When Non-Expanding Horizons met Near Horizon Geometries

Jerzy Lewandowski, Adam Szereszewski, Piotr Waluk
Uniwersytet Warszawski

A long time ago in a galaxy far, far away….

A peculiar solution to Einstein's Equations was constructed

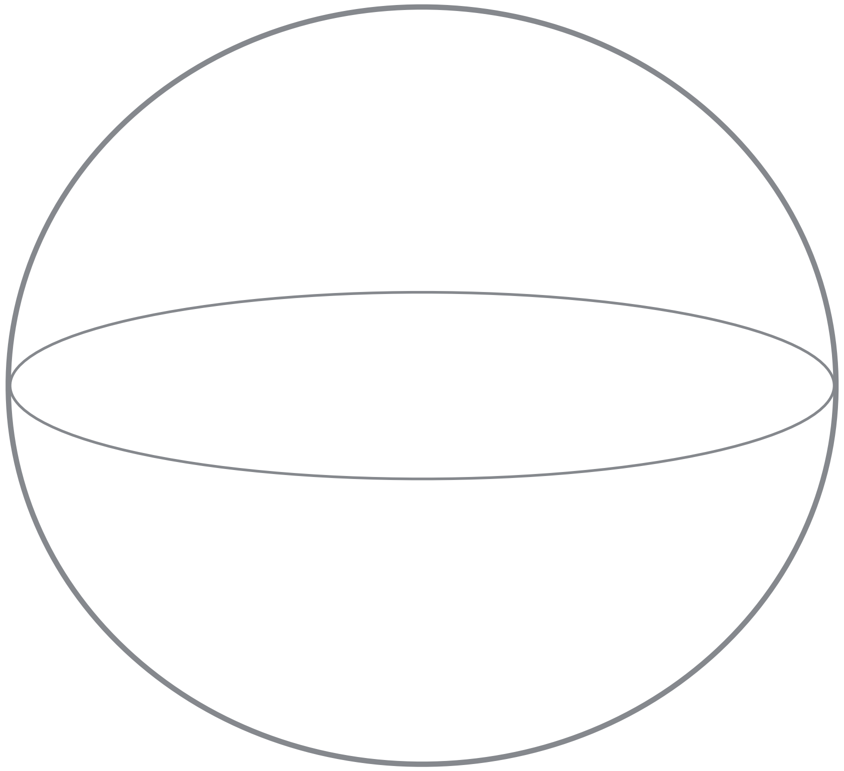
Pawlowski, Lewandowski, Jezierski - 2003

Spacetime foliated by the Killing horizons …

Ingredients

$$S^2, \quad g_{AB}, \quad \omega_A$$

a 2-sphere,
a metric tensor, a 1-form



Such that :

$$^2\nabla_A \omega_B \ + \ ^2\nabla_B \omega_A \ + \ 2\omega_A \omega_B \ - \ ^2R_{AB} \ = \ 0$$

Ashtekar, Beetle, JL 2001; JL, Pawłowski 2002;
Chruściel, Reall, Tod 2005; Jezierski 2008, Jezierski, Kamiński 2012

$$S^2, \quad g_{AB}, \quad \omega_A$$

$$^2\nabla_A \omega_B \; + \; ^2\nabla_B \omega_A \; + \; 2\omega_A \omega_B \; - \; ^2R_{AB} \; = \; 0$$

The solution

On: $S^2 \times \mathbb{R} \times \mathbb{R}$

parametrized by: (x^A, u, v)

$$g_{\mu\nu} dx^\mu dx^\nu \quad :=$$

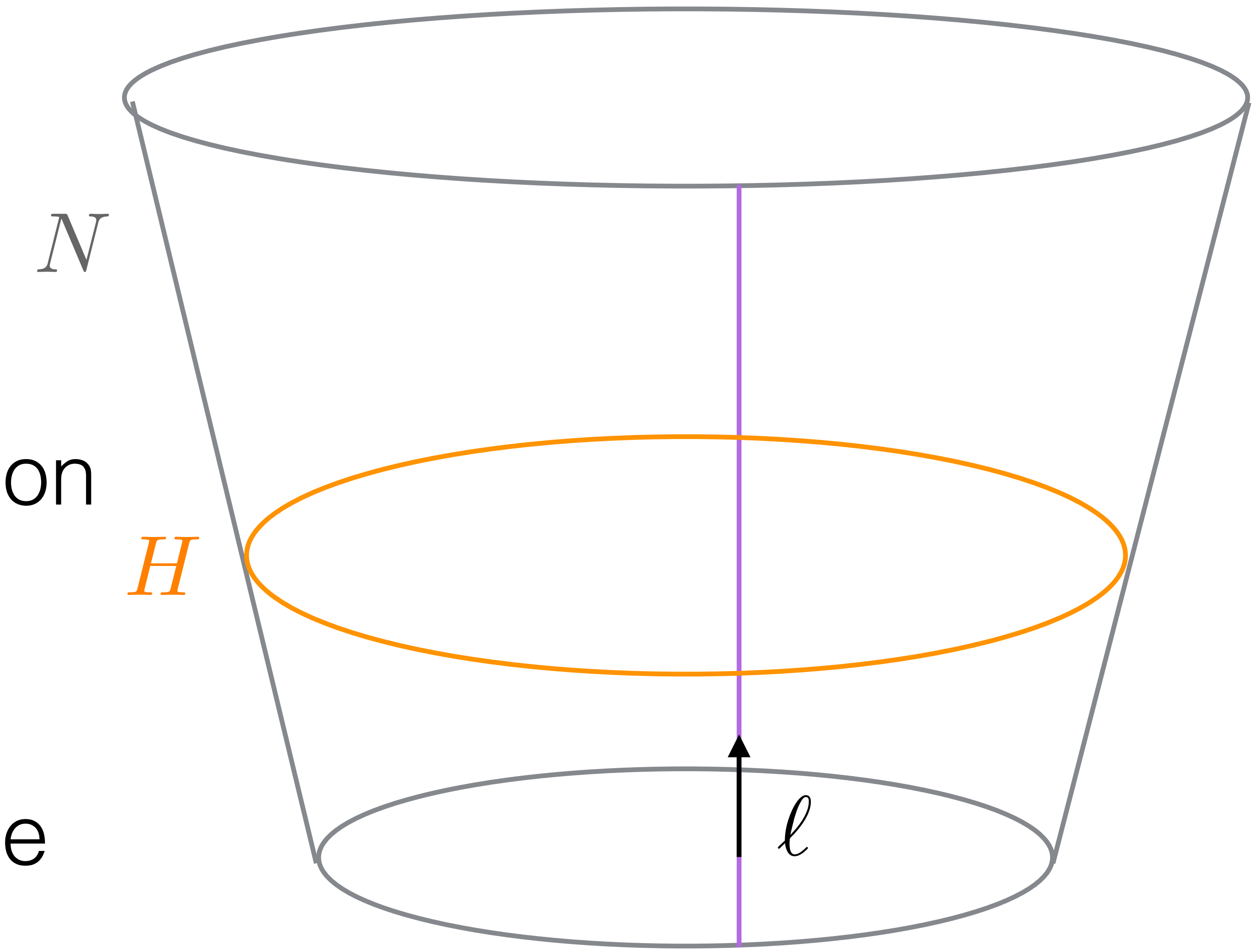
$$g_{AB} dx^A dx^B \; - \; 2du \left(dv - 2v\omega - \frac{1}{2} v^2 (\mathrm{div}\omega + 2\omega^2) du \right)$$

$$R_{\mu\nu} \; = \; 0$$

The present day ….

Non-expanding horizons: definition

- a codimension 1
- null surface
- non-expanding
- compact space-like section



Assumptions about spacetime

$$M, g_{\mu\nu} \qquad - + \ldots +$$

$$\dim M \;=\; n \geq 3$$

$$G_{\mu\nu} \;=\; T_{\mu\nu}$$

$$\ell^\mu \ell_\mu = \; 0$$

$$T_{\ell\ell} \;\geq\; 0$$

$$T_{\ell\alpha} T_\ell{}^\alpha \;\leq\; 0$$

Higher dimensional Raychaudhuri equation

JL, Pawłowski 2004

Every null, geodesic vector field tangent to a null surface satisfies:

$$\ell^\mu \nabla_\mu \theta \; = \; -\frac{1}{n-2}\theta^2 \; - \sigma_{AB}\sigma^{AB} - T_{\ell\ell}$$

Therefore:

$$\theta = 0 \quad \Rightarrow \quad \mathcal{L}_\ell \, g_{ab} \; = \; 0$$

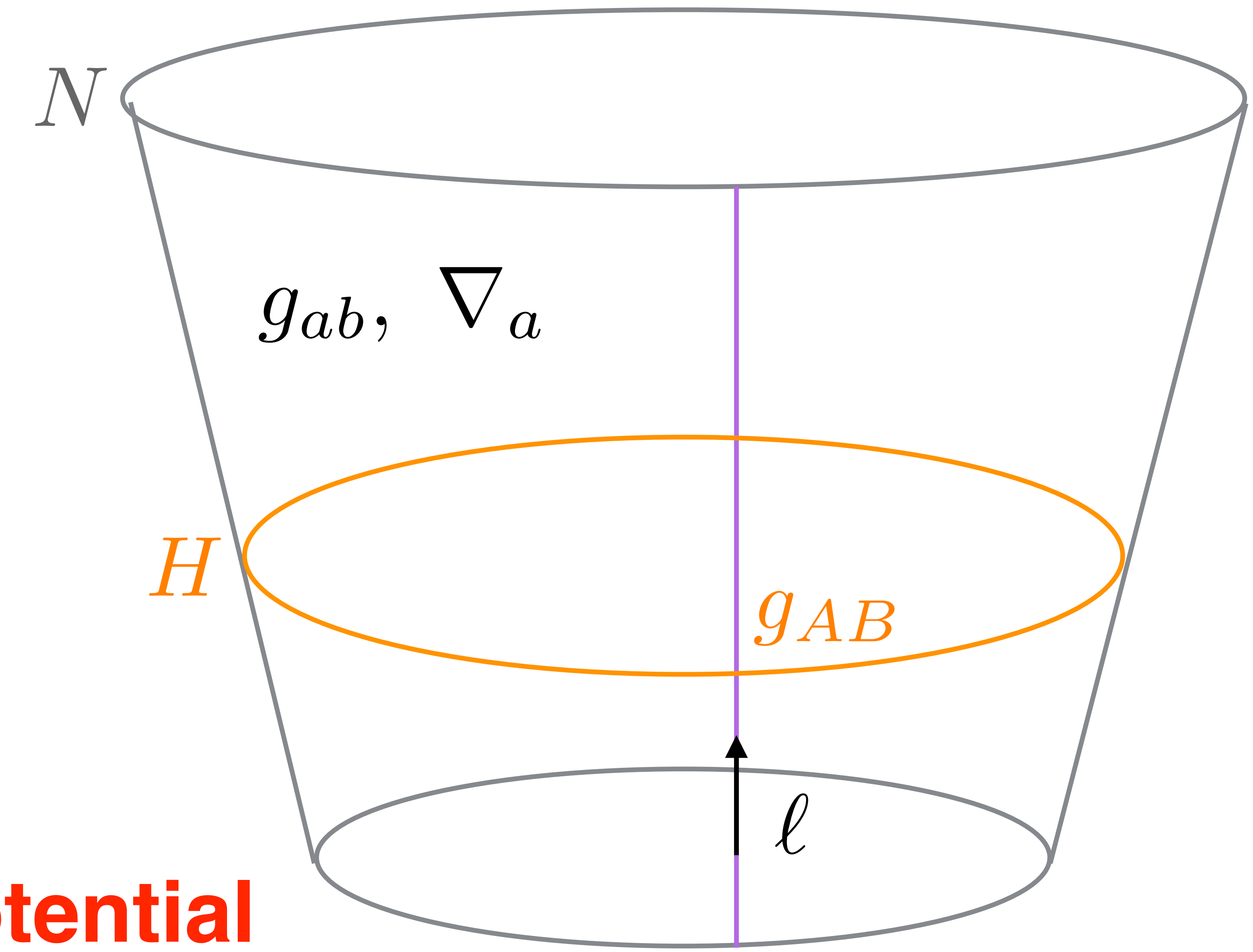
Non-expanding horizons: geometry

$g_{\mu\nu},\ \nabla_\mu$ - spacetime geometry

$\mu,\nu\ -\ M$

$a,b\ -\ N$

$A,B\ -\ H$



$g_{ab},\ \nabla_a$ - **horizon geometry**

$\nabla_a \ell^b\ =\ \omega^\ell_a\, \ell^b$ - **rotation potential**

$\kappa^\ell\ :=\ \ell^a \omega^\ell_a$ - **surface gravity**

$\nabla_a \kappa^\ell\ -\ \mathcal{L}_\ell\, \omega^\ell_a\ =\ 0$ - **the zeroth law**

Non-expanding horizons: symmetric

Symmetric NEH:

$$\begin{aligned} \mathcal{L}_X g_{ab} &= 0 \\ [\mathcal{L}_X, \nabla_a] &= 0 \end{aligned}$$

Isolated Horizon:

$$[\mathcal{L}_\ell, \nabla_a] = 0$$

$$\begin{aligned} \mathcal{L}_\ell \, \omega_a^\ell &= 0 \\ \kappa^\ell &= \text{const} \end{aligned}$$

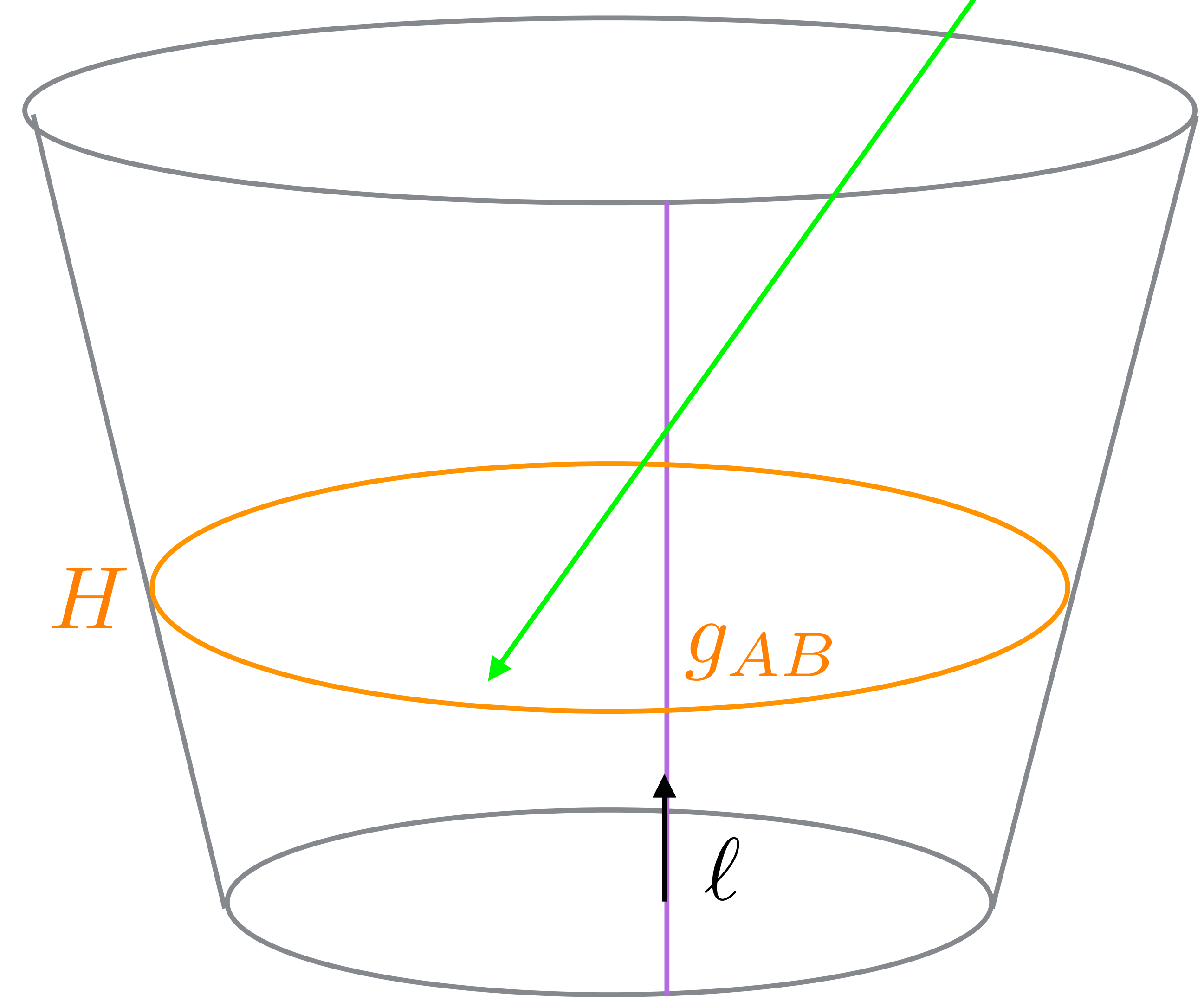
Extremal Isolated Horizons

$T_{\mu\nu}$

$$\kappa^{\ell} = 0$$

$${}^{(n-2)}\nabla_A \omega^{\ell}_B + {}^{(n-2)}\nabla_B \omega^{\ell}_A + 2\omega^{\ell}_A \omega^{\ell}_B - {}^{(n-2)}R_{AB} + R_{AB} = 0 \quad (1)$$

The constraint satisfied on every *n-2*-slice



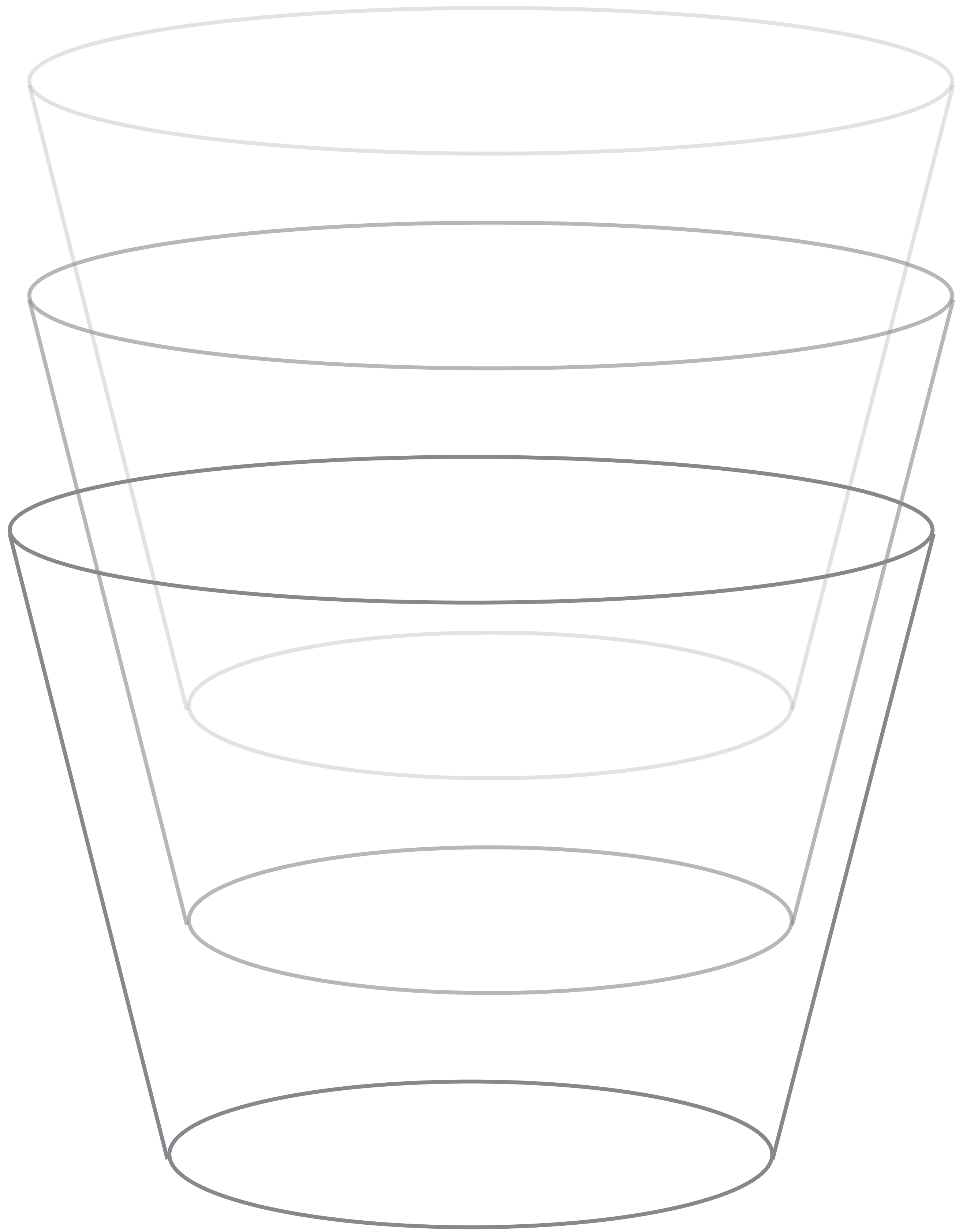
Conversely, if a non-expanding horizon, admits a slice and a null vector such that the condition (1) is satisfied, then it is an extremal isolated horizon provided:

$$\mathcal{L}_{\ell} R_{ab} = 0$$

References to the extremal isolated horizons:

Ashtekar, JL, Beetle 2001;
JL, Pawłowski 2002;
JL, Pawłowski 2004;
JL, Szereszewski, Waluk 2016

Spacetimes foliated by non-expanding horizons



Spacetimes foliated by non-expanding horizons

Let

$$u = \text{const}$$

define the foliation.

$$\ell' = -g^{\mu\nu}\nabla_\nu u$$

On each space-like section of every non-expanding horizon the following condition is satisfied

$$-{}^{(n-2)}\nabla_A \omega^{\ell'}_B - {}^{(n-2)}\nabla_B \omega^{\ell'}_A + 2\omega^{\ell'}_A \omega^{\ell'}_B - {}^{(n-2)}R_{AB} + R_{AB} = 0$$

Compare the very condition satisfied by foliating horizons

$$-{}^{(n-2)}\nabla_A \omega^{\ell'}_B \;-\; {}^{(n-2)}\nabla_B \omega^{\ell'}_A \;+\; 2\omega^{\ell'}_A \omega^{\ell'}_B \;-\; {}^{(n-2)}R_{AB} \;+\; R_{AB} \;=\; 0$$

with the extremal horizon condition

$${}^{(n-2)}\nabla_A \omega^{\ell}_B \;+\; {}^{(n-2)}\nabla_B \omega^{\ell}_A \;+\; 2\omega^{\ell}_A \omega^{\ell}_B \;-\; {}^{(n-2)}R_{AB} \;+\; R_{AB} \;=\; 0$$

The transformation

$$\omega^{\ell} \;= -\omega^{\ell'}$$

would do. But where from?

Intersecting non-expanding horizons

Suppose two horizons intersect:

$$N \cap N' \; = \; H$$

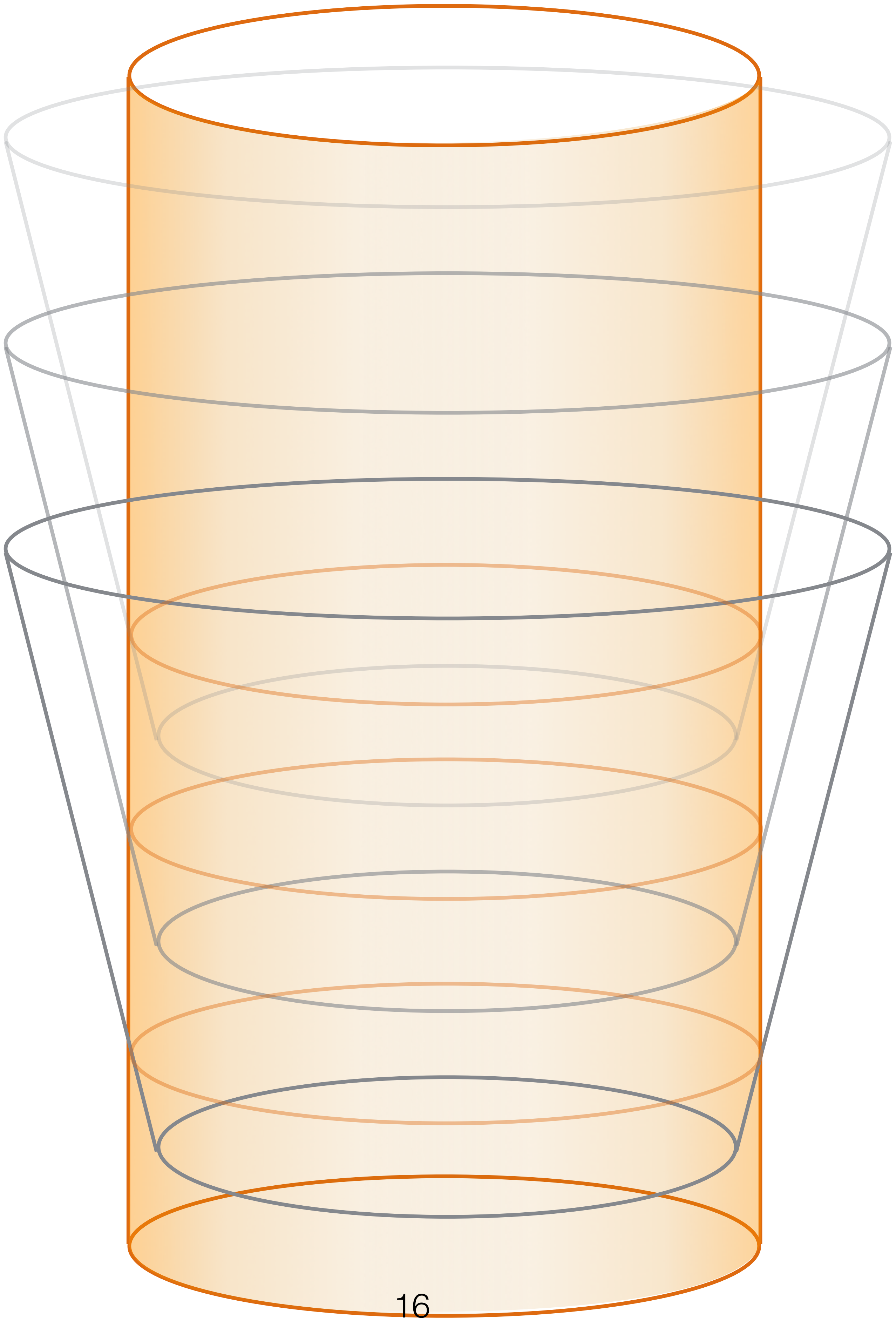
and

$$\ell^\mu \ell'_\mu \; = \; -1$$

Then

$$\omega^\ell_A \; = \; -\ell'_\mu \nabla_A \ell^\mu \; = \; \ell_\mu \nabla_A \ell'^\mu \; = \; -\omega^{\ell'}_A$$

Non-expanding horizon foliation and a transversal horizon



Results valid for vacuum

1. Suppose spacetime is foliated by non-expanding horizons and there exists a transversal non-expanding horizon. Then, the transversal horizon is extremal. Still true with non-zero cosmological constant.

2. Given two intersecting non-expanding horizons, the Cauchy problem is well posed given

$$g_{AB},\ \omega_{AB}$$

on the intersection, hence this data determines a solution

3. A general solution to 1 above is:

Given an n-1 dimensional manifold, a metric and a 1-form thereon:

$$H, \quad g_{AB}, \quad \omega_A$$

Such that:

$$^{(n-2)}\nabla_A \omega_B \; + \; ^{(n-2)}\nabla_B \omega_A \; + \; 2\omega_A \omega_B \; - \; ^{(n-2)}R_{AB} \; = \; 0$$

On:

$$H \times \mathbb{R} \times \mathbb{R}$$
$$(x^A, u, v)$$

Define:

$$g_{\mu\nu} dx^\mu dx^\nu \quad :=$$
$$g_{AB} dx^A dx^B \; - \; 2du \left(dv - 2v\omega - \frac{1}{2} v^2 (\mathrm{div}\omega + 2\omega^2) du \right)$$

$$R_{\mu\nu} \; = \; 0$$

The Spacetime Killing vectors

$$g_{AB}dx^A dx^B \ - \ 2du\left(dv - 2v\omega - \frac{1}{2}v^2(\mathrm{div}\omega + 2\omega^2)du\right)$$

Killing vectors:

$$K \ = \ \partial_u \qquad\qquad \xi \ = \ v\partial_v \ - \ u\partial_u$$

Killing horizons of $u_0 K + \xi$

$$u \ = \ u_0 \quad \textbf{or} \quad v \ = \ 0$$

Extremal Killing horizon of K

$$v \ = \ 0$$

Near horizon geometries

Geometry in a neighborhood of an extremal Killing horizon

$$v = 0$$

$$g_{AB}(x,v)dx^A dx^B - 2du\left(dv + vW_A(x,v)dx^A + v^2 H(x,v)du\right)$$

Horowitz transformation

$$v \mapsto \epsilon v, \quad u \mapsto \frac{u}{\epsilon}$$

Limit at $\epsilon \mapsto 0$ **a near horizon geometry:**

$$g_{AB}(x,0)dx^A dx^B - 2du(dv + vW_A(x,0)dx^A + v^2 H(x,0)du)$$

Horowitz 1999, Reall 2002; Kunduri, Lucietti 2013

Summary

1. We investigated *n* dimensional spacetimes foliated by non-expanding horizons. We found out that:
   a) each foliating horizon satisfies a constraint dual to the extremal isolated horizon constraint
   b) the duality can be geometrically interpreted as a condition allowing (but not implying) the existence of a transversal extremal horizon.
   c) if we assume that there actually exists a transversal extremal horizon, than we can solve vacuum Einstein's equation completely.
   d) The general solution to our problem (foliation by NEH and a transversal NEH) turned out to be a general **near horizon geometry.**

Summary

e) In the *n=4* case we derived all them already in the earlier paper by *Pawlowski, JL, Jezierski of 2003*

f) Remarkably in the two major Living Reviews: *Ashtekar, Krishnan 2004 Isolated and Dynamical Horizons* our paper is mentioned as “another exotic case” *Kunduri, Lucietti Living Reviews 2013 Near Horizon Geometries* the paper is unnoticed

2. Open questions:

a) Are there other solutions?

b) Does our “master equation” on 2-sphere have other solutions than those defined by the extremal Kerr? If you find a solution you have a new exact solution to Einsteins Equations.

Thank You