SPINFOAM ON LEFSCHETZ-THIMBLE:

Markov Chain Monte-Carlo Computation of Lorentzian Spinfoam Propagator

Zichang Huang, Department of Physics, Fudan University
arXiv: 2012.11515

Feb. 23rd @ILQGS

Collaborators: Muxin Han (Florida Atlantic University), Hongguang Liu (University of Erlangen-Nuremberg), Dongxue Qu (Florida Atlantic University), Yidun Wan (Fudan University)

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INTRODUCTION

Spinfoam model
& the sign problem

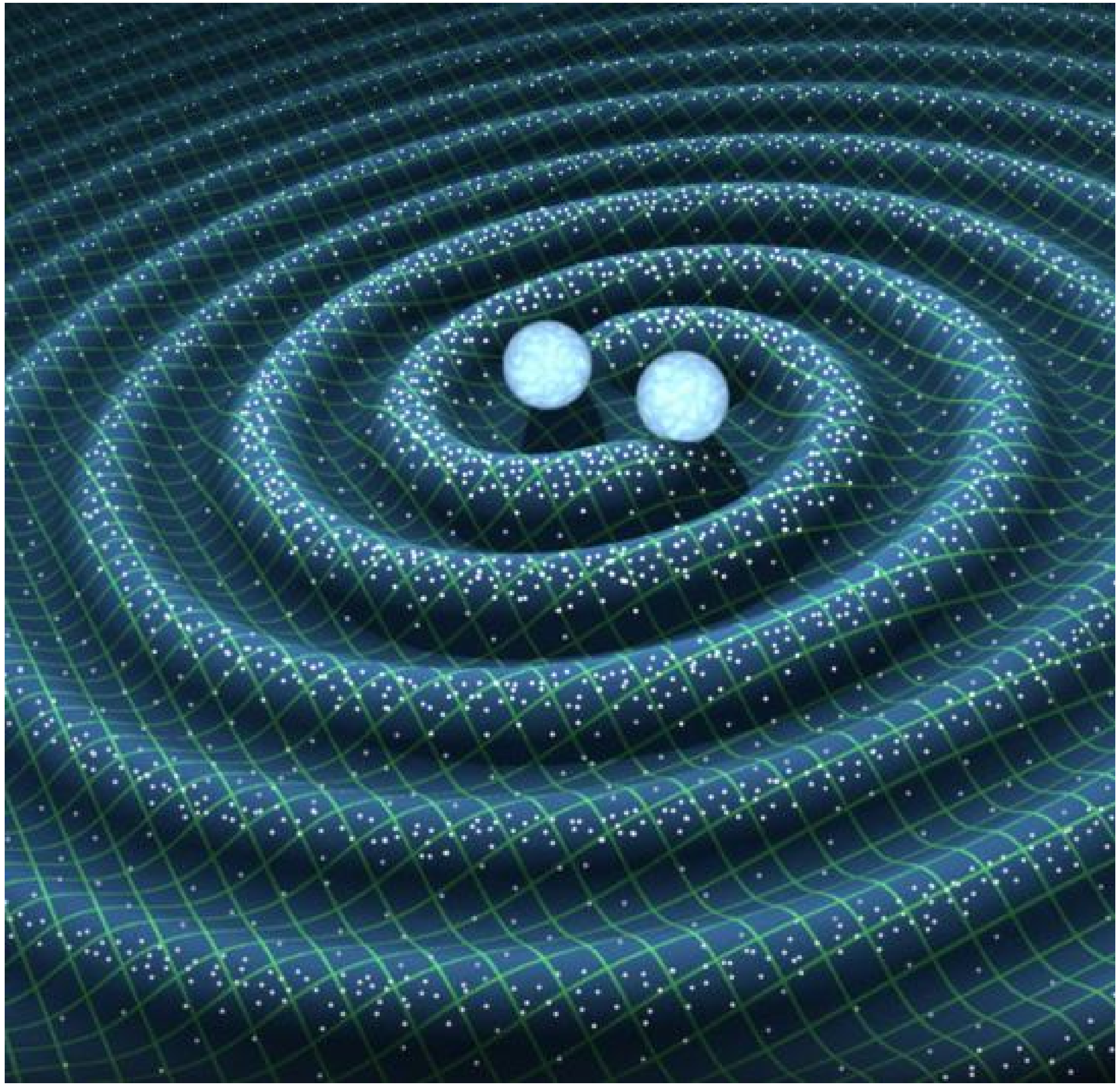
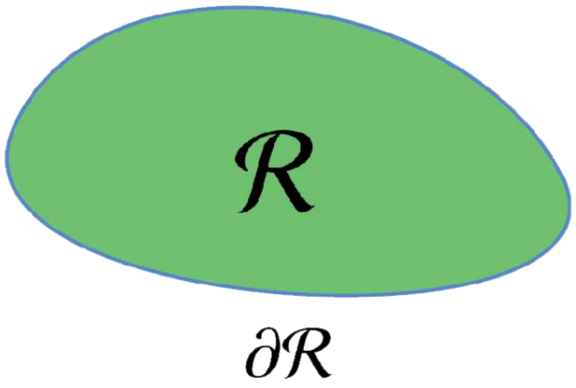
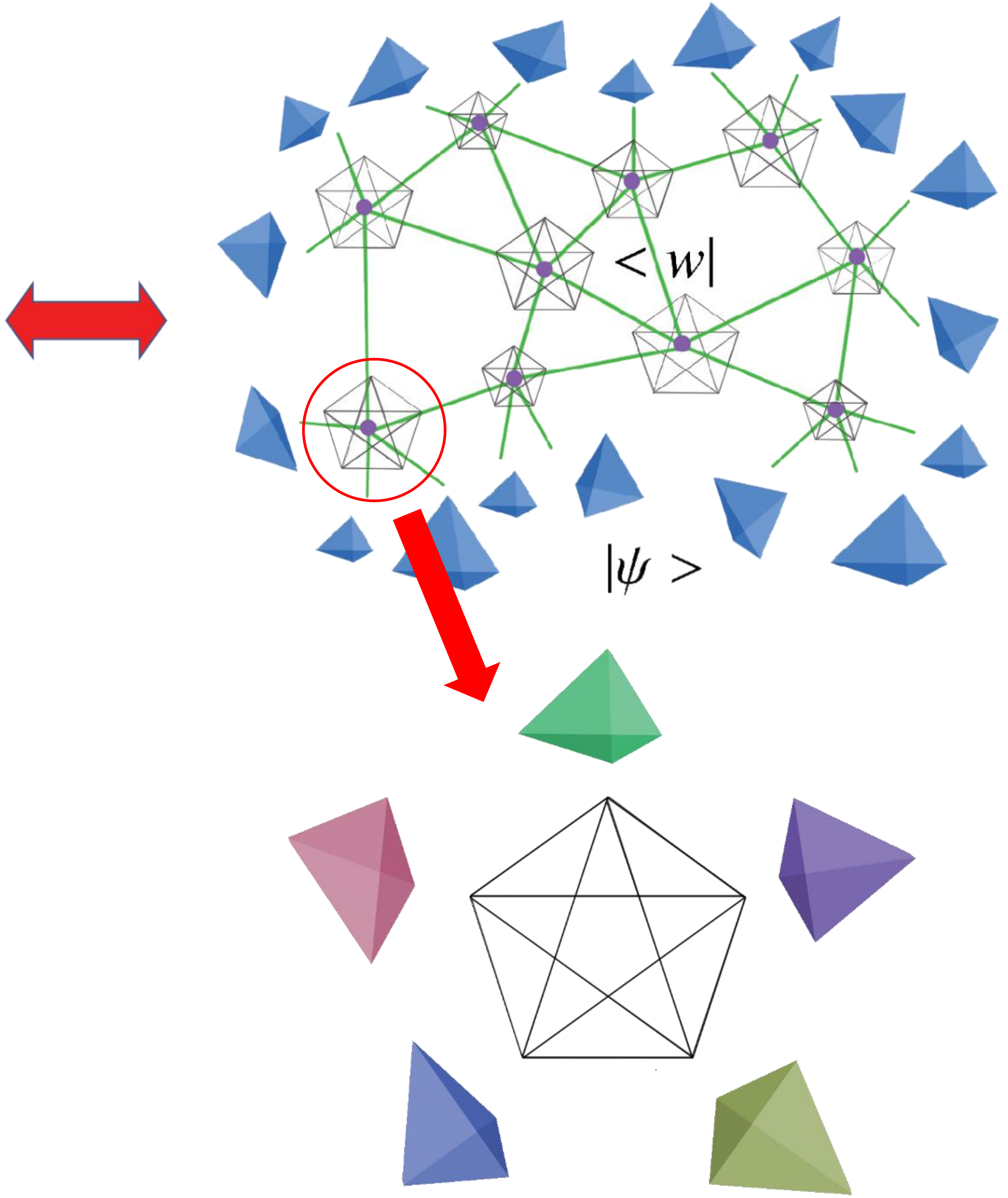


Photo Credit: R. HURT - CALTECH / JPL

SPINFOAM MODEL





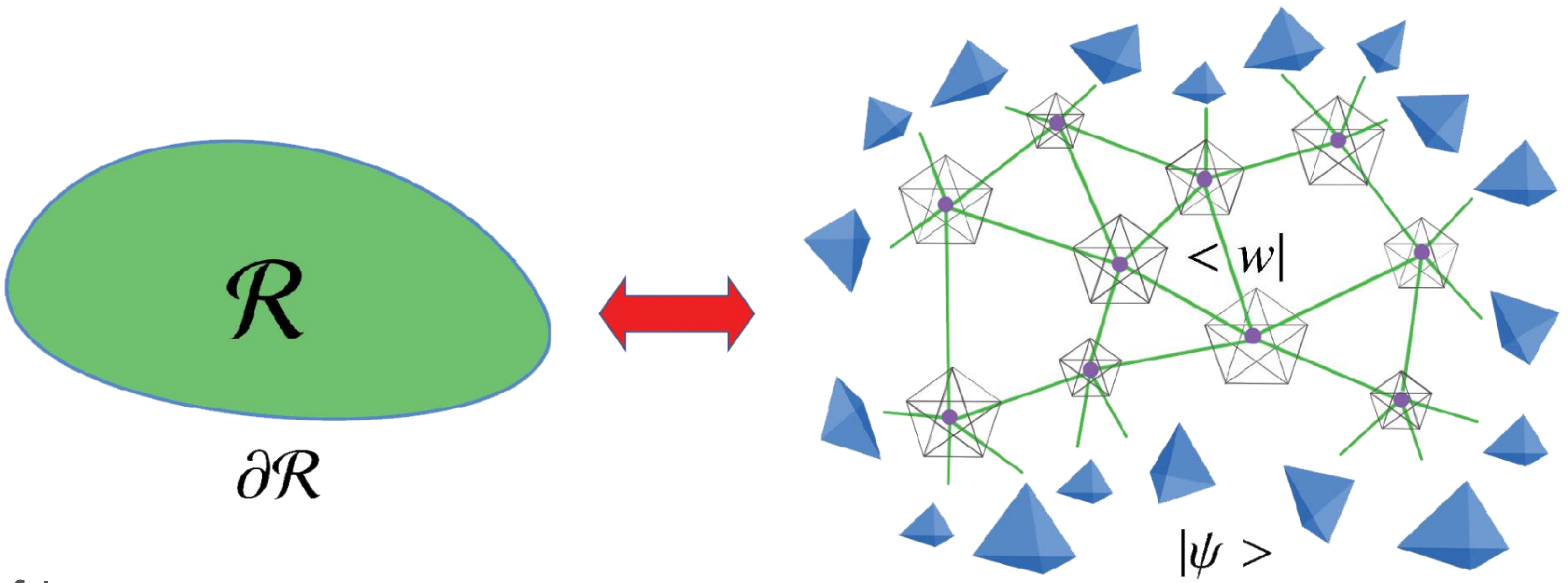
- Covariant formulation of Loop Quantum Gravity in 4D
- Quantum space time
  - Boundary: spin-network states
  - Bulk: map

$$\langle W| : |\psi\rangle \mapsto \mathcal{A}$$

  - Discretized spacetime:
    - Minimum unit: 4-simplex amplitude (space time atom)
    - Connection = Entanglement

[T.Thiemann 2007, A.Ashtekar, J. Lewandoski 2004, C.Roveill F.Vidotto 2015, A. perez 2003, J.Engle, E.Livine, R.Pereira, C.Roveill 2008, L.Freidel K.Karsnov 2008, etc]

SPINFOAM MODEL



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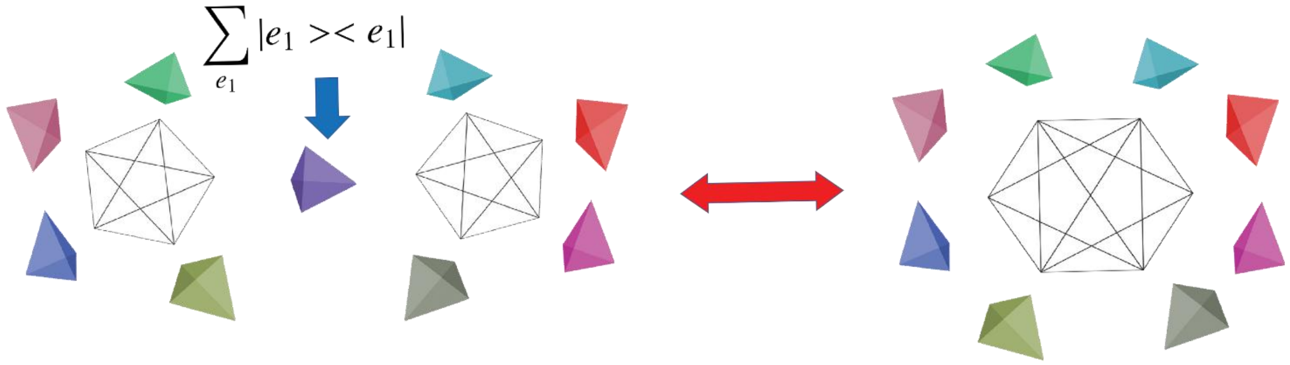

$$\langle x_f, t_f | x_i, t_i \rangle$$
$$= \int \mathrm{d}x_{N-1} \mathrm{d}x_{N-2} \cdots \mathrm{d}x_1$$
$$\times \langle x_f, t_f | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_2, t_2 | x_1, t_1 \rangle$$

SPINFOAM MODEL

- Spinfoam amplitude: (finite dimensional integral)

$$\mathcal{A} = \langle W|\psi\rangle = \int D[\phi]W[\phi]\psi[\phi] = \int D\phi \mathrm{e}^{-S[\phi]}$$ Path-integral formulation of LQG

- Semi-classical limit → Discrete gravity(Regge calculus)
- Observables:

$$\langle\hat{\mathcal{O}}\rangle = \frac{\langle W|\hat{\mathcal{O}}|\psi\rangle}{\langle W \mid \psi\rangle} = \frac{\int D\phi \mathcal{O}[\phi]\mathrm{e}^{-S[\phi]}}{\int D\phi \mathrm{e}^{-S[\phi]}}$$

Penrose metric:
$$q^{ab}(x) = \delta^{ij} E_i^a(x) E_j^b(x)$$

e.g., 2-point connected correlation function:

$$G^{abcd}(x,y) = \left\langle q^{ab}(x) q^{cd}(y)\right\rangle - \left\langle q^{ab}(x)\right\rangle \left\langle q^{cd}(y)\right\rangle$$

↓ Semi-classical limit

graviton propagator

[C.Roveill 2006, A. perez 2003, E. Bianchi, L. Modesto, C. Rovelli, and S. Speziale 2006, E. Bianchi, E. Magliaro, and C. Perini 2009, E. Bianchi and Y. Ding 2012, M. Han, Z. Huang, and A. Zipfel 2013]

SPINFOAM MODEL

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We compute G^{abcd} numerically in this work

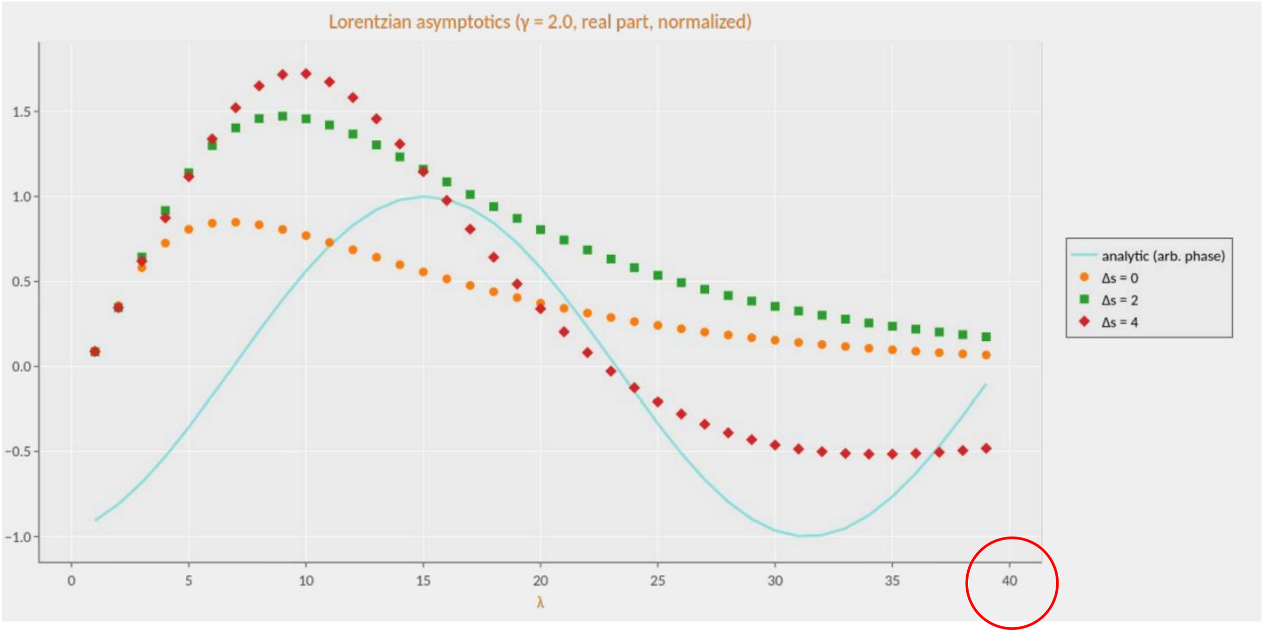
REQUIREMENTS

- Reliability
  - Can go to high spin limit ⟺ Asymptotic results
  - Can adapt to multiple situation
    - Euclidean/Lorentzian models
    - Different values of γ
    - Different boundary states
- Utility
  - Can compute different observables:

$$\langle q^{ab}(x)q^{cd}(y)\rangle\,,\ \langle q^{ab}(x)\rangle\,,\ \langle q^{cd}(y)\rangle\cdots$$

SL2CFOAM

- sl2cfoam package
  - Can only compute amplitude (so far)
  - Boundary spin cannot be very large



Spin smaller than 100

Not suitable for the task

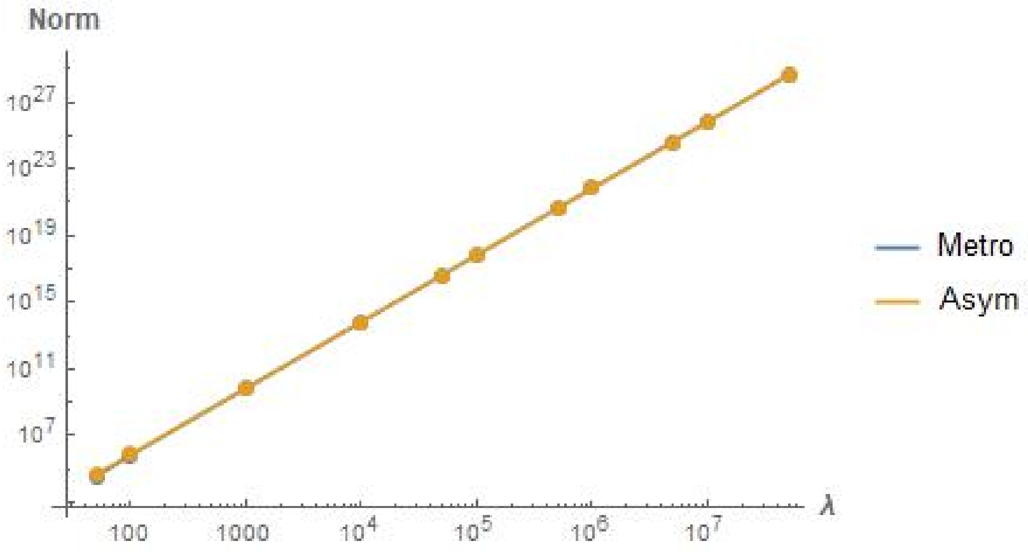
[P.Dona, S. Speziale, F. Gozzini, G. Sarno, etc]

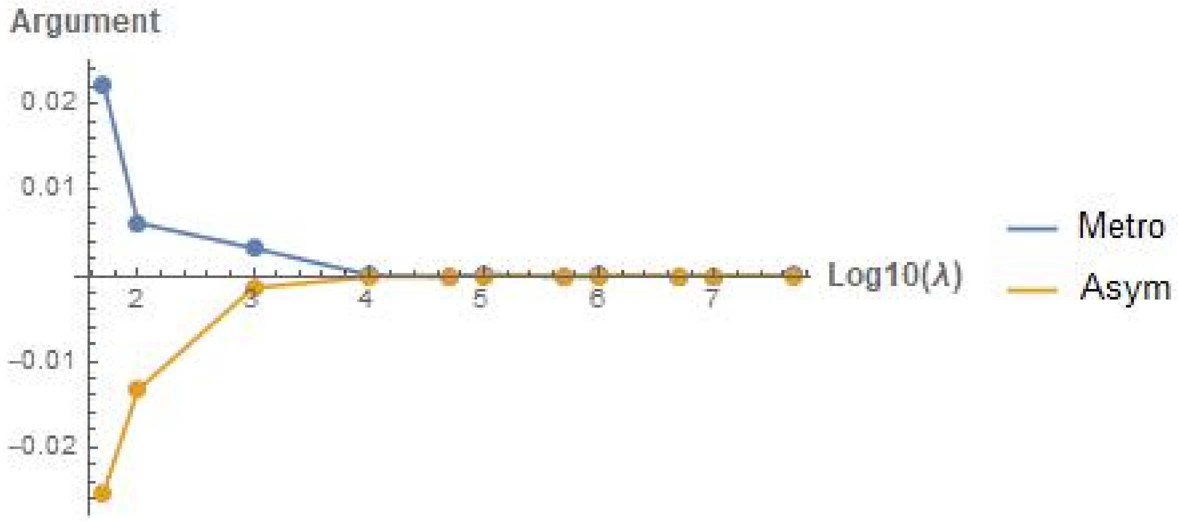
Can we find an algorithm to do the computation?

Yes, we can!

PREVIEW THE RESULTS

- Results of Expectation values $\langle E_1^2 \cdot E_1^3 E_4^1 \cdot E_4^5 \rangle$





Leading order in $1/\lambda$ expansion

Thimble + Metropolis

λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	8.71	0.79	0.12	0.052	0.036	0.017	0.0062	0.0018	0.00037	0.00069

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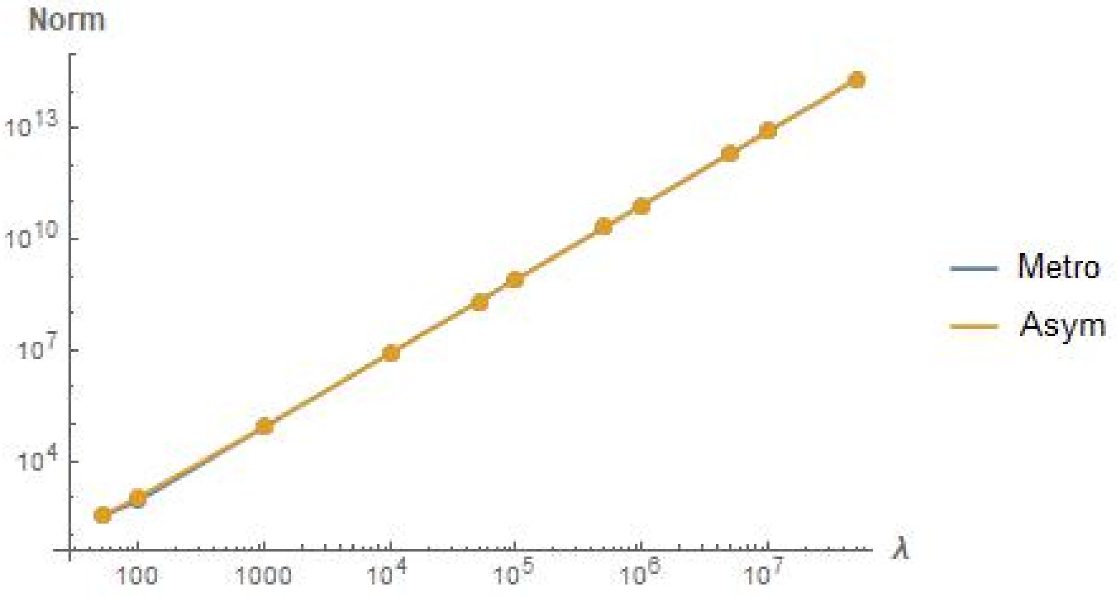
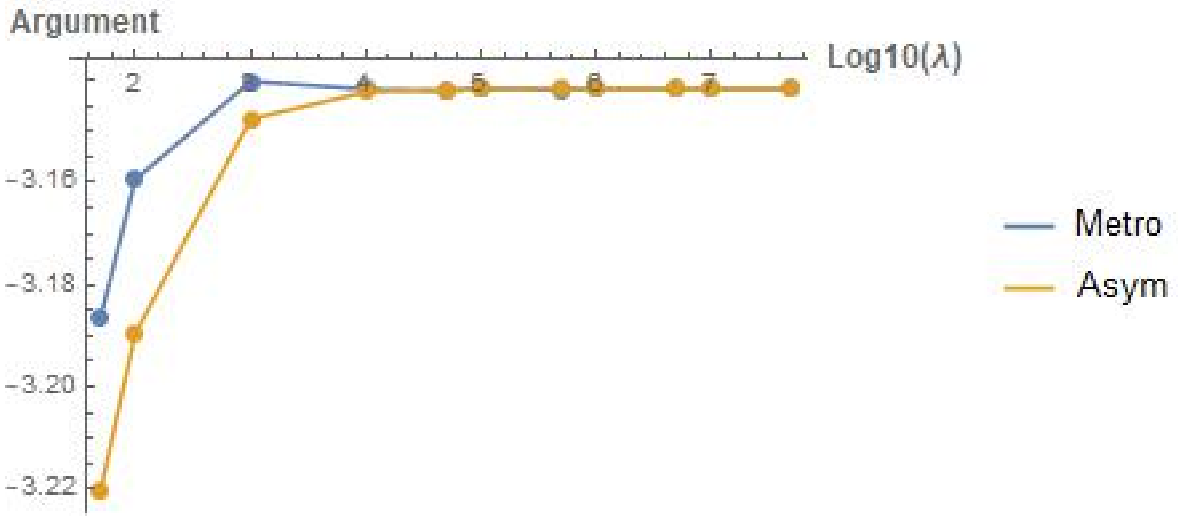
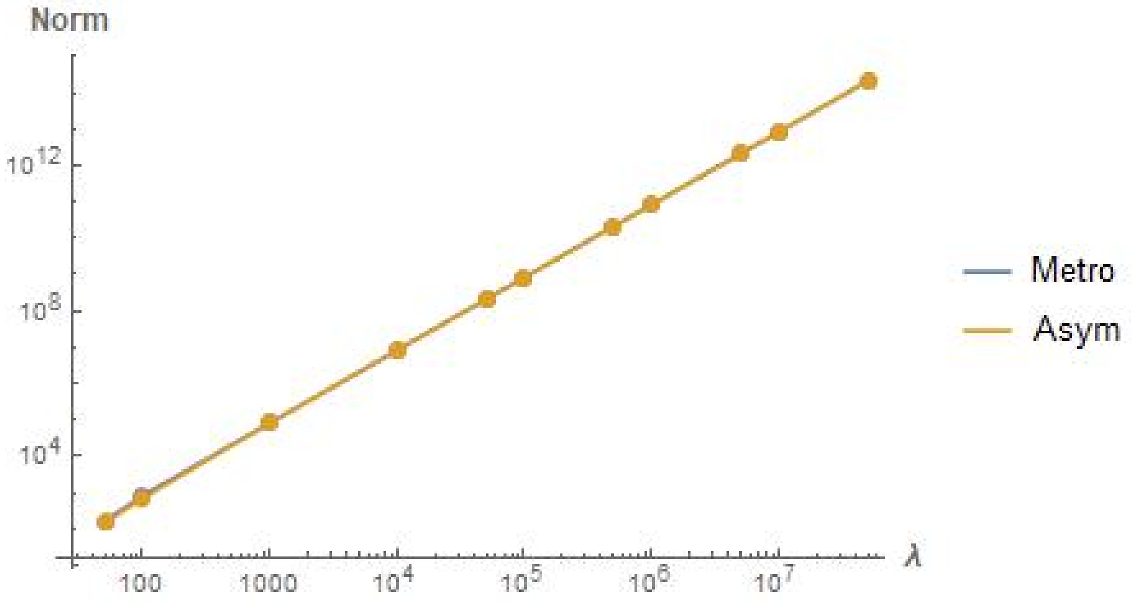


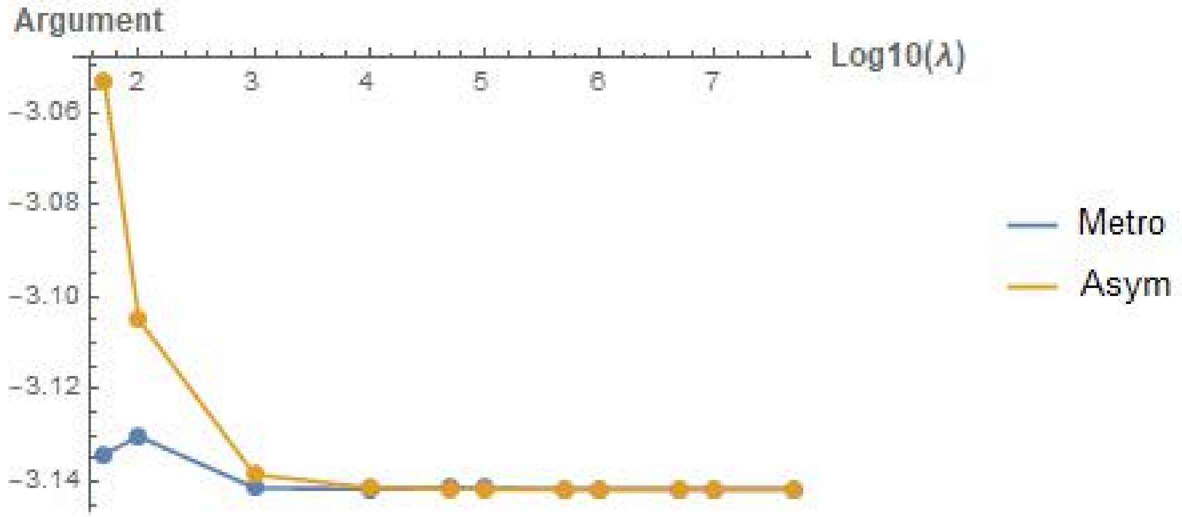


λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	22.32	2.00	0.31	0.078	0.022	0.016	0.012	0.0022	0.000047	0.0016

PREVIEW THE RESULTS

- Results of Expectation values $\langle E_4^1 \cdot E_4^5 \rangle$

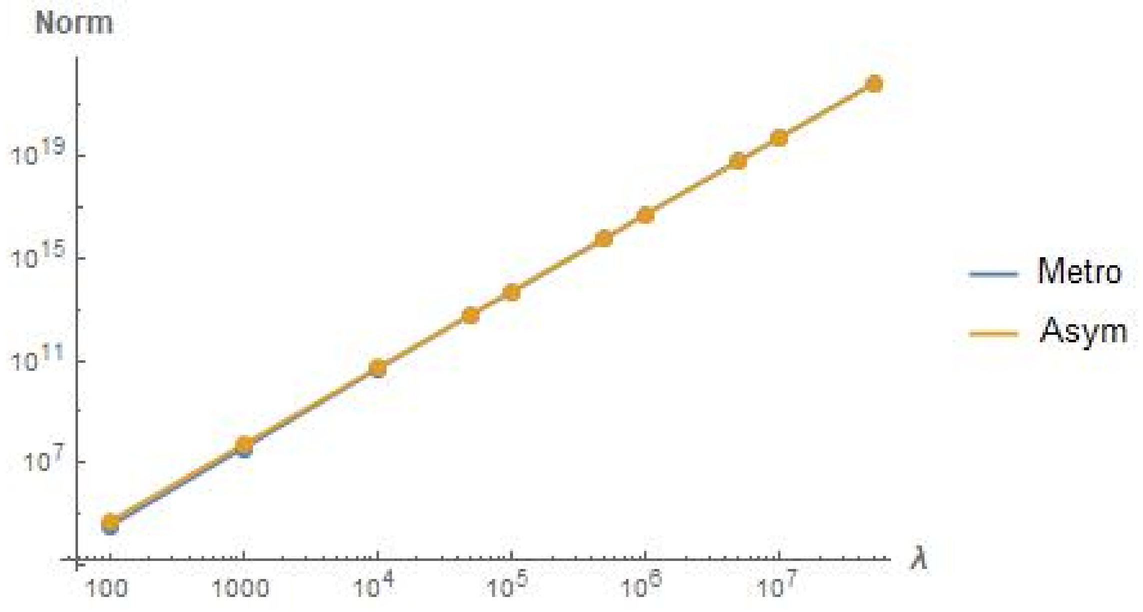


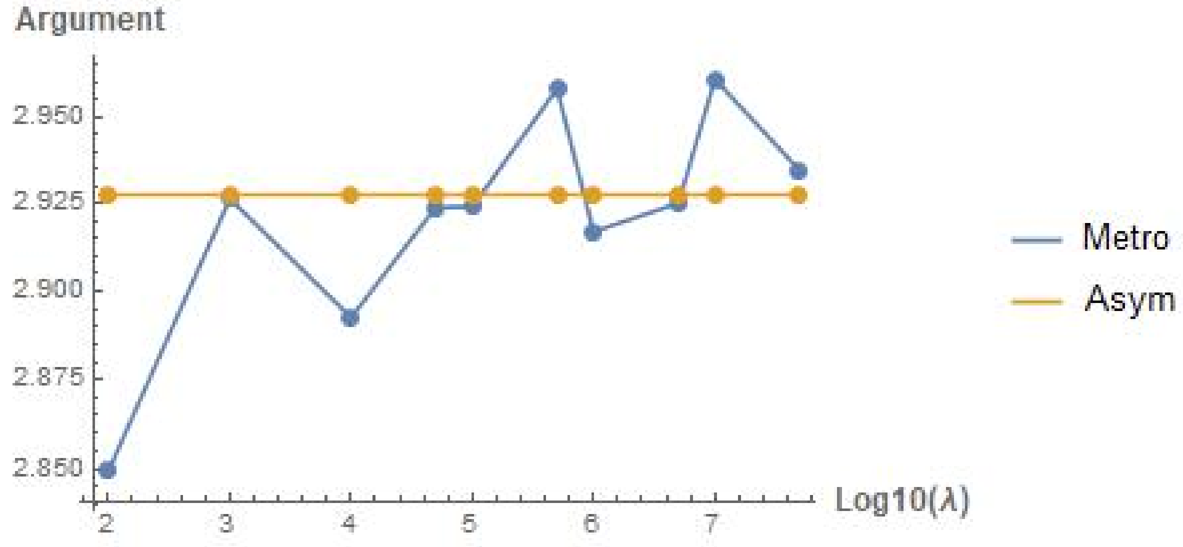


λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	18.66	1.18	0.18	0.026	0.017	0.00054	0.0037	0.00035	0.00036	0.00083

PREVIEW THE RESULTS

- Results of propagator component G_{14}^{2315}





λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	37.90	27.00	13.22	2.76	10.09	8.86	1.89	1.13	3.90	2.06

How to ...... ?

Do the integral directly!

$$\langle \hat{\mathcal{O}} \rangle = \frac{\langle W|\hat{\mathcal{O}}|\psi\rangle}{\langle W \mid \psi\rangle} = \frac{\int D\phi \mathcal{O}[\phi] \mathrm{e}^{-S[\phi]}}{\int D\phi \mathrm{e}^{-S[\phi]}}$$

THE SIGN PROBLEM

- Complex valued Action:

$$S(x) \in \mathbb{C} \quad \Longleftrightarrow \quad \mathrm{e}^{-S(x)} \text{ oscillatory}$$

- N-dimensional case:

$$\langle \hat{\mathcal{O}} \rangle = \frac{\int Dx \mathcal{O}(x) \mathrm{e}^{-S(x)}}{\int D(x) \mathrm{e}^{-S(x)}} \text{ should be } O(1)$$

Oscillatory: $\int Dx \mathcal{O}(x) \mathrm{e}^{-S(x)} \sim \mathrm{e}^{-O(N)}$

Monte-Carlo: $O\left(1/\sqrt{N_{\mathrm{conf}}}\right)$

$$\langle \mathcal{O} \rangle \sim \frac{e^{-O(N)} \pm O\left(1/\sqrt{N_{\mathrm{conf}}}\right)}{e^{-O(N)} \pm O\left(1/\sqrt{N_{\mathrm{conf}}}\right)} \quad \longrightarrow \quad N_{\mathrm{conf}} = e^{O(N)}$$

[M. Fukuma and N. Matsumoto 2020]

THE SIGN PROBLEM

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$$\langle \mathcal{O}(x) \rangle \sim \frac{e^{-O(N)} \pm O\left(1/\sqrt{N_{\mathrm{conf}}}\right)}{e^{-O(N)} \pm O\left(1/\sqrt{N_{\mathrm{conf}}}\right)} \quad \Longrightarrow \quad \boxed{N_{\mathrm{conf}} = e^{O(N)}}$$

Conventional Monte-Carlo is inefficient!

ONE SOLUTION!

- Lefschetz-Thimble integration:

$$\int Dx\mathcal{O}(x)\mathrm{e}^{-S(x)} = \int_{\mathbb{R}^N} Dz\mathcal{O}(z)\mathrm{e}^{-S(z)} = \int_{\Sigma} Dz\mathcal{O}(z)\mathrm{e}^{-S(z)}$$

$$\forall z \in \Sigma \text{ s.t. } \mathrm{Im}(S(z)) = \text{constant}$$

⬇

$$\forall z \in \Sigma,\ \mathcal{O}(z)e^{-S(z)} \text{ non-oscillatory}$$

[E.Witten 2010 , 2011]

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LEFSCHETZ-THIMBLE

A cure of the sign problem

LEFSCHETZ-THIMBLE

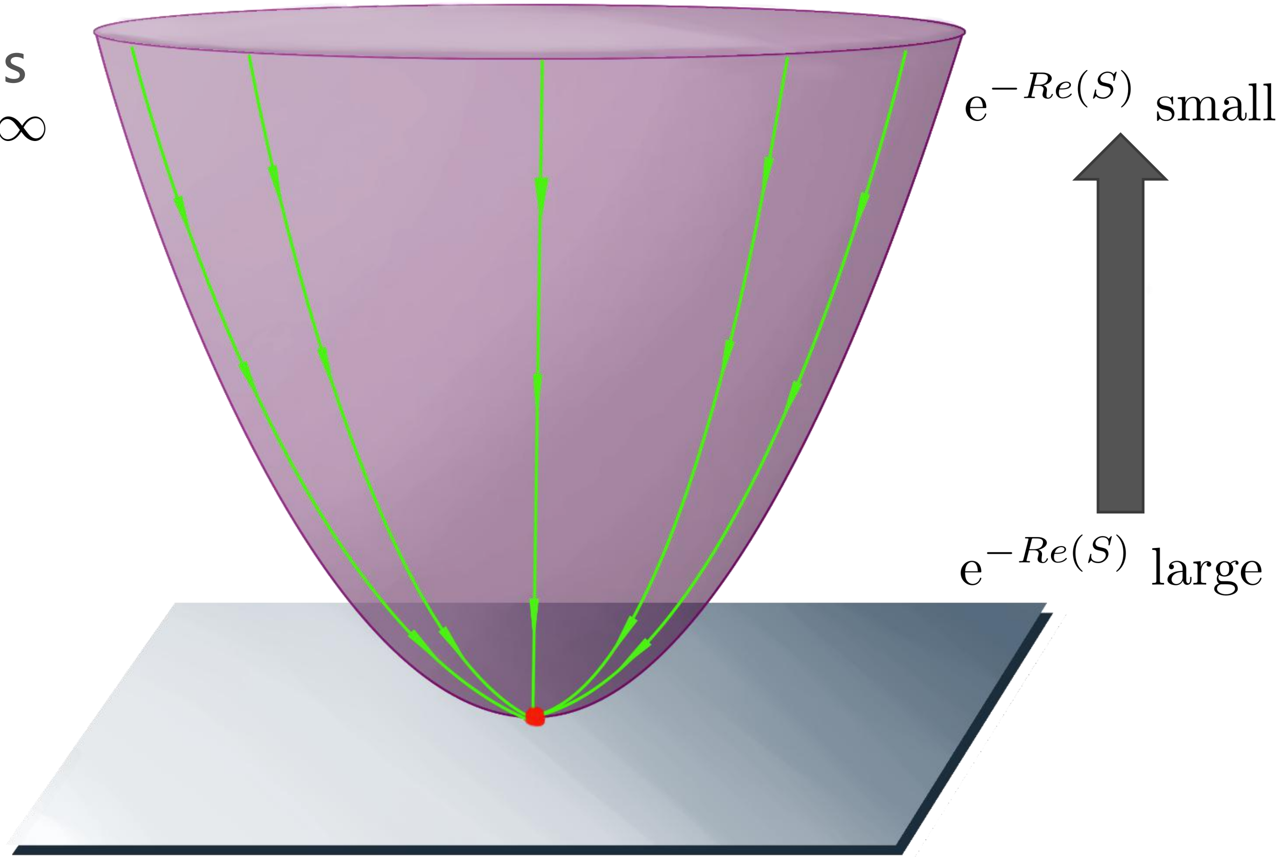
- Lefschetz-Thimble $\mathcal{J}_\sigma$
  - Union of steepest decent (SD) paths falling to critical point σ when $t \to \infty$
  - Steepest decent equation:

$$\frac{\mathrm{d}z^a}{\mathrm{d}t} = -\frac{\partial \overline{S(\vec{z})}}{\partial \overline{z^a}}$$

  - Imaginary part of action:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial S}{\partial z^a}\frac{\mathrm{d}z^a}{\mathrm{d}t} = -\left|\frac{\partial S}{\partial z^a}\right|^2$$

$$\mathrm{Im}(S(z)) = \text{constant}$$



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Solution

  - Imaginary part of action:

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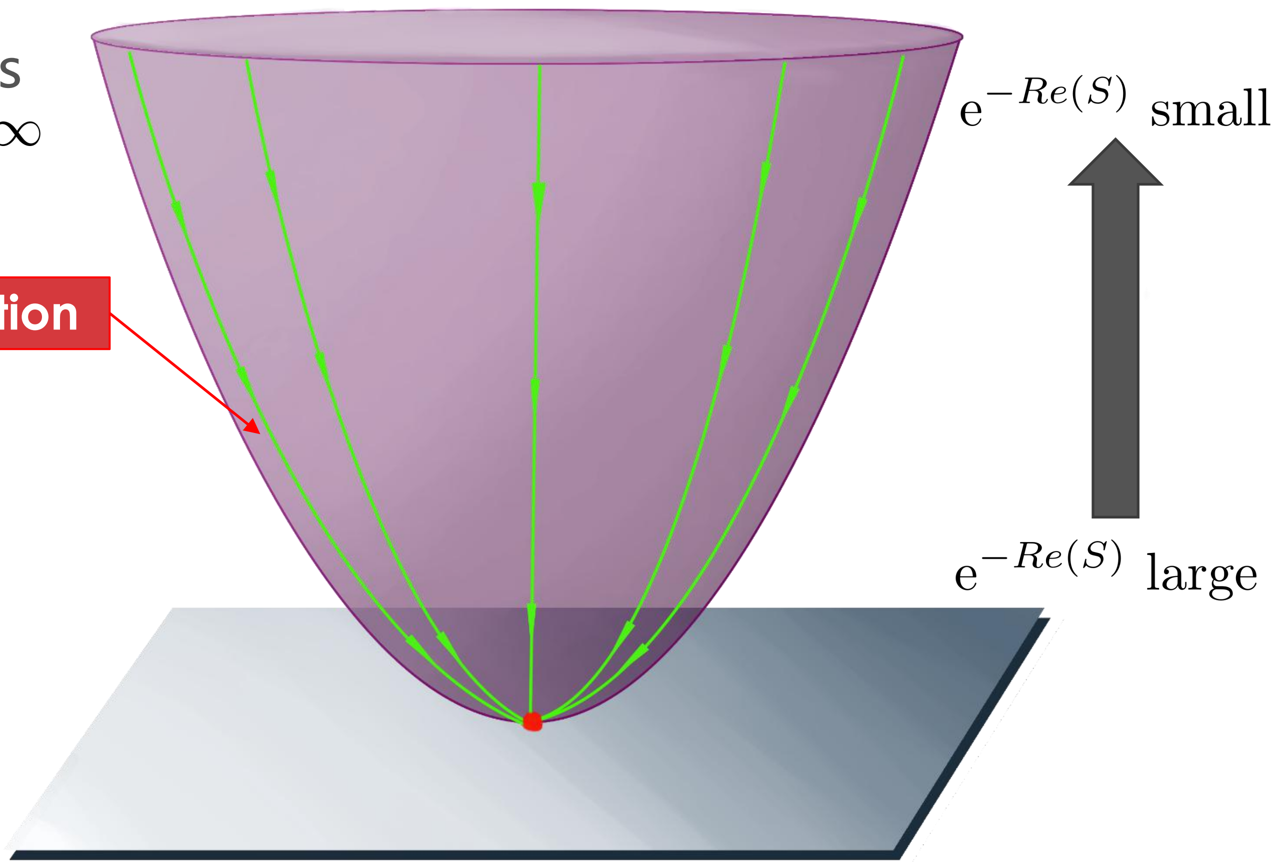

SD equation is not the equation of motion!

LEFSCHETZ-THIMBLE

- Picard-Lefschetz Theory:

$$\int_{\mathbb{R}^n} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-S(\vec{z})} = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-S(z)}$$

$$\mathbb{R}^n \text{ homologically equivalent to } \sum_{\sigma} n_{\sigma} \mathcal{J}_{\sigma}$$

LEFSCHETZ-THIMBLE

- Expectations values:

$$\langle f \rangle = \frac{\int_{\mathbb{R}^n} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-\hat{S}(\vec{z})}}{\int_{\mathbb{R}^n} \mathrm{d}^n z \mathrm{e}^{-\hat{S}(\vec{z})}} = \frac{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-\hat{S}(\vec{z})}}{\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}^n z \mathrm{e}^{-\hat{S}(\vec{z})}}$$

- If one thimble dominate the integral:

$$\langle f \rangle \simeq \frac{n_{\sigma'} \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(S(p_{\sigma'}))} \int_{\mathcal{J}_{\sigma'}} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-\mathrm{Re}(\hat{S}(\vec{z}))}}{n_{\sigma'} \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(S(p_{\sigma'}))} \int_{\mathcal{J}_{\sigma'}} \mathrm{d}^n z \mathrm{e}^{-\mathrm{Re}(\hat{S}(z))}} = \frac{\int_{\mathcal{J}_{\sigma'}} \mathrm{d}^n z \hat{f}(\vec{z}) \mathrm{e}^{-\mathrm{Re}(\hat{S}(\vec{z}))}}{\int_{\mathcal{J}_{\sigma'}} \mathrm{d}^n z \mathrm{e}^{-\mathrm{Re}(\hat{S}(z))}}$$

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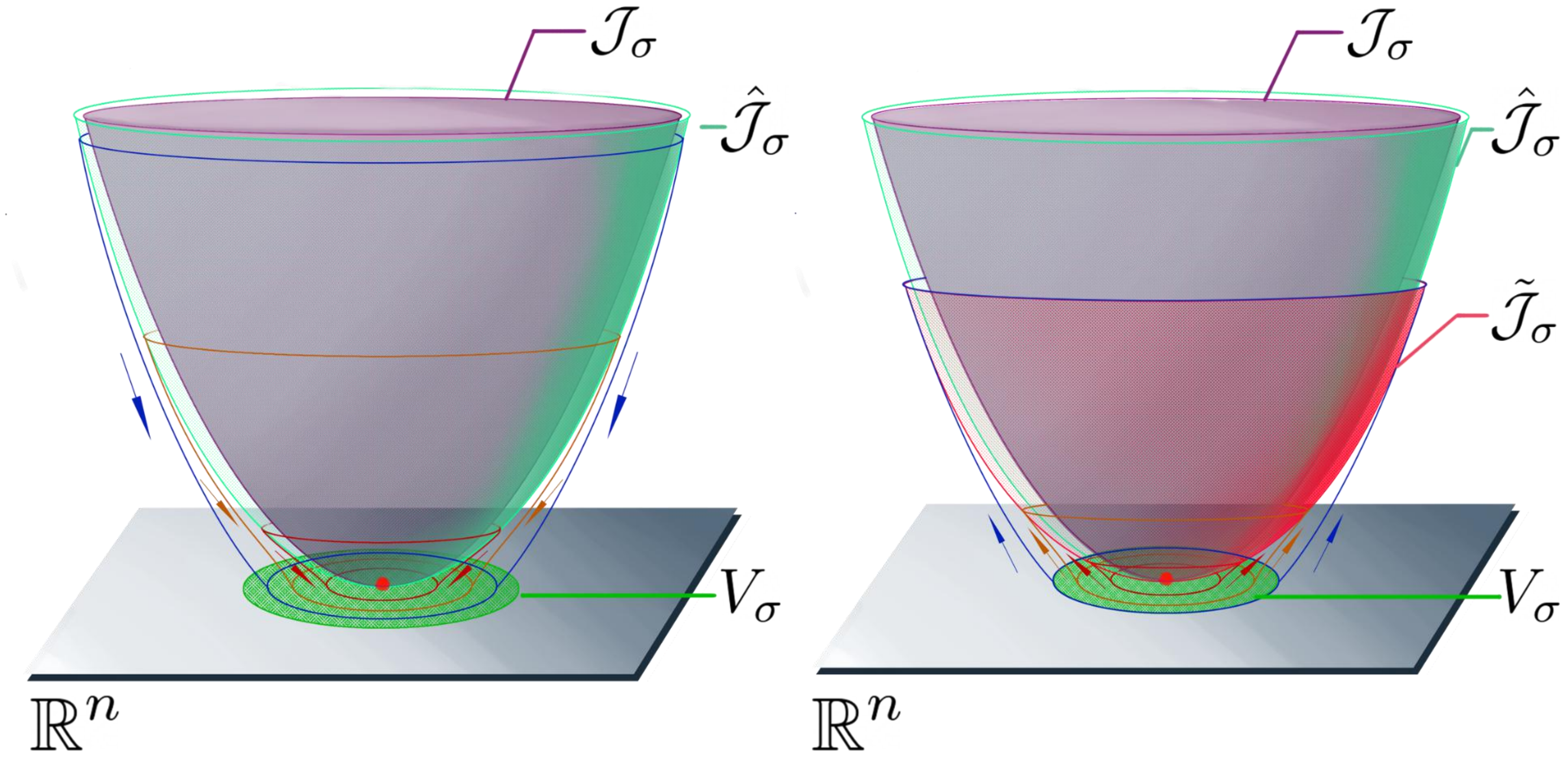
Spinfoam propagator satisfies

APPROXIMATED THIMBLE

SA flow:

$$\frac{\mathrm{d}z^a}{\mathrm{d}t} = \frac{\overline{\partial S(\vec{z})}}{\overline{\partial z^a}}$$



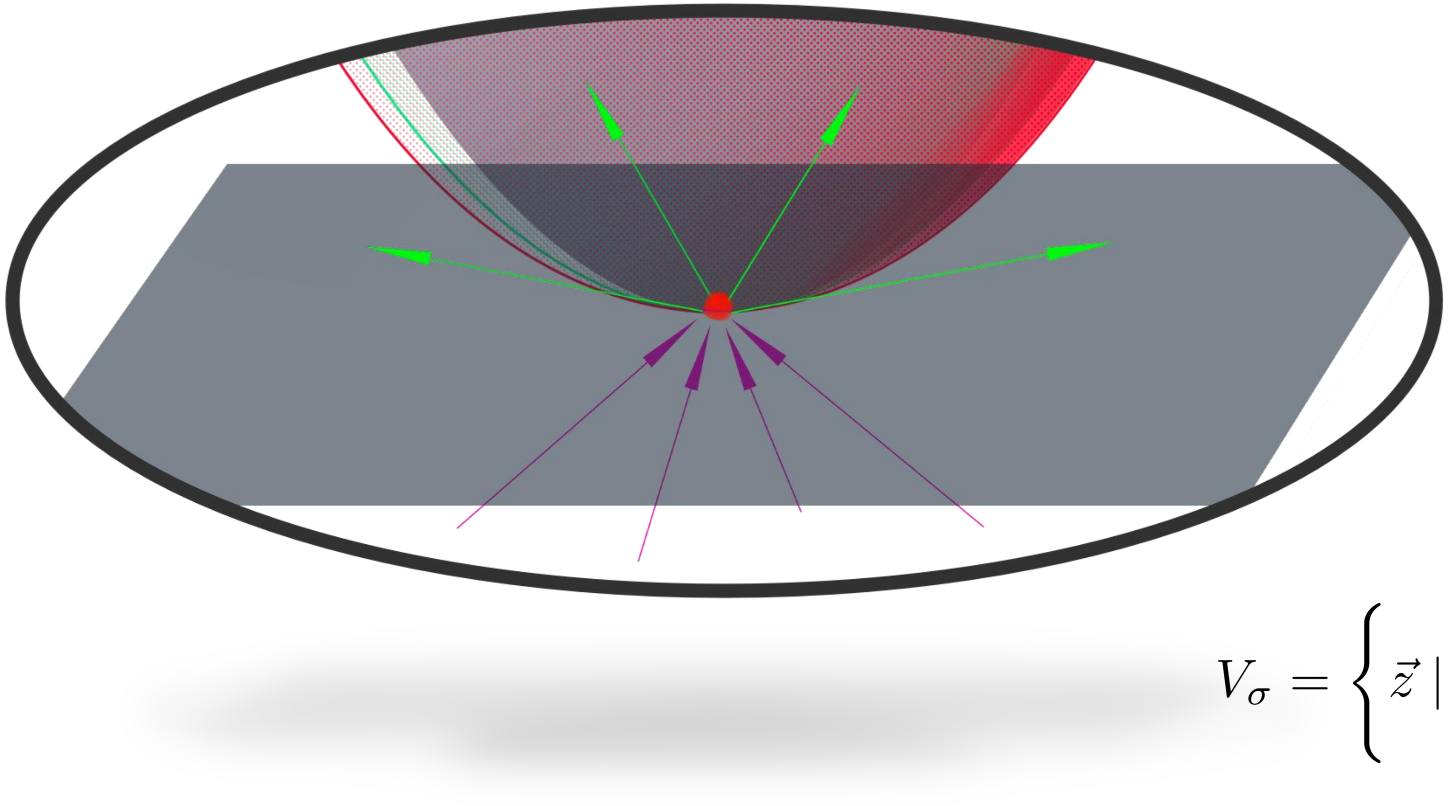
1) Steepest decent falling to V_σ

2) Steepest ascent from V_σ

- Not initiate from the far away point
- Fluctuation of $\mathrm{Im}(S(z))$ is small when V_σ is small
- Only need a portion of the thimble around the critical point

$e^{-\mathrm{Re}(S)}$ decays very fast

OPTIMAL CHOICE OF V_σ



- Generalized eigenvalue equation:

$$\mathbf{H}\omega = \lambda\bar{\omega}.$$

- Eigenvectors with positive eigen-values indicate the directions of the perturbation that can generate the approx. thimble.

- The V_σ is a good choice of $\hat{\mathcal{J}}_\sigma$ generating

$$V_\sigma = \left\{ \vec{z} \mid \vec{z} = \sum_{a=1}^{n} \hat{\omega}_i x^i + \vec{z}_\sigma, \text{ each } x^i \in \mathbb{R} \text{ is small} \right\}$$

FLOW OF THE JACOBIAN

- Volume element on V_σ: $\mathrm{d}^n x \det{(J_i^k)_0}$

$$(J_i^k)_0 \equiv \partial z^k / \partial x^i$$

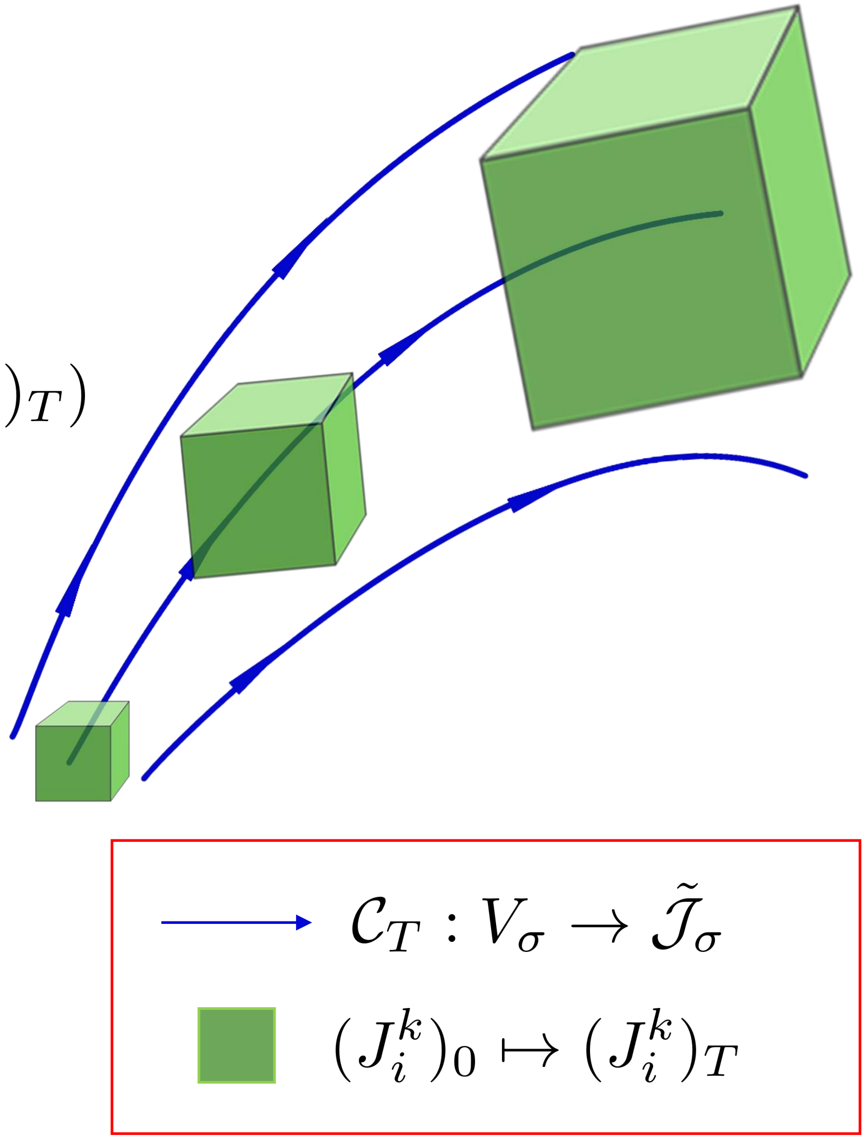
- Volume element on $\tilde{\mathcal{J}}_\sigma$: $\mathrm{d}^n z = \mathrm{d}^n x \det((J_i^k)_T)$

$$(J_i^k)_T \equiv \partial \mathcal{C}_T(x)^k / \partial x^i$$

  - Linearized SA flow:

$$\frac{\mathrm{d}\left(J_i^k\right)_t}{\mathrm{d}t} = \sum_{l=1}^{n} \overline{\frac{\partial^2 \hat{S}}{\partial z_k \partial z_l}} \, \overline{\left(J_i^l\right)_t}$$

$$(J_i^k)_0 \mapsto (J_i^k)_T$$



INTEGRATION ON THE THIMBLE

- Integration on the thimble

$$\int_{\tilde{\mathcal{J}}_\sigma} \mathrm{d}^n z\, \psi(z) = \int_{V_\sigma} \mathrm{d}^n x\ \det(J(x))\, \psi(z(x))$$

- Convert to the integral on $\tilde{\mathcal{J}}_\sigma$ to V_σ

$$\langle f \rangle \simeq \frac{\int_{\tilde{\mathcal{J}}_\alpha} \mathrm{d}^n z \hat{f}(z) \mathrm{e}^{-\hat{S}(z)}}{\int_{\tilde{\mathcal{J}}_\alpha} \mathrm{d}^n z \mathrm{e}^{-\hat{S}(z)}} = \frac{\int_{V_\sigma} \mathrm{d}^n x \det(J(x)) \hat{f}\left(\mathcal{C}_T(x)\right) \mathrm{e}^{-\hat{S}(\mathcal{C}_T(x))}}{\int_{V_\sigma} \mathrm{d}^n x \det(J(x)) \mathrm{e}^{-\hat{S}(\mathcal{C}_T(x))}}$$

$$= \frac{\left\langle \mathrm{e}^{i\theta_{res}} \hat{f} \right\rangle_{eff}}{\left\langle \mathrm{e}^{\mathrm{i}\theta_{res}} \right\rangle_{eff}}$$

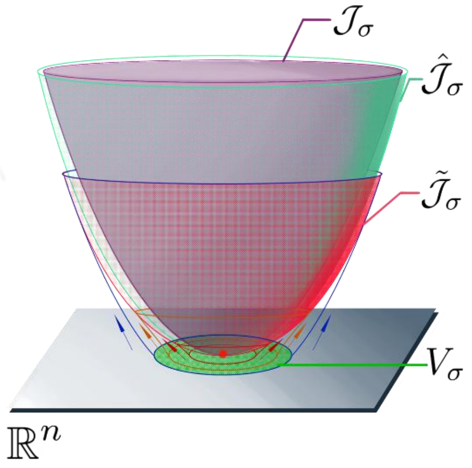
$$\mathrm{Re}(\hat{S}) - \log(\det(J)) \equiv S_{eff}$$

$$\arg(\det(J)) - \mathrm{Im}(\hat{S}) \equiv \theta_{res}$$

$$\langle \mathcal{O} \rangle_{eff} = \frac{\int_{\hat{V}_\sigma} \mathrm{d}^n x\, \mathcal{O}\, \mathrm{e}^{-S_{eff}}}{\int_{\hat{V}_\sigma} \mathrm{d}^n x\, \mathrm{e}^{-S_{eff}}}$$

[A. Alexandru, G. Basar, P. F. Bedaque 2020 2017 2016, M. Cristoforetti, F. Di Renzo, A. Mukherjee, and L. Scorzato 2014 2012, T. Kanazawa and Y. Tanizaki 2015 2016, etc]

INTEGRATE IN THE THIMBLE



- Convert to the integral on $\tilde{\mathcal{J}}_\sigma$ to V_σ

$$\langle f\rangle \simeq \frac{\int_{\tilde{\mathcal{J}}_\alpha} \mathrm{d}^n z \hat{f}(z) \mathrm{e}^{-\hat{S}(z)}}{\int_{\tilde{\mathcal{J}}_\alpha} \mathrm{d}^n z \mathrm{e}^{-\hat{S}(z)}} = \frac{\int_{V_\sigma} \mathrm{d}^n x \det(J(x)) \hat{f}\left(\mathcal{C}_T(x)\right) \mathrm{e}^{-\hat{S}(\mathcal{C}_T(x))}}{\int_{V_\sigma} \mathrm{d}^n x \det(J(x)) \mathrm{e}^{-\hat{S}(\mathcal{C}_T(x))}}$$

$$= \frac{\left\langle \mathrm{e}^{i\theta_{res}} \hat{f} \right\rangle_{eff}}{\left\langle \mathrm{e}^{\mathrm{i}\theta_{res}} \right\rangle_{eff}}$$

Sampling from the Boltzmann factor $\mathrm{e}^{-S_{eff}}$

Can be computed by Markov-chain Monte Carlo Method

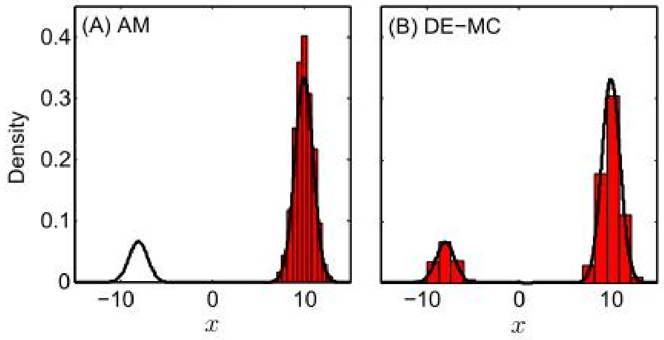
$$\mathrm{Re}(\hat{S}) - \log(\det(J)) \equiv S_{eff}$$

$$\arg(\det(J)) - \mathrm{Im}(\hat{S}) \equiv \theta_{res}$$

$$\langle \mathcal{O} \rangle_{eff} = \frac{\int_{\hat{V}_\sigma} \mathrm{d}^n x\, \mathcal{O}\, \mathrm{e}^{-S_{eff}}}{\int_{\hat{V}_\sigma} \mathrm{d}^n x\, \mathrm{e}^{-S_{eff}}}$$

DIFFERENTIAL EVOLUTION ADAPTIVE METROPOLIS ALGORITHM(DREAM)

- Advantages of DREAM Algorithm:
  - Good balance between acceptance and progression (Good for high-dimensional cases)
  - Adapt to multimodal distribution
  - Scan different region simultaneously
  - Run in parallel, adapt to multi-core computer processor



[J. Vrugt, C. ter Braak, C. Diks, B. Robinson, J. Hyman, and D. Higdon 2009]

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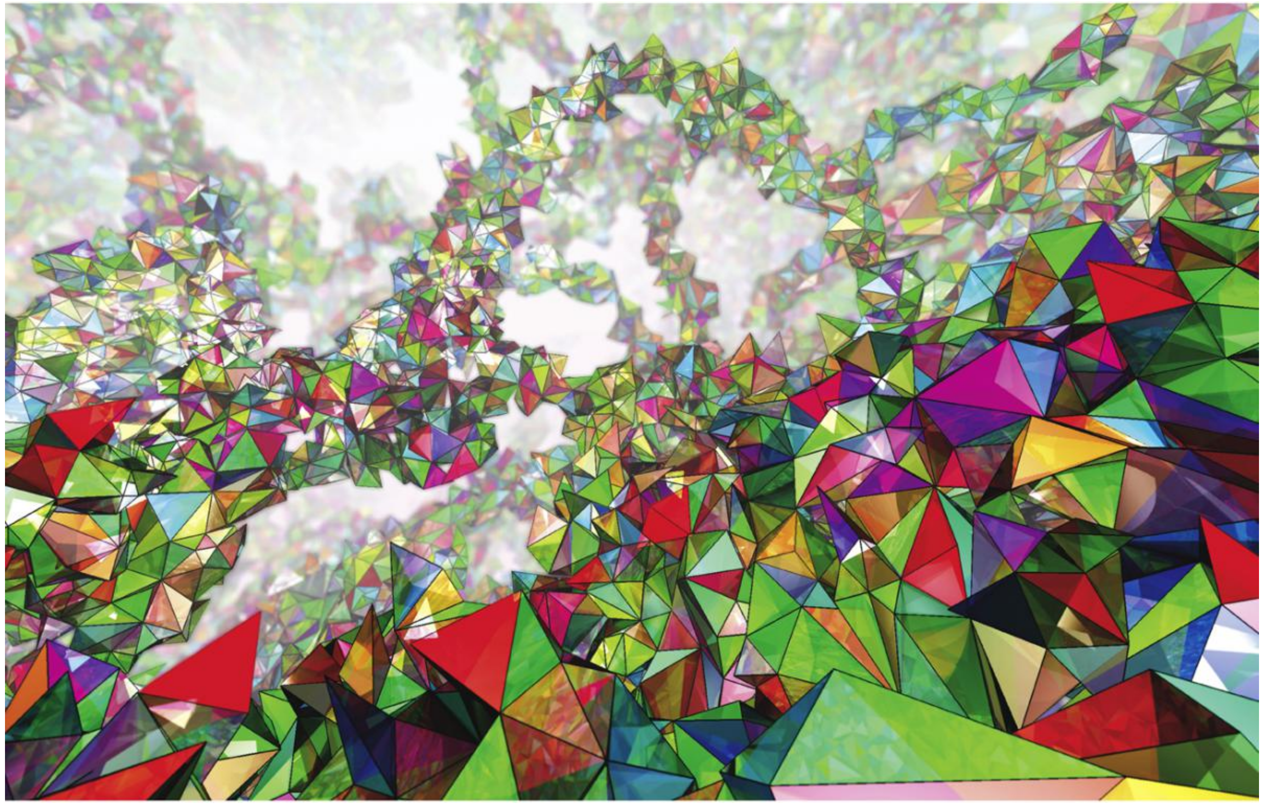
SPINFOAM PROPAGATOR

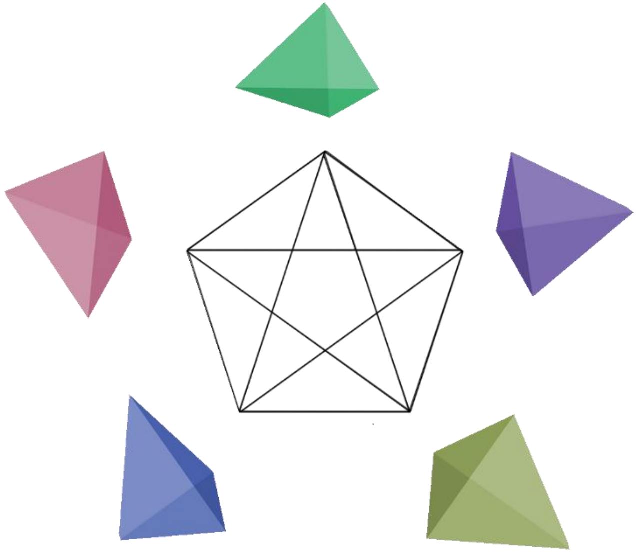
Graviton Propagator in Loop Quantum Gravity



Artist's impression of quantum space in loop quantum gravity

Picture from Thiemann's book

EPRL SPINFOAM MODEL



- 4-simplex amplitude

$$\langle W|\Psi_0\rangle = \sum_{j_{ab}} \psi_{j_0,\zeta_0} \int_{SL(2,\mathbb{C})^5} \prod_a \mathrm{d}g_a \prod_{a>b} P_{ab}(g)$$

$$P_{ab}(g) = \left\langle \lambda j_{ab}, -\vec{n}_{ab} \left| Y^\dagger g_a^{-1} g_b Y \right| \lambda j_{ab}, \vec{n}_{ba} \right\rangle$$

Y maps the spin-j $SU(2)$ irreducible representation $\mathcal{H}_j$ to the lowest level in $SL(2,\mathbb{C})$ $(j,\gamma j)$-irreducible representation $\mathcal{H}_{(j,\gamma j)} = \oplus_{k=j}^{\infty} \mathcal{H}_k$

We take Barbero-Immirzi parameter γ as 0.1.

[J.Engle, E.Livine, R.Pereira, C.Roveill 2008, L.Freidel K.Karsnov 2008]

EPRL SPINFOAM MODEL

- 4-simplex amplitude

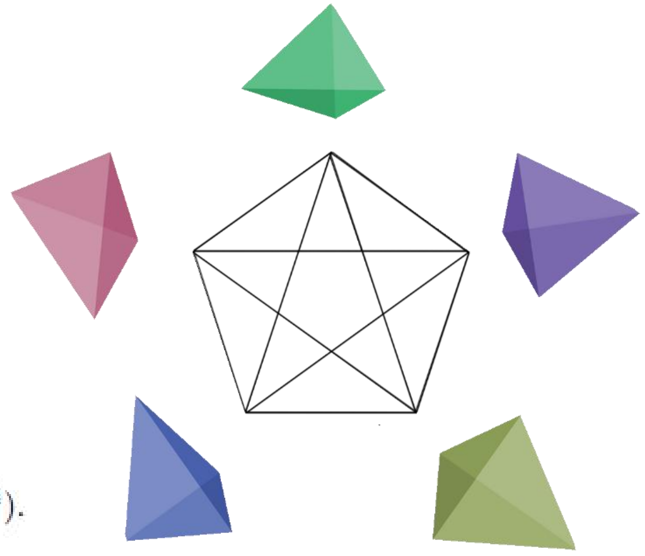
$$\langle W \mid \Psi_0 \rangle = \sum_{\lambda j_{ab}} \psi_{\lambda j_0, \zeta_0} \int_{SL(2,\mathbb{C})^s} \prod_a \mathrm{d}g_a \int \left(\prod_{a>b} \frac{d_{\lambda j_{ab}}}{\pi} \mathrm{d}\tilde{\mathbf{z}}_{ab} \right) \mathrm{e}^{\lambda S}$$

$$S(j,g,\mathbf{z}) = \sum_{a>b} \left[2j_{ab} \log \left(\langle J\xi_{ab}, Z_{ab} \rangle \langle Z_{ba}, \xi_{ba} \rangle \right) - (1+\mathrm{i}\gamma) j_{ab} \log \langle Z_{ab}, Z_{ab} \rangle - (1-\mathrm{i}\gamma) j_{ab} \log \langle Z_{ba}, Z_{ba} \rangle \right]$$

$$d_j = 2j+1,\ Z_{ab} = g_a^\dagger z_{ab},\ \text{and}\ Z_{ba} = g_b^\dagger z_{ab}$$

$\xi_{ab}, \psi_{\lambda j_0, \zeta_0}, j_a$ given by the boundary state

EPRL SPINFOAM MODEL



- Boundary Geometry

$$P_1=(0,0,0,0),\ P_2=(0,0,0,-2\sqrt{5}/3^{1/4}),\ P_3=(0,0,-3^{1/4}\sqrt{5},-3^{1/4}\sqrt{5}),$$
$$P_4=(0,-2\sqrt{10}/3^{3/4},-\sqrt{5}/3^{3/4},-\sqrt{5}/3^{1/4}),\ P_5=(-3^{-1/4}10^{-1/2},-\sqrt{5/2}/3^{3/4},-\sqrt{5}/3^{3/4},-\sqrt{5}/3^{1/4}).$$

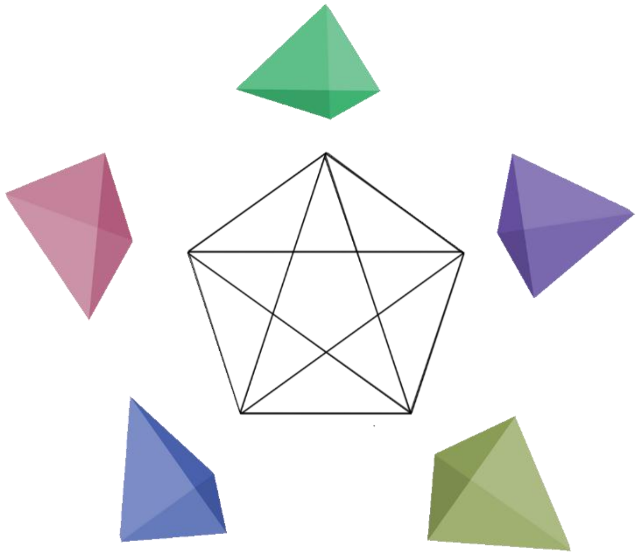
area j_{0ab} (a \ b)	2	3	4	5
1	5	5	5	5
2	⟍	2	2	2
3	⟍	⟍	2	2
4	⟍	⟍	⟍	2

normal $\vec{n}_{ab}$ (a \ b)	1	2	3	4	5
1	⟍	(1,0,0)	(-0.33,0.94,0)	(-0.33,-0.47,0.82)	(-0.33,-0.47,-0.82)
2	(-1,0,0)	⟍	(0.83,0.55,0)	(0.83,-0.28,0.48)	(0.83,-0.28,-0.48)
3	(0.33,-0.94,0)	(0.24,0.97,0)	⟍	(-0.54,0.69,0.48)	(-0.54,0.69,-0.48)
4	(0.33,0.47,-0.82)	(0.24,-0.48,0.84)	(-0.54,0.068,0.84)	⟍	(-0.54,-0.76,0.36)
5	(0.33,0.47,0.82)	(0.24,-0.48,-0.84)	(-0.54,0.068,-0.84)	(-0.54,-0.76,-0.36)	⟍

$$N_1=(-1,0,0,0),\ N_2=\left(\frac{5}{\sqrt{22}},\sqrt{\frac{3}{22}},0,0\right),\ N_3=\left(\frac{5}{\sqrt{22}},-\frac{1}{\sqrt{66}},\frac{2}{\sqrt{33}},0\right),$$
$$N_4=\left(\frac{5}{\sqrt{22}},-\frac{1}{\sqrt{66}},-\frac{1}{\sqrt{33}},\frac{1}{\sqrt{11}}\right),\ N_5=\left(\frac{5}{\sqrt{22}},-\frac{1}{\sqrt{66}},-\frac{1}{\sqrt{33}},-\frac{1}{\sqrt{11}}\right)$$

[P.Dona, S.Speziale, S.Giorgio, M.Fanizza (2019)]

EPRL SPINFOAM MODEL



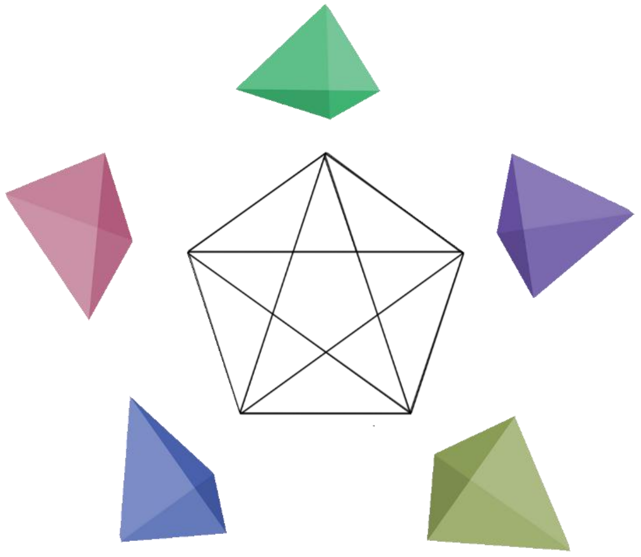
- Boundary Condition

$$|\Psi_0\rangle = \sum_{\lambda j_{ab}} \psi_{\lambda j_0, \zeta_0} ||\lambda j_{ab}, \vec{n}_{ab}\rangle$$

$$\psi_{\lambda j_0, \zeta_0} = \exp\left(-i \sum_{ab} \zeta_0^{ab}(\lambda j_{ab} - \lambda j_{0ab})\right) \exp\left(-\sum_{ab,cd} \alpha^{(ab)(cd)} \frac{\lambda j_{ab} - \lambda j_{0ab}}{\sqrt{\lambda j_{0ab}}} \frac{\lambda j_{cd} - \lambda j_{0cd}}{\sqrt{\lambda j_{0cd}}}\right)$$

ζ_0^{ab} \ b / a	2	3	4	5
1	-3.14+0.36γ	0.68+0.36γ	5.05+0.36γ	5.05+0.36γ
2	\	5.05-0.59γ	-5.93-0.59γ	-3.20-0.59γ
3	\	\	-2.81-0.59γ	-5.54-0.59γ
4	\	\	\	-4.37-0.59γ

EPRL SPINFOAM MODEL



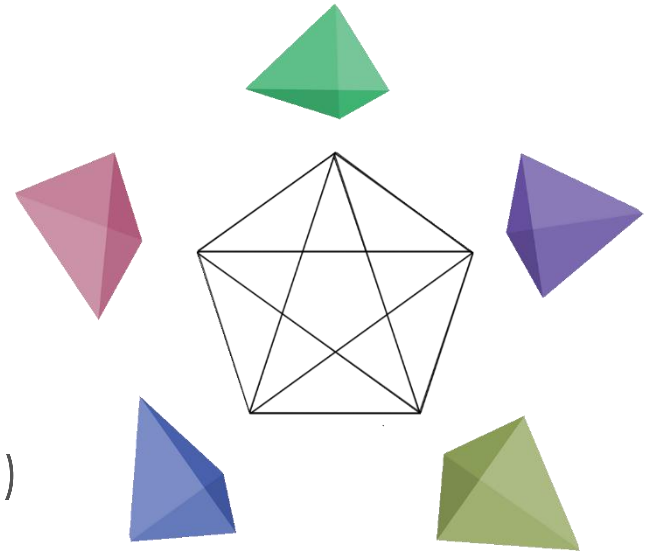
- Boundary Condition

$$|\Psi_0\rangle = \sum_{\lambda j_{ab}} \psi_{\lambda j_0,\zeta_0} ||\lambda j_{ab}, \vec{n}_{ab}\rangle$$

$$\psi_{\lambda j_0,\zeta_0} = \exp\left(-i\sum_{ab}\zeta_0^{ab}(\lambda j_{ab} - \lambda {j_0}_{ab})\right)\exp\left(-\sum_{ab,cd}\alpha^{(ab)(cd)}\frac{\lambda j_{ab} - \lambda {j_0}_{ab}}{\sqrt{\lambda {j_0}_{ab}}}\frac{\lambda j_{cd} - \lambda {j_0}_{cd}}{\sqrt{\lambda {j_0}_{cd}}}\right)$$

In order to get a right propagator limit, α has to take specific value. But, since we just want to justify the reliability of the algorithm, we can randomly choose the α and compare the results with the one gotten from asymptotic expansion based on the same α

CRITICAL POINT



- Semi-classical expansion $= 1/\lambda$ expansion

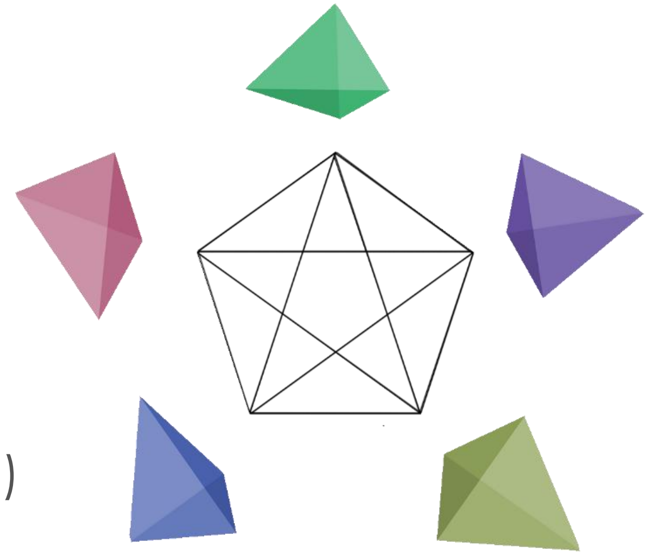
  (stationary phase approximation)

- In our calculation, there is only one critical point corresponding to the geometry of the 4-simplex

a	1	2	3	4	5
g_{0a}	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.18i & 1.01i \\ 1.01i & 0.18i \end{pmatrix}$	$\begin{pmatrix} 0.18i & 0.96-0.34i \\ -0.96-0.34i & 0.18i \end{pmatrix}$	$\begin{pmatrix} 1.01i & -0.48-0.34i \\ 0.48-0.34i & -0.65i \end{pmatrix}$	$\begin{pmatrix} -0.65i & -0.48-0.34i \\ 0.48-0.34i & 1.01i \end{pmatrix}$

$\vert z_{0ab}\rangle$ b / a	1	2	3	4	5
1	\	(1,1)	(1,-0.333+0.942i)	(1,-0.184-0.259i)	(1,-1.817-2.569 i)
2	(1,1)	\	(1,0.685-0.729i)	(1, 1.857 + 0.989 i)	(1, 0.420 + 0.223 i)
3	(1, 0.333 - 0.943 i)	(1, 0.685 - 0.729 i)	\	(1, 0.313 + 2.080 i)	(1, 0.071 + 0.470 i)
4	(1, -0.184-0.259i)	(1, 1.857 + 0.989 i)	(1,0.313 + 2.080 i)	\	(1, 0.058+0.082i)
5	(1, -1.817-2.569 i)	(1, 0.420 + 0.223 i)	(1,0.071 + 0.470 i)	(1, 0.058+0.082i)	\

CRITICAL POINT



- Semi-classical expansion $= 1/\lambda$ expansion

  (stationary phase approximation)

- In our calculation, there is only one critical point corresponding to the geometry of the 4-simplex

$$\frac{\partial S}{\partial g}[g_0, z_0] = 0 \qquad \frac{\partial S}{\partial z}[g_0, z_0] = 0 \qquad Re(S[g_0, z_0]) = 0$$

$$\frac{\partial S}{\partial j_{ab}}[g_0, z_0] = \lambda \mathrm{i} \zeta_{ab}$$

$$\frac{\partial S_{tot}}{\partial j_{ab}}[j_0, g_0, z_0] = 0$$

SPINFOAM PROPAGATOR

- 2-point correlation function:

$$G_{mn}^{abcd} = \frac{\langle W|E_n^a \cdot E_n^b E_m^c \cdot E_m^d|\Psi_0\rangle}{\langle W|\Psi_0\rangle} - \frac{\langle W|E_n^a \cdot E_n^b|\Psi_0\rangle}{\langle W|\Psi_0\rangle}\frac{\langle W|E_m^c \cdot E_m^d|\Psi_0\rangle}{\langle W|\Psi_0\rangle}$$

Penrose metric:

$$q^{ab}(x) = \delta^{ij} E_i^a(x) E_j^b(x)$$

$$\left\langle \lambda j_{ab}, -\vec{n}_{ab} \left| Y^\dagger g_a^{-1} g_b Y \left(E_b^a\right)^i \right| \lambda j_{ab}, \vec{n}_{ba} \right\rangle$$
$$= \left\langle \lambda j_{ab}, -\vec{n}_{ab} \left| Y^\dagger g_a^{-1} g_b Y \right| \lambda j_{ab}, \vec{n}_{ba} \right\rangle \lambda j_{ab} \gamma \frac{\langle \sigma_i Z_{ba}, \xi_{ba}\rangle}{\langle Z_{ba}, \xi_{ba}\rangle}$$

$$\left\langle \lambda j_{ab}, -\vec{n}_{ab} \left| \left(E_b^a\right)^{i\dagger} Y^\dagger g_a^{-1} g_b Y \right| \lambda j_{ab}, \vec{n}_{ba} \right\rangle$$
$$= \left\langle \lambda j_{ab}, -\vec{n}_{ab} \left| Y^\dagger g_a^{-1} g_b Y \right| \lambda j_{ab}, \vec{n}_{ba} \right\rangle (-\lambda j_{ab} \gamma) \frac{\langle J\xi_{ab}, \sigma_i Z_{ab}\rangle}{\langle J\xi_{ab}, Z_{ab}\rangle}$$

[E. Bianchi, Y. Ding, etc]

SPINFOAM PROPAGATOR

- 2-point correlation function:

$$\left\langle W \left| E_n^a \cdot E_n^b E_m^c \cdot E_m^d \right| \Psi_0 \right\rangle = \sum_{j_{ab}} \psi_{\lambda j_0, \zeta_0} \int \mathrm{d}\phi U(j,\phi) \left[A_{an}(j,\phi) \cdot A_{bn}(j,\phi) \right] \left[A_{cm}(j,\phi) \cdot A_{dm}(j,\phi) \right] \mathrm{e}^{\lambda S(j,\phi)}$$

$$\left\langle W \left| E_n^a \cdot E_n^b \right| \Psi_0 \right\rangle = \sum_{j_{ab}} \psi_{\lambda j_0, \zeta_0} \int \mathrm{d}\phi U(j,\phi) A_{an}(j,\phi) \cdot A_{bn}(j,\phi) \mathrm{e}^{\lambda S(j,\phi)}$$

$$\left\langle W \left| E_m^c \cdot E_m^d \right| \Psi_0 \right\rangle = \sum_{j_{ab}} \psi_{\lambda j_0, \zeta_0} \int \mathrm{d}\phi U(j,\phi) A_{cm}(j,\phi) \cdot A_{dm}(j,\phi) \mathrm{e}^{\lambda S(j,\phi)}$$

$$A_{ab}^i = \gamma \lambda j_{ab} \frac{\langle \sigma^i Z_{ba}, \xi_{ba} \rangle}{\langle Z_{ba}, \xi_{ba} \rangle}, \quad A_{ba}^i = -\gamma \lambda j_{ab} \frac{\langle J\xi_{ba}, \sigma_i Z_{ba} \rangle}{\langle J\xi_{ba}, Z_{ba} \rangle},$$

INTEGRAL OF J

- Poisson re-summation:

$$\sum_{J\in\frac{\mathbb{Z}^+}{2}\cup 0} f(J) = \frac{1}{2}\sum_{J\in\mathbb{Z}} f(|J/2|) + \frac{1}{2}f(0) = 2\sum_{k\in\mathbb{Z}}\int_0^\infty dJ f(J)\mathrm{e}^{4\pi \mathrm{i}kJ} + \frac{1}{2}f(0)$$

Remind: $\psi_{\lambda j_0,\zeta_0} = \exp\left(-i\sum_{ab}\zeta_0^{ab}(\lambda j_{ab} - \lambda j_{0ab})\right)\exp\left(-\sum_{ab,cd}\alpha^{(ab)(cd)}\frac{\lambda j_{ab} - \lambda j_{0ab}}{\sqrt{\lambda j_{0ab}}}\frac{\lambda j_{cd} - \lambda j_{0cd}}{\sqrt{\lambda j_{0cd}}}\right)$

The term $\frac{1}{2}f(0)$ is negligible when λ is large

INTEGRAL OF J

- Poisson resummation:

$$\langle W \mid \Psi_0\rangle = (2\lambda)^{10} \sum_{\{k_{ab}\}\in\mathbb{Z}^{10}} \int_0^\infty \mathrm{d}^{10}j \int \mathrm{d}\phi U \mathrm{e}^{-\lambda S_{tot}^{(k)}}$$

$$\left\langle W \left| E_n^a \cdot E_n^b E_m^c \cdot E_m^d \right| \Psi_0\right\rangle = (2\lambda)^{10} \sum_{\{k_{ab}\}\in\mathbb{Z}^{10}} \int_0^\infty \mathrm{d}^{10}j \int \mathrm{d}\phi U \mathrm{e}^{-\lambda S_{tot}^{(k)}} \left(A_{an} \cdot A_{bn}\right)\left(A_{cm} \cdot A_{dm}\right)$$

$$\left\langle W \left| E_n^a \cdot E_n^b \right| \Psi_0\right\rangle = (2\lambda)^{10} \sum_{\{k_{ab}\}\in\mathbb{Z}^{10}} \int_0^\infty \mathrm{d}^{10}j \int \mathrm{d}\phi U \mathrm{e}^{-\lambda S_{tot}^{(k)}} \left(A_{an} \cdot A_{bn}\right)$$

$$\left\langle W \left| E_m^c \cdot E_m^d \right| \Psi_0\right\rangle = (2\lambda)^{10} \sum_{\{k_{ab}\}\in\mathbb{Z}^{10}} \int_0^\infty \mathrm{d}^{10}j \int \mathrm{d}\phi U \mathrm{e}^{-\lambda S_{tot}^{(k)}} \left(A_{cm} \cdot A_{dm}\right)$$

with

$$S_{tot}^{(k)} = S_{tot} + 4\pi\mathrm{i} \sum_{a>b} j_{ab} k_{ab}$$

$$S_{tot} = \mathrm{i} \sum_{ab} \zeta_0^{ab}\left(j_{ab} - j_{0b}\right) + \sum_{ab,cd} \alpha^{(ab)(cd)} \frac{j_{ab} - j_{0ab}}{\sqrt{j_{0ab}}} \frac{j_{cd} - j_0 cd}{\sqrt{j_{0cd}}} - S(j,\phi),$$

INTEGRAL OF J

- The integral around the critical point dominate the whole integral.
- By our choice of ζ_{ab} , critical point can only be found when $k_{ab} = 0$
- The $k_{ab} \neq 0$ terms in the summation are exponentially suppressed when λ is large.

INTEGRAL OF J

$$\langle W \mid \Psi_0\rangle \simeq \int_{-\infty}^{\infty} \mathrm{d}^{10} j \int \mathrm{d}\phi \tilde{U} \mathrm{e}^{-\lambda S_{tot}}$$

$$\left\langle W \left| E_n^a \cdot E_n^b E_m^c \cdot E_m^d \right| \Psi_0\right\rangle \simeq \int_{-\infty}^{\infty} \mathrm{d}^{10} j \int \mathrm{d}\phi \tilde{U} \mathrm{e}^{-\lambda S_{tot}} \left(A_{an} \cdot A_{bn}\right)\left(A_{cm} \cdot A_{dm}\right)$$

$$\left\langle W \left| E_n^a \cdot E_n^b \right| \Psi_0\right\rangle \simeq \int_{-\infty}^{\infty} \mathrm{d}^{10} j \int \mathrm{d}\phi \tilde{U} \mathrm{e}^{-\lambda S_{tot}} \left(A_{an} \cdot A_{bn}\right)$$

$$\left\langle W \left| E_m^c \cdot E_m^d \right| \Psi_0\right\rangle \simeq \int_{-\infty}^{\infty} \mathrm{d}^{10} j \int \mathrm{d}\phi \tilde{U} \mathrm{e}^{-\lambda S_{tot}} \left(A_{cm} \cdot A_{dm}\right)$$

$$S_{tot} = \mathrm{i} \sum_{ab} \zeta_0^{ab} \left(j_{ab} - j_{0ab}\right) + \sum_{ab,cd} \alpha^{(ab)(cd)} \frac{j_{ab} - j_{0ab}}{\sqrt{j_{0ab}}} \frac{j_{cd} - j_{0cd}}{\sqrt{j_{0cd}}} - S(j, \phi),$$

Our algorithm can be applied

EPRL-SPINFOAM MODEL

- The action depends on 54 real variables
- The action is complex-valued
- The observables we want to compute are

$$\frac{\langle W|E_n^a \cdot E_n^b E_m^c \cdot E_m^d|\Psi_0\rangle}{\langle W|\Psi_0\rangle}, \quad \frac{\langle W|E_n^a \cdot E_n^b|\Psi_0\rangle}{\langle W|\Psi_0\rangle}, \quad \frac{\langle W|E_m^c \cdot E_m^d|\Psi_0\rangle}{\langle W|\Psi_0\rangle}$$

and

$$G_{mn}^{abcd}$$

PERTURBATIVE AND NON-PERTURBATIVE CORRECTIONS

- Analytical continuation leads to more critical points.
- These critical points provide exponentially suppressed corrections to the integral at large λ .
- Only compute the observables on the thimble attached to dominant critical point.
- The computation contains the perturbative $1/\lambda$ corrections to all orders on the dominant thimble and neglects all the exponentially suppressed corrections

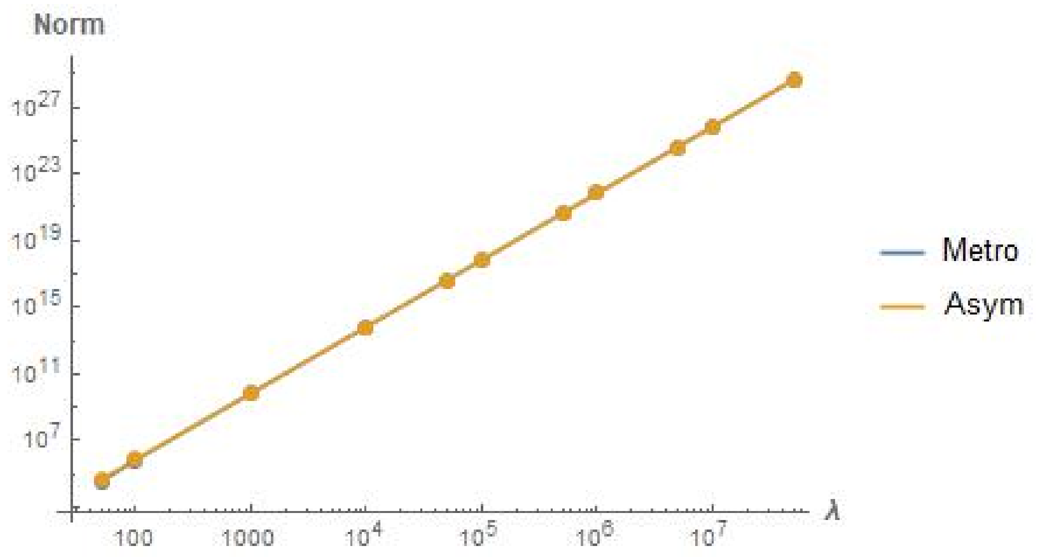
Suppressed corrections are caused by 1) other thimbles, 2) $k_{ab} \neq 0$ terms, 3) $f(0)$ term, 4) the integral $\int_{-\infty}^{0} dj$

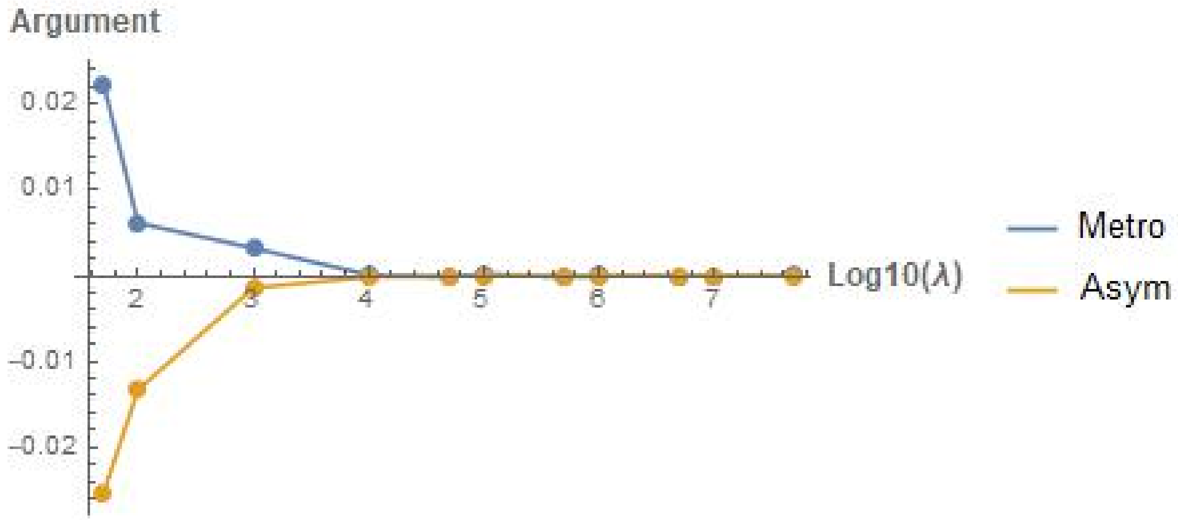
OPTIMIZATIONS

- Numerical solver of SA equation (stiffness problem)
  - Time rescaling: $t \to \frac{t}{\lambda}$
  - Tolerance of the error
- DREAM optimization (sampling on thimble)
  - Initial point optimization (same energy scheme)
  - Flow time optimization (average among convergent results)
  - Burn-in stage (adaptively adjust CR and Beta)

SPINFOAM PROPAGATOR

- Results of Expectation values $\langle E_1^2 \cdot E_1^3 E_4^1 \cdot E_4^5 \rangle$





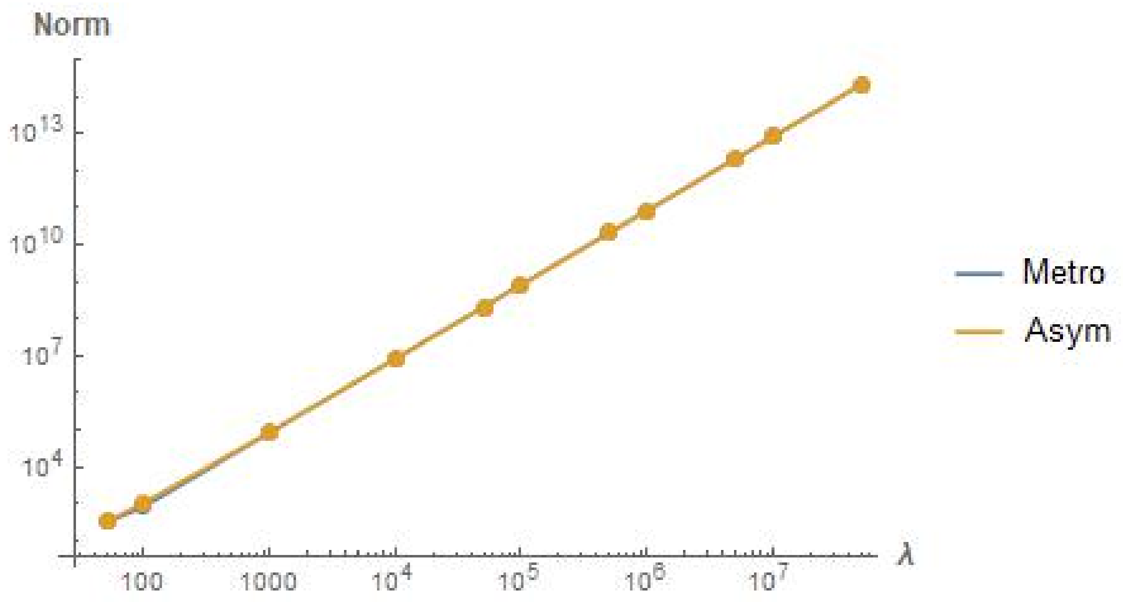
Leading order in $1/\lambda$ expansion

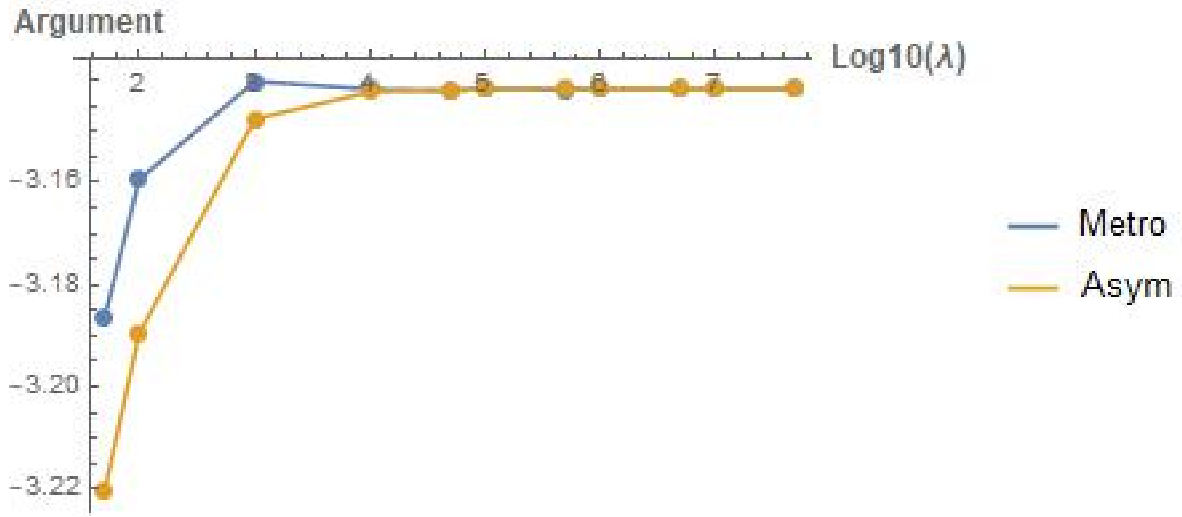
Thimble + Metropolis

λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	8.71	0.79	0.12	0.052	0.036	0.017	0.0062	0.0018	0.00037	0.00069

SPINFOAM PROPAGATOR

- Results of Expectation values $\langle E_1^2 \cdot E_1^3 \rangle$

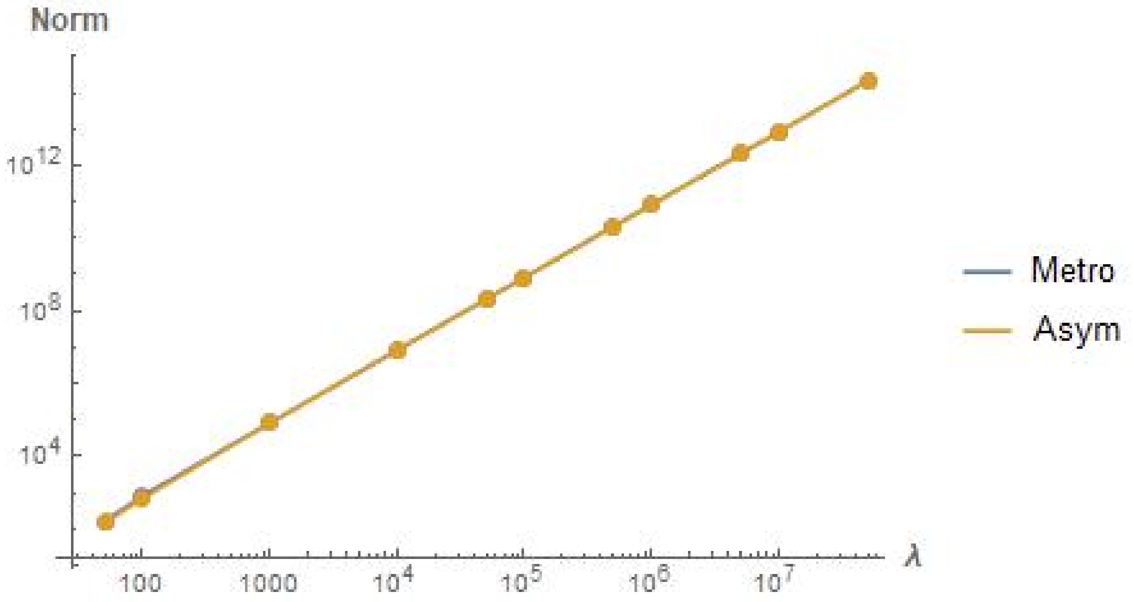


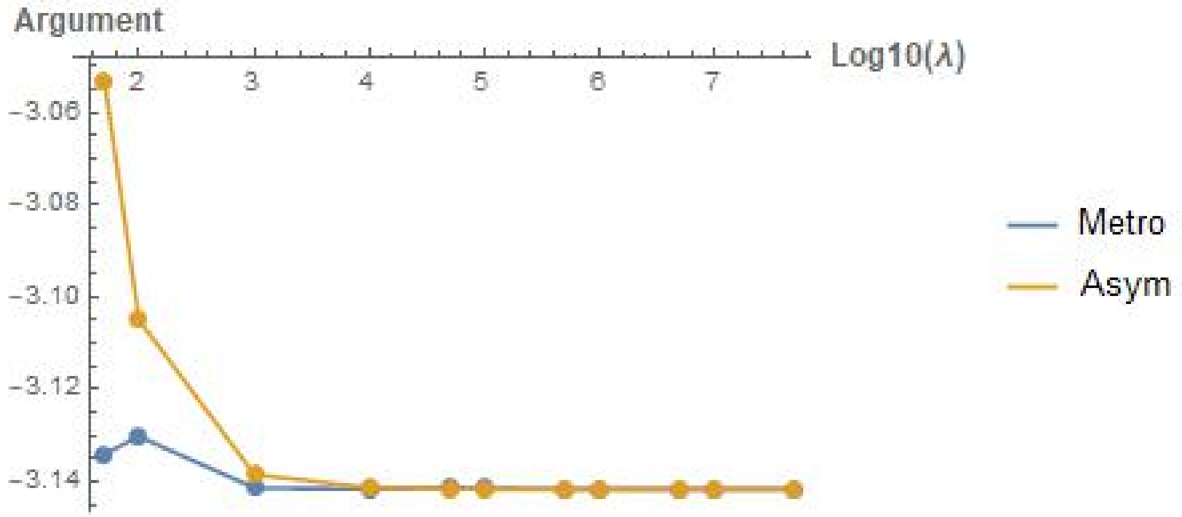


λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	22.32	2.00	0.31	0.078	0.022	0.016	0.012	0.0022	0.000047	0.0016

SPINFOAM PROPAGATOR

- Results of Expectation values $\langle E_4^1 \cdot E_4^5 \rangle$

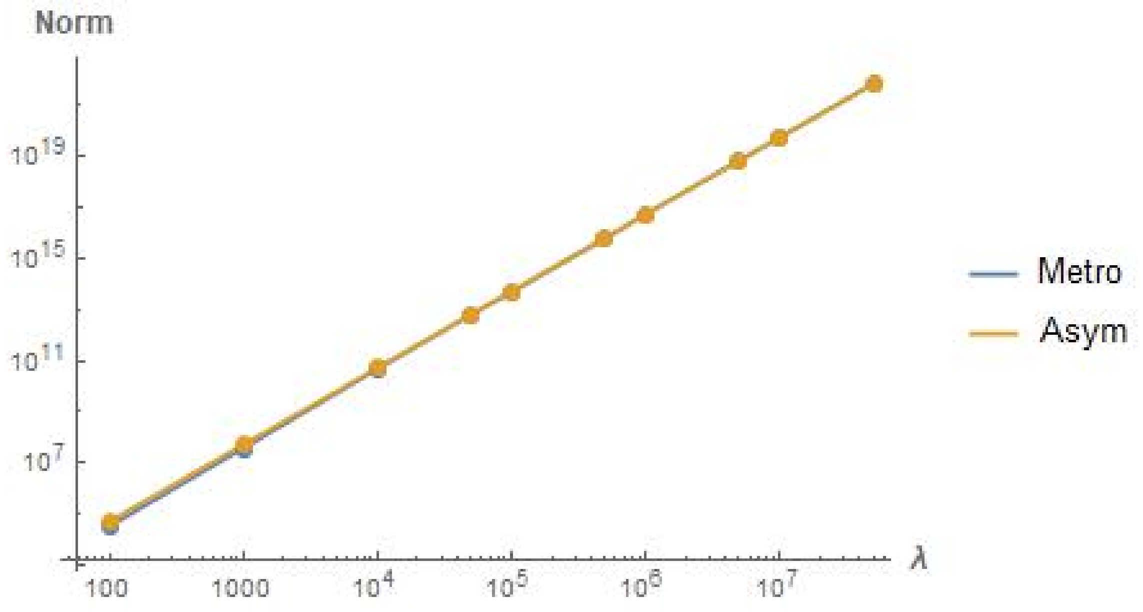


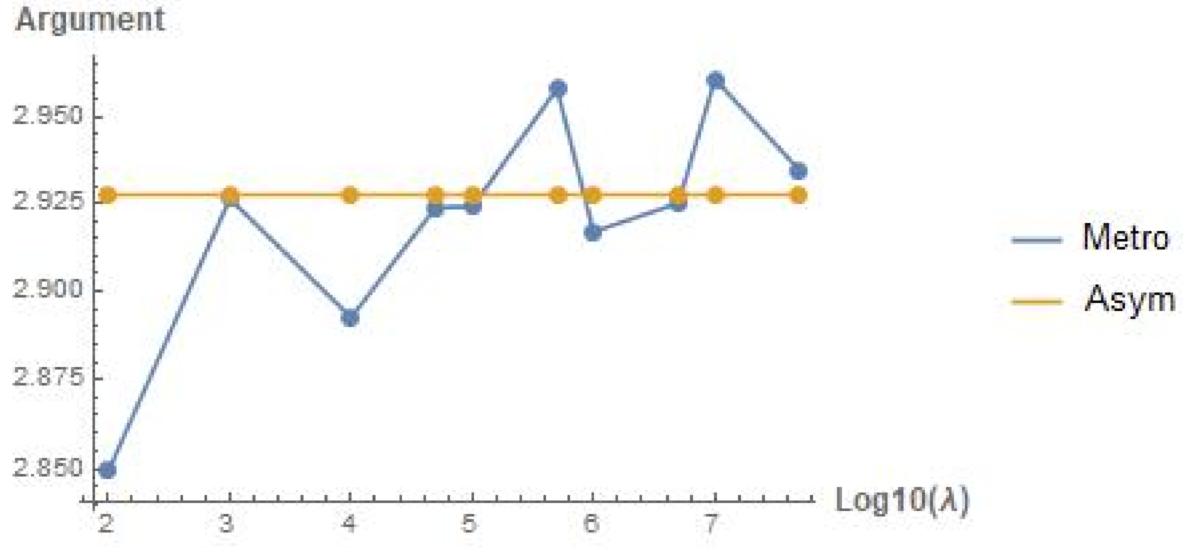


λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	18.66	1.18	0.18	0.026	0.017	0.00054	0.0037	0.00035	0.00036	0.00083

SPINFOAM PROPAGATOR

- Results of propagator component G_{14}^{2315}

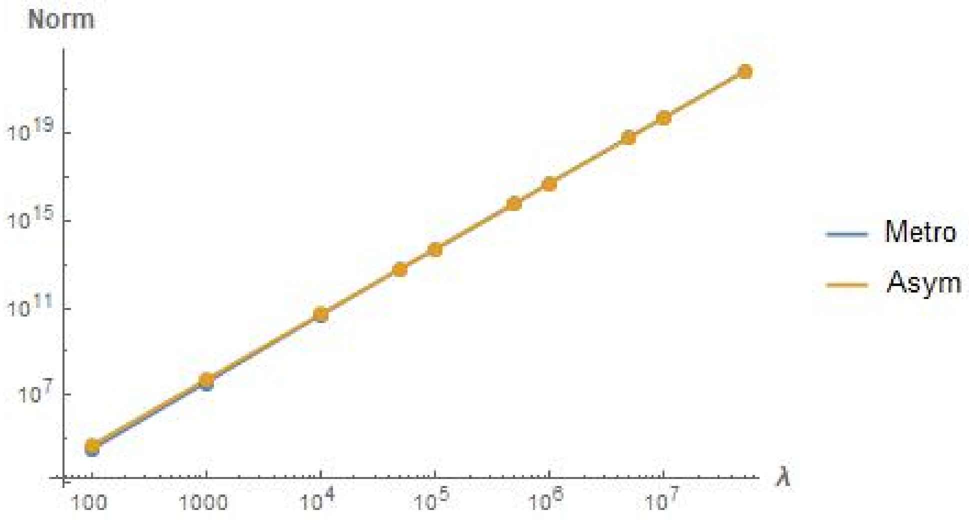


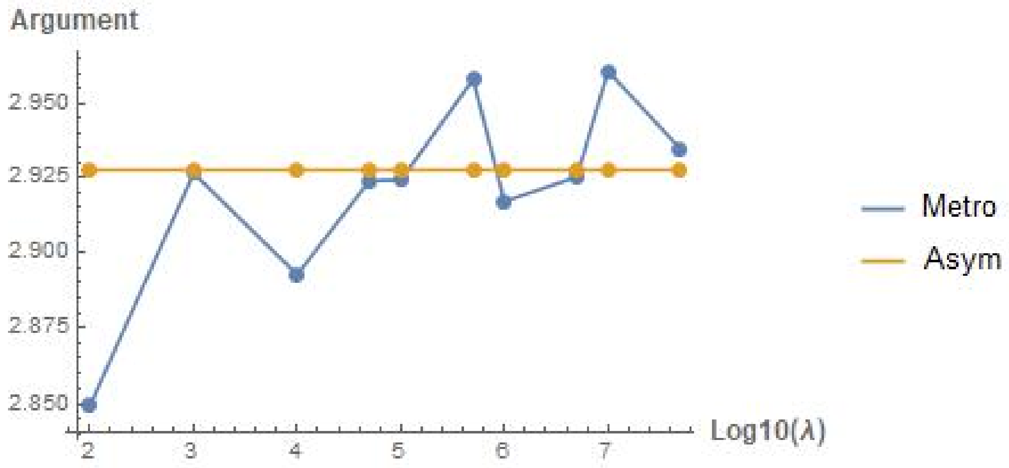


λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	37.90	27.00	13.22	2.76	10.09	8.86	1.89	1.13	3.90	2.06

SPINFOAM PROPAGATOR

- Results of propagator component G_{14}^{2315}





λ	10^2	10^3	10^4	5×10^4	10^5	5×10^5	10^6	5×10^6	10^7	5×10^7
Difference (%)	37.90	27.00	13.22	2.76	10.09	8.86	1.89	1.13	3.90	2.06

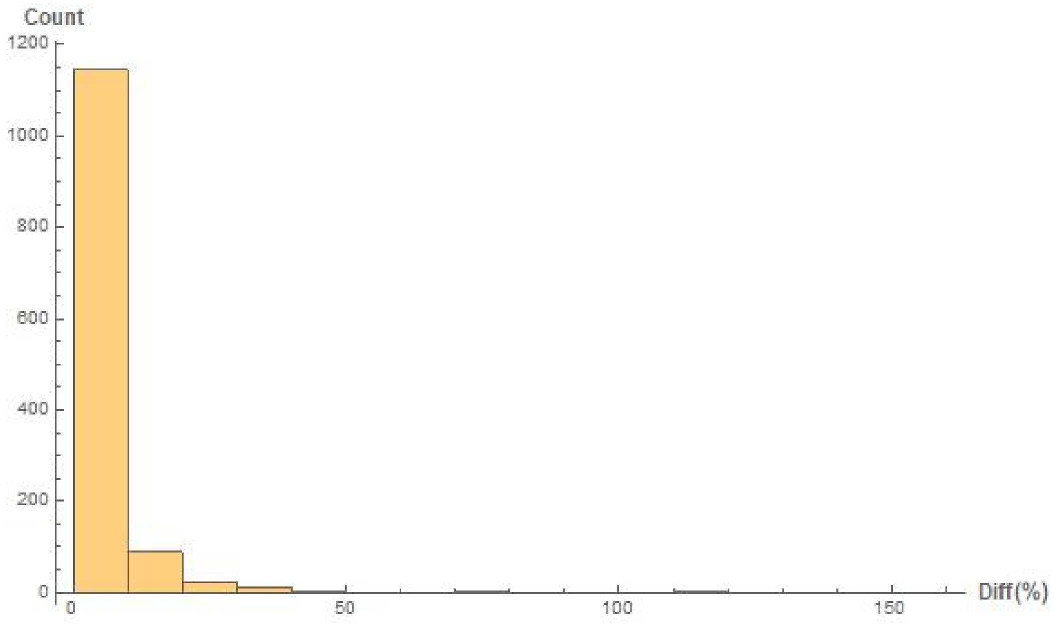
$$G^{abcd}(x,y) = \langle q^{ab}(x)q^{cd}(y)\rangle - \langle q^{ab}(x)\rangle \langle q^{cd}(y)\rangle$$

Propagator is of high order of $1/\lambda$

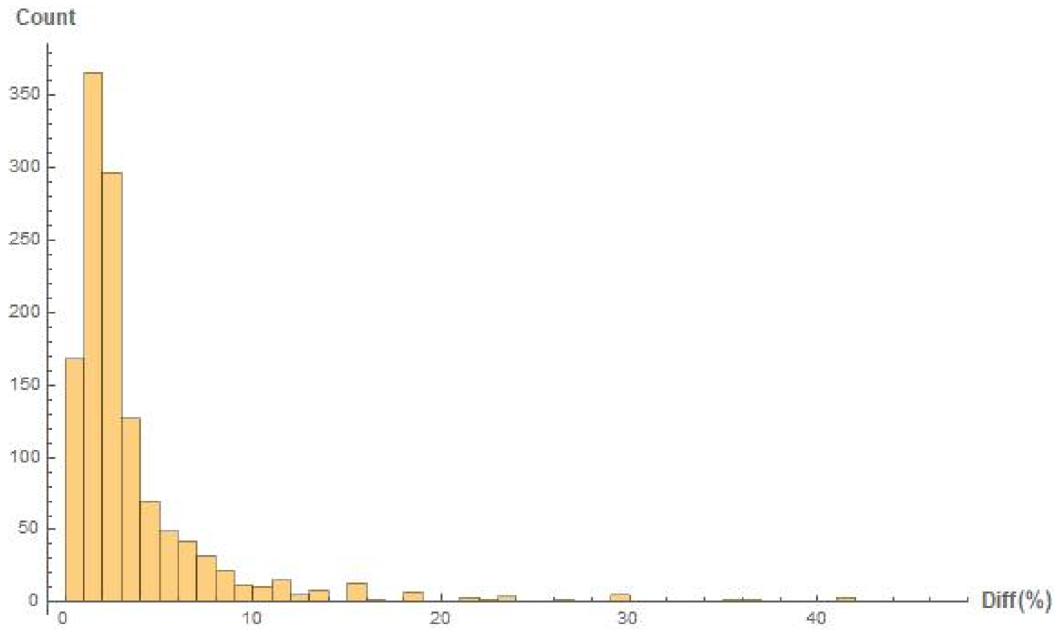
SPINFOAM PROPAGATOR

- The propagator has 1275 components

Histogram of percentage difference to leading order in $1/\lambda$expansion



$\lambda = 10^6$



$\lambda = 10^7$

OUTLOOK

Future Plans

OUTLOOK

- Spin foam propagators in more complicated cases (Pachner 1-5 move)
- Computing other operators (flatness problem)
- Improvement of the algorithm (hybrid monte-carlo, etc)
- Use SD flow and Metropolis algorithm to find the critical point

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Thanks!