pl}\,, \\
f_3 &= \left[\frac{\alpha_0}{2}-\nu_0\left(\frac{\sigma}{6}+1\right)\right]\delta_{\rm Pl} \\
&= \frac{1}{2}\frac{3\alpha_0}{\sigma-3}\delta_{\rm Pl}
\end{aligned}
$$

Counterterms. II

Consistency condition:

$$2\frac{\mathrm{d}f_3}{\mathrm{d}\ln p}+3(f_3-f)=0 \;\Rightarrow\; \boxed{\alpha_0\left(\frac{\sigma}{6}-1\right)-\nu_0\left(\frac{\sigma}{6}+1\right)\left(\frac{\sigma}{3}-1\right)=0}$$

Inflationary and de Sitter background solutions exist for

$$0 \lesssim \sigma \lesssim 3$$

Minisuperspace parametrization incompatible with consistency condition and inflationary solutions!

Mini-superspace: Fiducial volume problem

Fiducial volume:

$$\int_\Sigma \mathrm{d}^3x = +\infty \qquad \to \qquad \int_{\Sigma(\mathcal{V}_0)} \mathrm{d}^3x = \mathcal{V}_0 < +\infty$$

Problem: typically,

$$\delta_{\rm Pl} \sim \left(\frac{\ell_{\rm Pl}^3}{\mathcal{V}_0}\right)^{\frac{\sigma}{3}} a^{-\sigma}\,.$$

Dependence of quantum corrections (and observables) on the fiducial volume.

Mini-superspace: Parameter ranges

FRW calculation:

$$\alpha_0 = \frac{(3q-\sigma)(6q-\sigma)}{2^2 3^4}, \quad \nu_0 = \frac{\sigma(2-l)}{54}, \quad \tfrac{1}{2} \leq l < 1 \quad \tfrac{1}{3} \leq q < \tfrac{2}{3}$$

Natural choice ("improved quantization" scheme [Ashtekar et al 2006]):

$$\sigma = 6\,, \qquad l = \tfrac{3}{4}\,, \qquad q = \tfrac{1}{2}$$

so that

$$\alpha_0 = \tfrac{1}{24} \approx 0.04\,, \qquad \nu_0 = \tfrac{5}{36} \approx 0.14\,.$$

Lattice refinement

Use of the "patch" volume $v=\mathcal{V}/\mathcal{N}$ of an underlying discrete state rather than the much larger volume $\mathcal{V}$ [Bojowald 2006]. Number of cells $\mathcal{N}=\mathcal{N}_0 a^{6n}$, where $0\leq n\leq 1/2$. ($\mathcal{N}$ must not decrease with the volume. Also, $v\sim a^{3(1-2n)}$ has a lower non-zero bound in a discrete geometrical setting.)

$$\delta_{\rm Pl}=\left(\frac{\ell_{\rm Pl}^3}{v}\right)^{m/3}=\left(\ell_{\rm Pl}^3\frac{\mathcal{N}}{\mathcal{V}}\right)^{m/3}=\left(\ell_{\rm Pl}^3\frac{\mathcal{N}_0}{\mathcal{V}_0}\right)^{\frac{m}{3}}a^{-(1-2n)m}\sim a^{-\sigma}$$

$$\boxed{\sigma\geq 0}$$

Calculations of inverse-volume operators and their spectra show that corrections approach the classical value always from above:

$$\alpha_0\geq 0\,,\qquad \nu_0\geq 0$$

Slow-roll parameters

$$\begin{aligned}
\epsilon &\equiv 1-\frac{\mathcal{H}'}{\mathcal{H}^2}\\
&= 4\pi G\frac{\varphi'^2}{\mathcal{H}^2}\left\{1+\left[\alpha_0+\nu_0\left(\frac{\sigma}{6}-1\right)\right]\delta_{\rm Pl}\right\}+\frac{\sigma\alpha_0}{2}\delta_{\rm Pl}\,,\\
\eta &\equiv 1-\frac{\varphi''}{\mathcal{H}\varphi'}\,,\\
\xi^2 &\equiv \frac{1}{\mathcal{H}^2}\left(\frac{\varphi''}{\varphi'}\right)'+\epsilon+\eta-1\,,
\end{aligned}$$

$$\begin{aligned}
\epsilon' &= 2\mathcal{H}\epsilon(\epsilon-\eta)-\sigma\mathcal{H}\tilde{\epsilon}\delta_{\rm Pl}\,, \qquad \eta'=\mathcal{H}(\epsilon\eta-\xi^2)\,,\\
\tilde{\epsilon} &\equiv \alpha_0\left(\frac{\sigma}{2}+2\epsilon-\eta\right)+\nu_0\left(\frac{\sigma}{6}-1\right)\epsilon\,.
\end{aligned}$$

Scalar perturbation equations

Two gauge-invariant scalar modes: $\Phi = (1+h)\Psi$.
Diffeomorphism constraint:

$$4\pi G\frac{\alpha}{\nu}\varphi'\delta\varphi = \Psi' + (1+f+h)\mathcal{H}\Psi\,.$$

Equation for Ψ:

$$\Psi'' + \mathcal{H}(2\eta + \sigma F_0\delta_{\rm Pl})\Psi' - \left(s^2\Delta + \mathcal{H}^2\left[2(\epsilon-\eta) - \sigma\mu_\Psi\delta_{\rm Pl}\right]\right)\Psi = 0.$$

Squared propagation speed:

$$s^2 = \alpha^2(1-f_3) = 1+\chi\delta_{\rm Pl}\,, \qquad \chi \equiv \frac{\sigma\nu_0}{3}\left(\frac{\sigma}{6}+1\right) + \frac{\alpha_0}{2}\left(5-\frac{\sigma}{3}\right)\,.$$

Perturbed Klein–Gordon equation:

$$\delta\varphi'' + 2\mathcal{H}(1+B_{10}\delta_{\rm Pl})\delta\varphi' - (s^2\Delta - \nu p V_{,\varphi\varphi})\delta\varphi - (4+B_{20}\delta_{\rm Pl})\varphi'\Psi'$$
$$+2(\eta - 3 + B_{30}\delta_{\rm Pl})\mathcal{H}\varphi'\Psi = 0$$

Is curvature perturbation conserved?

$$\begin{aligned}\mathcal{R} &= \Psi + \frac{\mathcal{H}}{\varphi'}(1+f-f_1)\,\delta\varphi \\ &= \Psi + \frac{\mathcal{H}}{\varphi'}\left(1-\frac{\sigma\nu_0}{6}\delta_{\rm Pl}\right)\delta\varphi\,.\end{aligned}$$

When constraint algebra is deformed, conservation equation for stress-energy is modified. $\mathcal{R}$ is no longer guaranteed to be conserved $\Rightarrow$ notable modifications of perturbation spectra expected.

Conservation of $\mathcal{R}$ (at large scales)

$$\mathcal{R}' = (\alpha\nu + f - f_1 - f_3)\frac{\mathcal{H}}{4\pi G\varphi'^2}\Delta\Psi + C\delta\varphi\,,$$

where

$$\begin{aligned} C &= \frac{\mathcal{H}^2}{\varphi'}\left[\frac{f' - f_1'}{\mathcal{H}} + \frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p} + \left(\frac{1}{3}\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p} - f + f_1\right)\epsilon \right. \\ &\left. -2\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln p} - 3(f - f_3)\right] \\ &= 0\,. \end{aligned}$$

$$\boxed{\mathcal{R}' = \left[1 + \left(\frac{\alpha_0}{2} + 2\nu_0\right)\delta_{\rm Pl}\right]\frac{\mathcal{H}}{4\pi G\varphi'^2}\Delta\Psi}$$

Mukhanov equation

Mukhanov variable:

$$u = z\mathcal{R}\,, \qquad z \equiv \frac{a\varphi'}{\mathcal{H}}\left[1+\left(\frac{\alpha_0}{2}-\nu_0\right)\delta_{\rm Pl}\right]$$

Via a tedious direct calculation or a diabolically fast trick:

$$\boxed{u'' - \left(s^2\Delta + \frac{z''}{z}\right)u = 0}$$

Very simple equation in closed form: expected from Hamilton–Jacobi method [Goldberg et al 1991; Langlois 1994], the reduced phase space after solving the constraints has one local d.o.f. as in GR.

Superluminal propagation of signals is avoided if

$$s^2 < \alpha^2 \qquad \Rightarrow \sigma \geq 6 \quad \text{for} \quad \alpha_0 > 0,\ \nu_0 \geq 0$$

Problem with small σ lattice parametrization?

Asymptotic solution

Power-law background $a = (-\tau)^n$, $n \lesssim -1$.
Horizon crossing:

$$k|\tau| = 1$$

Small scales:

$$u_{k\gg\mathcal{H}}(\tau) = \frac{\mathrm{e}^{-\mathrm{i}k\tau}}{\sqrt{2k}}[1 + y(k,\tau)\delta_{\rm Pl}]\,, \qquad y = \frac{\chi}{2(\sigma n - 1)}(1 + \mathrm{i}k\tau)$$

Large scales:

$$|u_{k\ll\mathcal{H}}| = \frac{1}{\sqrt{2k}}\left[1 + \frac{\chi}{2(\sigma n - 1)}\delta_{\rm Pl}(k)\right]\frac{z}{z(k)}$$

Scalar spectrum and index

Spectrum:

$$\mathcal{P}_{\rm s} \equiv \frac{k^3}{2\pi^2 z^2} \left\langle |u_{k\ll\mathcal{H}}|^2 \right\rangle \Big|_{k|\tau|=1} = \frac{G}{\pi} \frac{\mathcal{H}^2}{a^2\epsilon} \left(1 + \gamma_{\rm s}\delta_{\rm Pl}\right)$$

where

$$\gamma_{\rm s} \equiv \nu_0 \left(\frac{\sigma}{6} + 1\right) + \frac{\sigma\alpha_0}{2\epsilon} - \frac{\chi}{\sigma + 1}\,.$$

Large-scale enhancement of power:

$\delta_{\rm Pl} \sim a^{-\sigma} \sim (1/|\tau|)^{-\sigma} \sim k^{-\sigma}$ at horizon crossing.

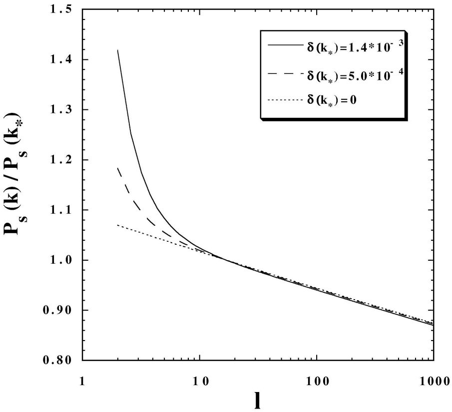
Index:

$$n_{\rm s} - 1 \equiv \frac{{\rm d}\ln\mathcal{P}_{\rm s}}{{\rm d}\ln k} = 2\eta - 4\epsilon + \sigma\gamma_{n_{\rm s}}\delta_{\rm Pl}$$

where

$$\gamma_{n_{\rm s}} = \alpha_0 - 2\nu_0 + \frac{\chi}{\sigma + 1}\,.$$

Power spectrum ($\sigma = 2$)



LQC effects at large scales (unobservable for $\sigma > 3$).

Scalar running

$$\alpha_{\rm s} \equiv \frac{{\rm d}n_{\rm s}}{{\rm d}\ln k} = 2(5\epsilon\eta - 4\epsilon^2 - \xi^2) + \sigma(4\tilde{\epsilon} - \sigma\gamma_{n_{\rm s}})\delta_{\rm Pl} = O(\epsilon^2) + O(\sigma\delta_{\rm Pl})$$

Departure from standard inflation: If large enough, quantum correction dominates and $\alpha_{\rm s} \sim \sigma\delta_{\rm Pl}$. Bounds on scalar running should be the main constraint on the parameters.

Mukhanov equation

Linear equation of motion for tensor mode h_k [Bojowald & Hossain 2008]:

$$h_k'' + 2\mathcal{H}\left(1 - \frac{\mathrm{d}\ln\alpha}{\mathrm{d}\ln p}\right) h_k' + \alpha^2 k^2 h_k = 0\,.$$

Defining

$$w_k \equiv \tilde{a} h_k\,, \qquad \tilde{a} \equiv a\left(1 - \frac{\alpha_0}{2}\delta_{\rm Pl}\right)\,,$$

we get the Mukhanov equation

$$\boxed{w_k'' + \left(\alpha^2 k^2 - \frac{\tilde{a}''}{\tilde{a}}\right) w_k = 0}$$

Identical to the scalar Mukhanov equation up to the substitutions $z \to \tilde{a}\,, \qquad \chi \to 2\alpha_0$.

Tensor spectrum, index and running

Spetrum:

$$\mathcal{P}_{\rm t} \equiv \frac{32G}{\pi}\frac{k^3}{\tilde{a}^2}\left\langle|w_{k\ll\mathcal{H}}|^2\right\rangle\Big|_{k|\tau|=1} = \frac{16G}{\pi}\frac{\mathcal{H}^2}{a^2}\left(1+\gamma_{\rm t}\delta_{\rm Pl}\right)$$

where

$$\gamma_{\rm t} \equiv \frac{\sigma-1}{\sigma+1}\alpha_0\,.$$

Index:

$$n_{\rm t} \equiv \frac{{\rm d}\ln\mathcal{P}_{\rm t}}{{\rm d}\ln k} = -2\epsilon - \sigma\gamma_{\rm t}\delta_{\rm Pl}$$

Running:

$$\alpha_{\rm t} \equiv \frac{{\rm d}n_{\rm t}}{{\rm d}\ln k} = -4\epsilon(\epsilon-\eta) + \sigma(2\tilde{\epsilon}+\sigma\gamma_{\rm t})\delta_{\rm Pl}\,.$$

Tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_{\rm t}}{\mathcal{P}_{\rm s}} = 16\epsilon[1 + (\gamma_{\rm t} - \gamma_{\rm s})\delta_{\rm Pl}]$$

Consistency relation:

$$r = -8\{n_{\rm t} + [n_{\rm t}(\gamma_{\rm t} - \gamma_{\rm s}) + \sigma\gamma_{\rm t}]\delta_{\rm Pl}\}$$

Here implicitly assumed that $\gamma_{\rm s}$ is not too large (either σ or α_0 or both should be small).

The idea

- Choose a potential and recast all observables as its functions.
- Choose a set of values for the potential and LQC parameters (≥ 2).
- Find upper bound for $\delta_{\rm Pl}$.

Potential slow-roll tower

$$\epsilon_V \equiv \frac{1}{16\pi G}\left(\frac{V_{,\varphi}}{V}\right)^2, \quad \eta_V \equiv \frac{1}{8\pi G}\frac{V_{,\varphi\varphi}}{V}, \quad \xi_V^2 \equiv \frac{V_{,\varphi}V_{,\varphi\varphi\varphi}}{(8\pi GV)^2}.$$

Relations $(\epsilon, \eta, \xi^2) \leftrightarrow (\epsilon_V, \eta_V, \xi_V^2)$ recast the observables as functions of the VSR tower. For $V(\phi) = \lambda\phi^n$:

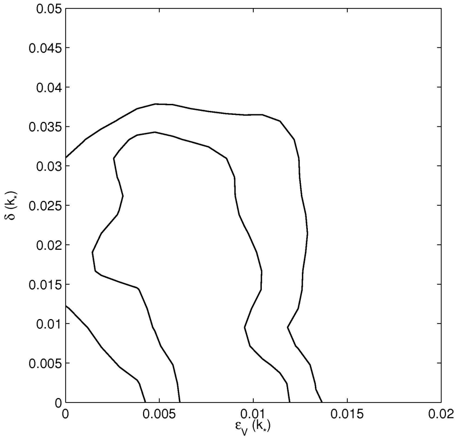
$$\epsilon_V = \frac{n^2}{2\kappa^2\phi^2}, \quad \eta_V = \frac{2(n-1)}{n}\epsilon_V, \quad \xi_V^2 = \frac{4(n-1)(n-2)}{n^2}\epsilon_V^2.$$

Likelihood analysis

- For fixed values of n and σ all the observables given above are written as functions of ϵ_V and $\delta = \alpha_0 \delta_{\rm Pl}$.
- CMB marginalized likelihood analysis performed by varying ϵ_V and $\delta_{\rm Pl}$ in CosmoMC. WMAP7+BAO+HST dataset used; plots for WMAP7+SDSS+HST (+BBN+SN IA) dataset are similar.

Combined marginalized distributions for δ and ϵ_V

WMAP7+BAO+HST dataset ($n = 2$, $\sigma = 1$)

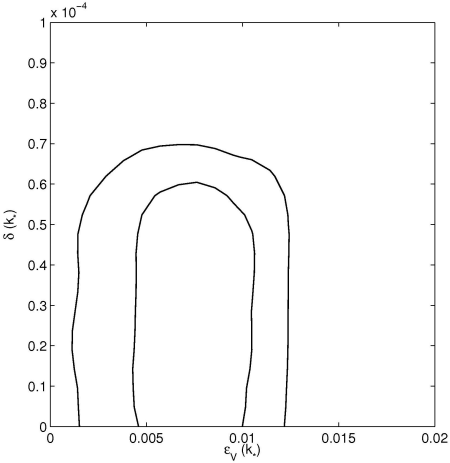


Discussion $n = 2$, $\sigma = 1$

- Upper bounds $\epsilon_V < 0.015$ and $\delta < 0.04$ (95% CL).
- LQC corrections in $n_s = 1 - 4\epsilon_V - (75/56 + 41/42\epsilon_V)\delta$ is not negligible relative to the SR term.
- $\delta \sim O(\epsilon_V)$ and 2nd-order observables (runnings) might be affected by unknown $O(\delta_{\rm Pl}^2)$ terms.
- This case is only marginally consistent with all approximations and assumptions.

Combined marginalized distributions for δ and ϵ_V

WMAP7+BAO+HST dataset ($n = 2$, $\sigma = 2$)



Discussion $n = 2$, $\sigma = 2$

- Upper bound $\delta < 7 \times 10^{-5} \ll \epsilon_V$. A posteriori important check that SR and $\delta_{\rm Pl}$ expansions are mutually consistent.
- For range $45 < N < 65$ of e-folds, $0.008 < \epsilon_V = 1/(2N) < 0.011$. The probability distribution of ϵ_V is consistent with this range, so the quadratic potential is compatible with observations (as in the standard case).
- For $\sigma > 2$, even tighter bounds (practically unobservable inverse-volume effects).

Conclusions

- Minisuperspace parametrization of FRW LQC ($\sigma > 1$) incompatible with anomaly cancellation in inhomogeneous LQC and power-law (inflationary) solutions.
- Lattice refinement parametrization overcomes these problems but has issue with superluminal propagation.
- Tight upper bound for quantum corrections.

Open issues

- Higher-order consistency of perturbative anomaly cancellation and parameter constraint?
- Issue of superluminal propagation.
- Holonomy corrections not yet implemented in scalar linear perturbations.
- Inverse-volume corrections also affect dispersion relations of waves propagating in a quantum spacetime. The theory can be constrained by a combination of cosmological and astroparticle observations.
- Cosmological constraints with WMAP7+BAO+HST dataset for various potentials and values of the parameters (in progress).