# The $1/N$ Expansion in Colored Tensor Models

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# Space-time and Scales

Space-time is one of the most fundamental notions is physics. In many theories (e.g. quantum mechanics) it appears as a fixed background. The distances and lapses of time are measured with respect to this fixed background.

Scales encode causality: effective physics at large distance is determined by fundamental physics at short distance.

General relativity promotes the metric to a dynamical variable, and the length scales become dynamical!

- How to define background independent scales separating fundamental and effective physics?
- How to obtain the usual space time as an effective phenomenon?

# Matrix Models

A success story: Matrix Models in two dimensions

- An ab initio combinatorial statistical theory.
- Have built in scales $N$.
- Generate ribbon graphs $\leftrightarrow$ discretized surfaces.
- They undergo a phase transition ("condensation") to a continuum theory of large surfaces.

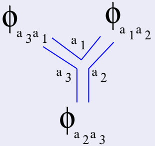
Physics: quantum gravity in $D = 2$, critical phenomena, conformal field theory, the theory of strong interactions, string theory, etc.
Mathematics: knot theory, number theory and the Riemann hypothesis, invariants of algebraic curves, enumeration problems, etc.

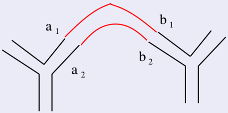
All these applications rely crucially on the "$1/N$" expansion!

# Ribbon Graphs as Feynman Graphs

Consider the partition function.

$$Z(Q) = \int [d\phi] \, e^{-N\left( \frac{1}{2} \sum \phi_{a_1 a_2} \delta_{a_1 b_1} \delta_{a_2 b_2} \phi^*_{b_1 b_2} + \lambda \sum \phi_{a_1 a_2} \phi_{a_2 a_3} \phi_{a_3 a_1} \right)}$$



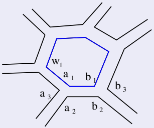

Ribbon vertex because the field $\phi$ has two arguments.
The lines conserve the two arguments (thus having two strands).
Strands close into faces.

## $Z(Q)$ is a sum over ribbon Feynman graphs.

# Amplitude of Ribbon Graphs

The Amplitude of a graph with $\mathcal{N}$ vertices is

$$A = \lambda^{\mathcal{N}} N^{-\mathcal{L}+\mathcal{N}} \sum \prod_{\text{lines}} \delta_{a_1 b_1} \delta_{a_2 b_2}$$

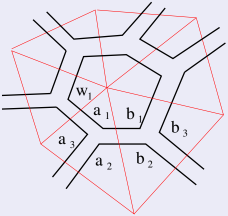



$$\sum \delta_{a_1 b_1} \delta_{b_1 c_1} \dots \delta_{w_1 a_1} = \sum \delta_{a_1 a_1} = N$$

$$A = \lambda^{\mathcal{N}} N^{\mathcal{N}-\mathcal{L}+\mathcal{F}} = \lambda^{\mathcal{N}} N^{2-2g(\mathcal{G})}$$

with $g_{\mathcal{G}}$ is the genus of the graph. $1/N$ expansion in the genus. Planar graphs ($g_{\mathcal{G}} = 0$) dominate in the large $N$ limit.

# Ribbon Graphs are Dual to Discrete Surfaces



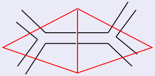

Place a point in the middle of each face. Draw a line crossing each ribbon line. The ribbon vertices correspond to triangles.

A ribbon graph encodes unambiguously a gluing of triangles.

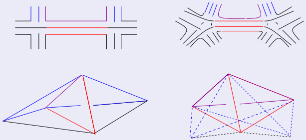
Matrix models sum over all graphs (i.e. surfaces) with canonical weights (Feynman rules). The dominant planar graphs represent spheres.

# From Matrix to COLORED Tensor Models

surfaces $\leftrightarrow$ ribbon graphs



$D$ dimensional spaces $\leftrightarrow$ colored stranded graphs



Matrix $M_{ab}$,
$$S = N\Big( M_{ab}\bar{M}_{ab} + \lambda M_{ab}M_{bc}M_{ca} \Big)$$

Tensors $T^{i}{}_{a_1 \dots a_D}$ with color $i$
$$S = N^{D/2}\Big( T^{i}_{\dots}\bar{T}^{i}_{\dots} + \lambda T^{0}_{\dots}T^{1}_{\dots}\dots T^{D}_{\dots} \Big)$$

$g(\mathcal{G}) \geq 0$ genus

$\omega(\mathcal{G}) \geq 0$ degree

$1/N$ expansion in the genus
$A(\mathcal{G}) = N^{2-2g(\mathcal{G})}$

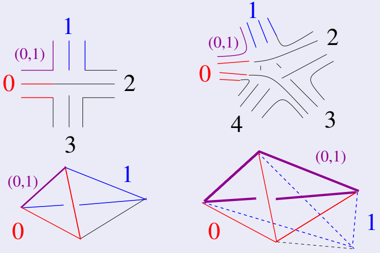
$1/N$ expansion in the degree
$A(\mathcal{G}) = N^{D-\frac{2}{(D-1)!}\omega(\mathcal{G})}$

leading order: $g(\mathcal{G}) = 0$, spheres.

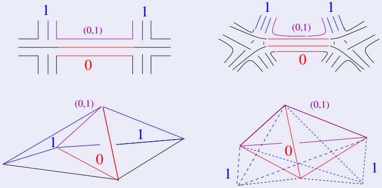
leading order: $\omega(\mathcal{G}) = 0$, spheres.

# Colored Stranded Graphs

Clockwise and anticlockwise turning colored vertices (positive and negative oriented $D$ simplices).




Lines have a well defined color and $D$ parallel strands ($D-1$ simplices).




Strands are identified by a couple of colors ($D-2$ simplices).

# Action

Let $T^i_{a_1\dots a_D}$, $\bar T^i_{a_1\dots a_D}$ tensor fields with color $i = 0\dots D$ .

$$S = N^{D/2}\Big( \sum_i \bar T^i_{a_1\dots a_D} T^i_{a_1\dots a_D} + \lambda \prod_i T^i_{a_{ii-1}\dots a_{i0} a_{iD}\dots a_{ii+1}} + \bar\lambda \prod_i \bar T^i_{a_{ii-1}\dots a_{i0} a_{iD}\dots a_{ii+1}} \Big)$$

## Topology of the Colored Graphs

Amplitude of the graphs:

- the $\mathcal{N} = 2p$ vertices of a graph bring each $N^{D/2}$
- the $\mathcal{L}$ lines of a graphs bring each $N^{-D/2}$
- the $\mathcal{F}$ faces of a graph bring each $N$

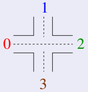
$$A^{\mathcal{G}} = (\lambda\bar\lambda)^p \, N^{-\mathcal{L}\frac{D}{2} + \mathcal{N}\frac{D}{2} + \mathcal{F}} = (\lambda\bar\lambda)^p \, N^{-p\frac{D(D-1)}{2} + \mathcal{F}}$$

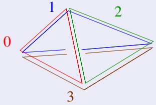
But $\mathcal{N}(D+1) = 2\mathcal{L} \Rightarrow \mathcal{L} = (D+1)p$

Compute $\mathcal{F}$ !

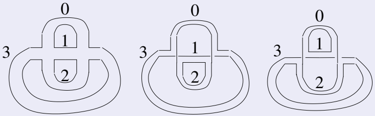
# Jackets 1

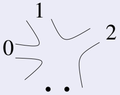
Define simpler graphs. Idea: forget the interior strands! Leads to a ribbon graph.



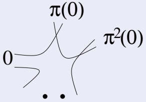




02 and 13: opposing edges of the tetrahedron. But 01, 23 and 12, 03 are perfectly equivalent. Three jacket (ribbon) graphs.




$0, 1, 2, \dots$




$0, \pi(0), \pi^2(0), \dots$

$\frac{1}{2}D!$ jackets. Contain all the vertices and all the lines of $\mathcal{G}$. A face belongs to $(D-1)!$ jackets.

The degree of $\mathcal{G}$ is $\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}}$.

# Jackets 2: Jackets and Amplitude

**Theorem**

*$\mathcal{F}$ and $\omega(\mathcal{G})$ are related by*

$$\mathcal{F} = \frac{1}{2}D(D-1)p + D - \frac{2}{(D-1)!}\omega(\mathcal{G})$$

**Proof:** $\mathcal{N} = 2p$, $\mathcal{L} = (D+1)p$
For each jacket $\mathcal{J}$, $2p - (D+1)p + \mathcal{F}_{\mathcal{J}} = 2 - 2g_{\mathcal{J}}$.
Sum over the jackets: $(D-1)!\mathcal{F} = \sum_{\mathcal{J}} \mathcal{F}_{\mathcal{J}} = \frac{1}{2}D!(D-1)p + D! - 2\sum_{\mathcal{J}} g_{\mathcal{J}}$ □

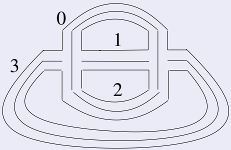
The amplitude of a graph is given by its degree

$$A^{\mathcal{G}} = (\lambda\bar{\lambda})^p \, N^{-p\frac{D(D-1)}{2} + \mathcal{F}} = (\lambda\bar{\lambda})^p \, N^{D - \frac{2}{(D-1)!}\omega(\mathcal{G})}$$

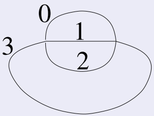
# Topology 1: Colored vs. Stranded Graphs

**THEOREM:** [M. Ferri and C. Gagliardi, '82] Any $D$-dimensional piecewise linear orientable manifold admits a colored triangulation.

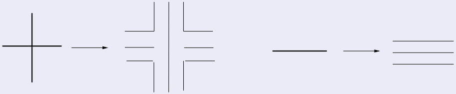
We have clockwise and anticlockwise turning vertices. Lines connect opposing vertices and have a color index. All the information is encoded in the colors




represented as



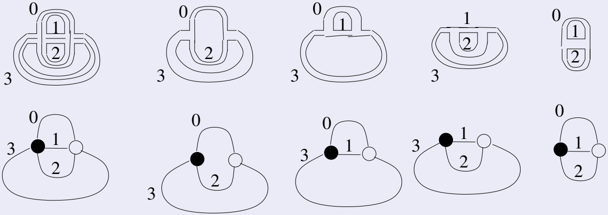

Conversely: expand the vertices into stranded vertices and the lines into stranded lines with parallel strands



# Topology 2: Bubbles

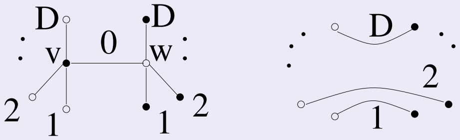
The vertices of $\mathcal{G}$ are subgraphs with 0 colors. The lines are subgraphs with exactly 1 color. The faces are subgraphs with exactly 2 colors.

The *n*-bubbles are the maximally connected subgraphs with n fixed colors (denoted $\mathcal{B}^{i_1\ldots i_n}_{(\sigma)}$, with $i_1 < \cdots < i_n$ the colors).



A colored graph $\mathcal{G}$ is dual to an orientable, normal, $D$ dimensional, simplicial pseudo manifold. Its $n$-bubbles are dual to the links of the $D-n$ simplices of the pseudo manifold.

# Topology 3: Homeomorphisms and 1-Dipoles




A 1-dipole: a line (say of color 0) connecting two vertices $v \in \mathcal{B}^{1\dots D}_{(\alpha)}$ and $w \in \mathcal{B}^{1\dots D}_{(\beta)}$ with $\mathcal{B}^{1\dots D}_{(\alpha)} \neq \mathcal{B}^{1\dots D}_{(\beta)}$.

A 1-Dipole can be contracted, that is the lines together with the vertices $v$ and $w$ can be deleted from $\mathcal{G}$ and the remaining lines reconnected respecting the coloring. Call the graph after contraction $\mathcal{G}/d$.

**THEOREM:** [M. Ferri and C. Gagliardi, '82] If either $\mathcal{B}^{1\dots D}_{(\alpha)}$ or $\mathcal{B}^{1\dots D}_{(\beta)}$ is dual to a sphere, then the two pseudo manifolds dual to $\mathcal{G}$ and $\mathcal{G}/d$ are homeomorphic.

It is in principle very difficult to check if a bubble is a sphere or not.

# Jackets, Bubbles, 1-Dipoles

The $D$-bubbles $\mathcal{B}^{\widehat{i}}_{(\rho)}$ of $\mathcal{G}$ are graphs with $D$ colors, thus they admit jackets and have a degree. The degrees of $\mathcal{G}$ and of its bubbles are not independent.

**Theorem**

$$\omega(\mathcal{G}) = \frac{(D-1)!}{2}\left(p + D - \mathcal{B}^{[D]}\right) + \sum_{i,\rho} \omega(\mathcal{B}^{\widehat{i}}_{(\rho)})$$

**Theorem**

*The degree of the graph is invariant under 1-Dipole moves,* $\omega(\mathcal{G}) = \omega(\mathcal{G}/d)$

# Degree 0 Graphs are Spheres

$\omega(\mathcal{G}) = \frac{(D-1)!}{2}\Big(p + D - \mathcal{B}^{[D]}\Big) + \sum_{i,\rho} \omega(\mathcal{B}^{\widehat{i}}_{(\rho)})$

In a graph $\mathcal{G}$ with $2p$ vertices and $\mathcal{B}^{[D]}$ $D$-bubbles I contract a full set of 1-Dipoles and bring it to $\mathcal{G}_f$ with $2p_f$ vertices and exactly one $D$-bubble for each colors $\widehat{i}$.
Every contraction: $p \to p-1$, $\mathcal{B}^{[D]} \to \mathcal{B}^{[D]} - 1$

$$p - p_f = \mathcal{B}^{[D]} - \mathcal{B}^{[D]}_f = \mathcal{B}^{[D]} - (D+1) \Rightarrow p + D - \mathcal{B}^{[D]} = p_f - 1 \geq 0$$

Thus $\omega(\mathcal{G}) = 0 \Rightarrow \omega(\mathcal{B}^{\widehat{i}}_{(\rho)}) = 0$.

### Theorem

*If $\omega(\mathcal{G}) = 0$ then $\mathcal{G}$ is dual to a $D$-dimensional sphere.*

**Proof:** Induction on $D$. $D = 2$: the colored graphs are ribbon graphs and the degree is the genus. In $D > 2$, $\omega(\mathcal{G}) = 0 \Rightarrow \omega(\mathcal{B}^{\widehat{i}}_{(\rho)}) = 0$ and all $\omega(\mathcal{B}^{\widehat{i}}_{(\rho)})$ are a spheres by the induction hypothesis. 1-Dipole contractions do not change the degree and are homeomorphisms. $\mathcal{G}_f$ is homeomorphic with $\mathcal{G}$ and has $p_f = 1$. The only graph with $p_f = 1$ is a sphere.

# From Matrix to COLORED Tensor Models

Tensors ${T^i}_{a_1 \dots a_D}$ with color $i$

$$S = N^{D/2}\Big( T^i_{\dots} \bar{T}^i_{\dots} + \lambda T^0_{\dots} T^1_{\dots} \dots T^D_{\dots} + \bar{\lambda} \bar{T}^0_{\dots} \bar{T}^1_{\dots} \dots \bar{T}^D_{\dots} \Big)$$

$\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}} \geq 0$ degree

$1/N$ expansion in the degree $A(\mathcal{G}) = N^{D - \frac{2}{(D-1)!}\omega(\mathcal{G})}$

colored stranded graphs $\leftrightarrow$ $D$ dimensional pseudo manifolds

leading order: $\omega(\mathcal{G}) = 0$ are spheres

# Conclusion: A To Do List

- Is the dominant sector summable?
- Does it lead to a phase transition and a continuum theory?
- What are the critical exponents?
- Multi critical points?
- More complex models, driven to the phase transition by renormalization group flow.

- Generalize the results obtained using matrix models in higher dimensions.