FOUR-DIMENSIONAL WALL-CROSSING FROM THREE-DIMENSIONAL FIELD THEORY

WORK DONE WITH

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NEW ENGLAND STRING MEETING

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OUTLINE

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1. INTRODUCTION

THIS TALK IS ABOUT THE BPS SPECTRUM OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

THE BPS SPECTRUM OF THE THEORY ON $\mathbb{R}^4$ IS A "PIECEWISE CONSTANT" FUNCTION OF THE BOUNDARY CONDITIONS AT ∞ OF VECTORMULTIPLET SCALARS.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF). THIS TALK WILL GIVE A PHYSICAL INTERPRETATION AND PROOF OF A FAMOUS WCF OF KONTSEVICH + SOIBELMAN.

2. REVIEW $\mathcal{N}=2, D=4$ WALL CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$
WITH $\mathcal{N}=2$ SUPERPOINCARÉ SYMMETRY

LET $\mathcal{H}$ BE THE ONE-PARTICLE
HILBERT SPACE.

AS A REPRESENTATION OF THE
$\mathcal{N}=2$ SUPERPOINCARÉ ALGEBRA $\mathcal{S}$, $\mathcal{H}$
DEPENDS ON THE BOUNDARY VALUES
OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED
IN THE MODULI SPACE OF VACUA: $\mathcal{M}_V$.

FOR $u \in \mathcal{M}_V$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{M}_V$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE SYMPLECTIC LATTICE Γ OF ELEC. + MAG. CHARGES. (OF RANK $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \mathcal{A}$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$

ON $\mathcal{H}_{\gamma,u}$ $\quad E \geq |Z_\gamma(u)|$

DEF. $\mathcal{H}^{BPS}_{\gamma,u}$ = SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_\gamma(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS

[Cecotti et.al. ; Seiberg & Witten]

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ SO

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|\right) \leq 0$$

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$$MS(\gamma_1,\gamma_2) := \{ u \,|\, Z_{\gamma_1}(u)/Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

WCF: BE MORE QUANTITATIVE ABOUT "HOW MANY" STATES DECAY

DEFINE THE BPS INDEX

$$\Omega^{Ph}(\gamma;u) := -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}}(2J_3)^2(-1)^{2J_3}$$

DENEF & MOORE GAVE FORMULAE FOR $\Delta\Omega$ FOR DECAYS $\gamma \to \gamma_1 + \gamma_2$ WHERE AT LEAST ONE OF γ_1, γ_2 ARE PRIMITIVE.

THE DERIVATION IS BASED ON DENEF'S MULTICENTERED SOLUTIONS OF $\mathcal{N}=2$ SUGRA AND QUIVER QUANTUM MECHANICS

THE METHODS ARE DIFFICULT TO USE WHEN <u>BOTH</u> γ_1, γ_2 ARE NON-PRIMITIVE

KONTSEVICH & SOIBELMAN PROPOSED A REMARKABLE W.C.F. FOR AN INDEX $\Omega^{DT}(\gamma;u)$,

"GENERALIZED DONALDSON-THOMAS INVARIANT OF A CALABI-YAU 3-FOLD"

THEIR FORMULA APPLIES TO ALL DECAYS $\gamma \to \gamma_1 + \gamma_2$.

WE EXPECT THAT

$$\Omega(\gamma;u) = \Omega^{Ph}(\gamma;u) = \Omega^{DT}(\gamma;u)$$

SO THE KS WCF APPLIES TO PHYSICAL BPS DEGENERACIES.

THIS TALK PROVES THE KS WCF FOR $\Omega^{Ph}(\gamma;u)$ IN $\mathcal{N}=2$ FIELD THEORIES

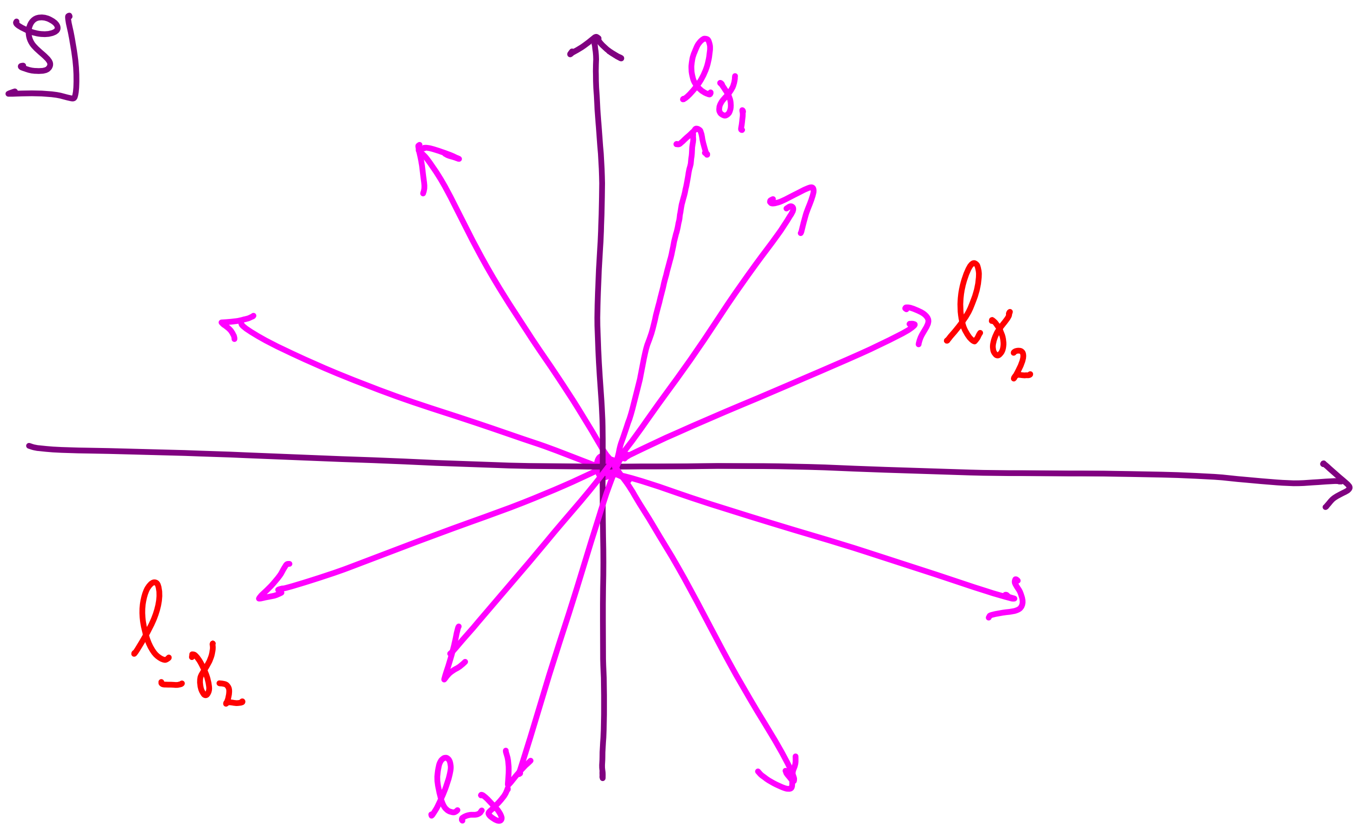
3. THE KONTSEVICH–SOIBELMAN FORMULA

DATA:

1. SYMPLECTIC LATTICE Γ
2. CENTRAL CHARGES $Z_\gamma(u)$, $u \in \mathcal{M}_V$
3. PIECEWISE CONSTANT $\Omega(\gamma;u) \in \mathbb{Z}$

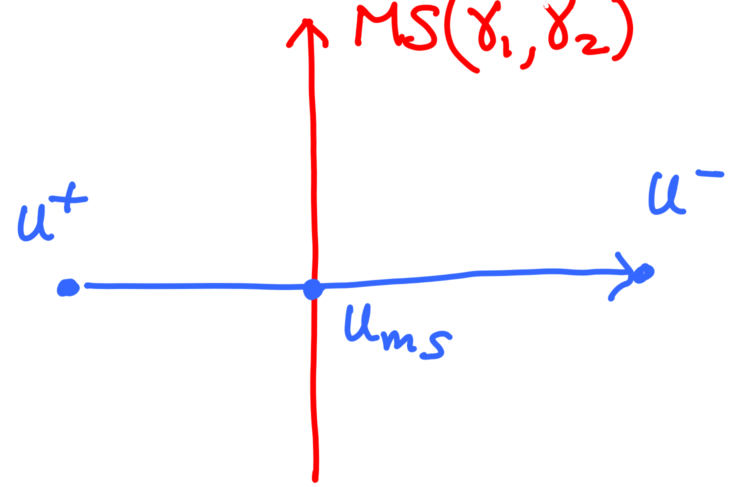
BPS RAYS: FOR $u \in \mathcal{M}_V$, $\gamma \in \Gamma$

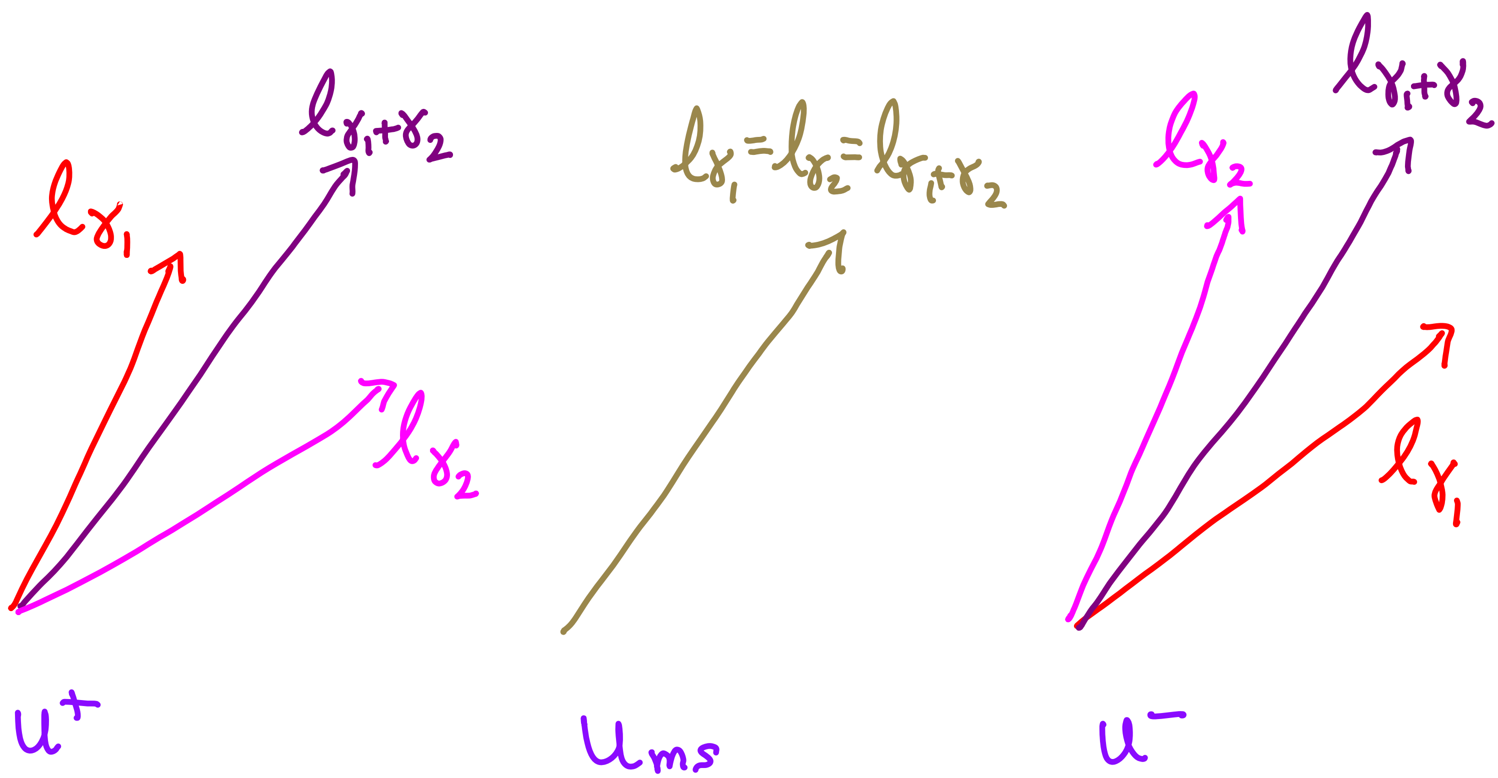
$$\ell_\gamma := Z_\gamma(u)\mathbb{R}_- = \left\{ \zeta \;\middle|\; \zeta / Z_\gamma(u) \in \mathbb{R}_- \right\}$$



AS u VARIES THE SLOPES OF THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1, \gamma_2)$ BPS RAYS WILL COALESCE





SECOND INGREDIENT: A SYMPLECTIC TORUS:

- INTRODUCE THE COMPLEX TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots\cdots \times \mathbb{C}^*}_{2r}$$

$\gamma \in \Gamma \implies$ FUNCTION $X_\gamma : T \to \mathbb{C}^*$

"HOLOMORPHIC FOURIER MODES"

CHOOSING A BASIS γ_i FOR $\Gamma \implies$

$$(e^{\theta_1}, \cdots\cdots, e^{\theta_{2r}}) \in T, \quad \theta_i \in \mathbb{C}$$

$$X_\gamma = \exp[\gamma \cdot \theta]$$

- HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^T := \frac{1}{2} \epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}} \qquad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A SYMPLECTOMORPHISM:

$$K_\gamma : \; X_{\gamma'} \rightarrow X_{\gamma'} \left(1 - X_\gamma\right)^{\langle \gamma', \gamma \rangle}$$

$$X_{\gamma'} \rightarrow X_{\gamma'} \exp\left[\langle \gamma, \gamma' \rangle \log\left(1 - X_\gamma\right)\right]$$

EXAMPLE

$$r=1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b),(a',b') \rangle = ab' - a'b$$

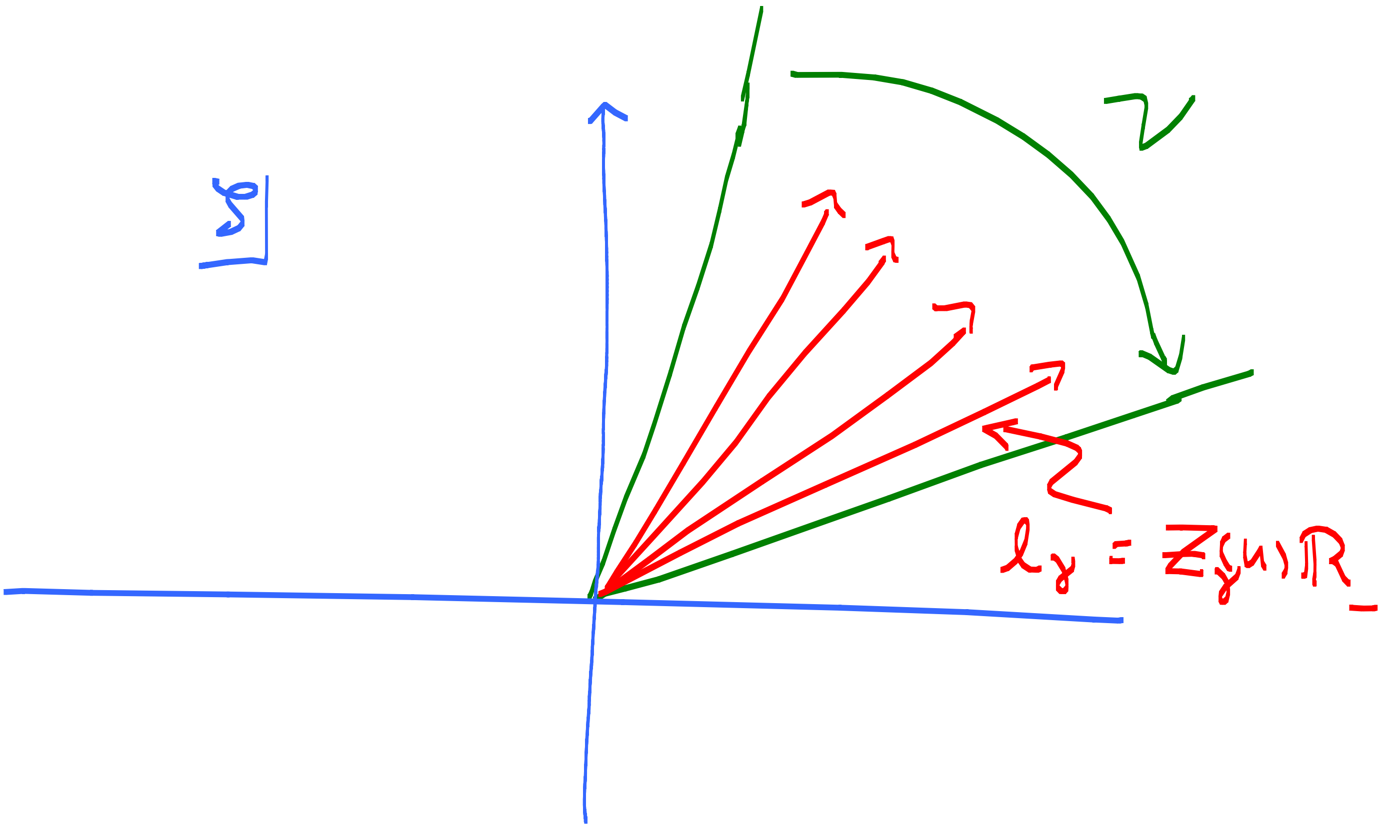
$$T \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \qquad y = X_{0,1}$$

$$\omega^T = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$K_{a,b} : \begin{cases} x \longrightarrow x\left(1-(-1)^{ab}x^a y^b\right)^b \\ y \longrightarrow y\left(1-(-1)^{ab}x^a y^b\right)^{-a} \end{cases}$$

NOW CHOOSE A CONVEX CONE $\mathcal{V}$



FOR EACH BPS RAY DEFINE

$$S_\gamma := \prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma';u)}$$

AND THEN DEFINE:

$$A_{\mathcal{V}} := \overrightarrow{\prod_{\ell_\gamma \subset \mathcal{V}}} S_\gamma$$

$$\mathcal{A}_V := \prod_{\ell_\gamma \subset V}^{\rightarrow} S_\gamma = \prod_{-Z_\gamma \in V}^{\rightarrow} K_\gamma^{\Omega(\gamma;u)}$$

THE PRODUCT IS TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

$\mathcal{A}_V$ DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma;u)$ DEPEND ON u ...

THE KS FORMULA STATES THAT

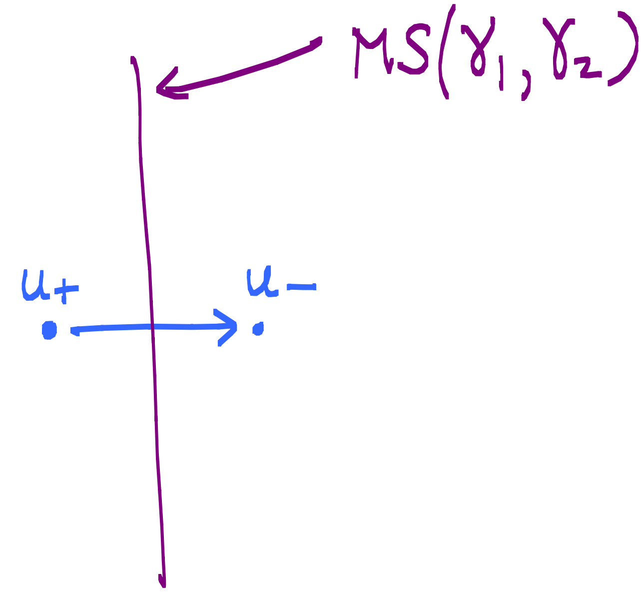
NEVERTHELES,

$$\mathcal{A}_{\mathcal{V}} = \prod_{\ell_\gamma \subset \mathcal{V}}^{\rightarrow} K_\gamma^{\Omega(\gamma;u)}$$

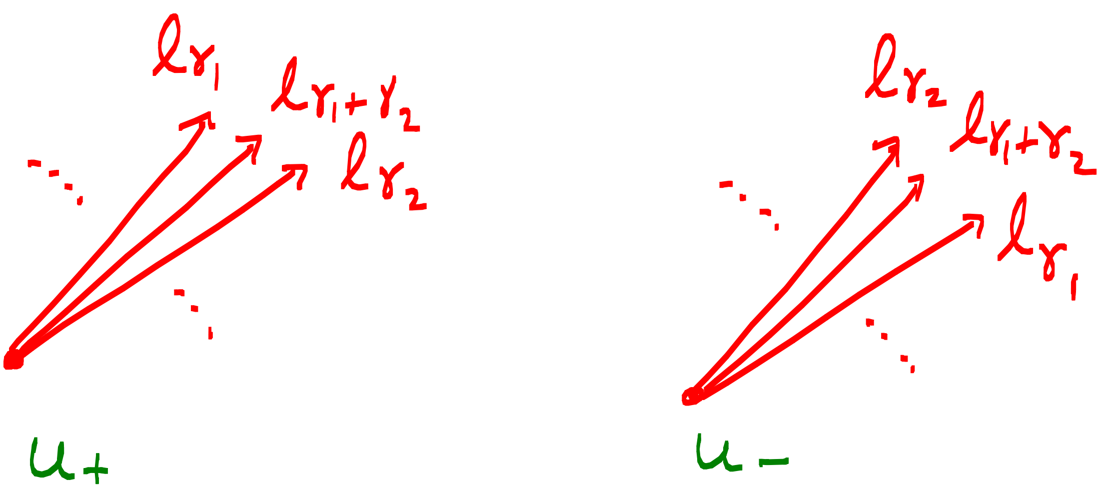
IS <u>CONSTANT IN u</u> AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR $\mathcal{V}$.

THIS IS A WALL-CROSSING FORMULA...

AS u CROSSES A WALL



$$\ell_\gamma = Z_\gamma(u) \cdot \mathbb{R}_- \qquad \text{ROTATES}$$



$\Longrightarrow$ EXCHANGE ORDER IN

$$\mathcal{A}_\vee = \overrightarrow{\prod_{\ell_\gamma \subset \vee}} K_\gamma^{\Omega(\gamma; u)}$$

$\Omega(\gamma; u)$ MAKES A COMPENSATING CHANGE

ONE CAN RECOVER THE PRIMITIVE & SEMI-PRIMITIVE W.C.F. FROM THIS FORMULATION...

[with Wu-yen Chuang]

4. COMPACTIFICATION OF $N=2, D=4$ FIELD THEORIES

A. SEIBERG-WITTEN SOLUTION

G - COMPACT S.S. GAUGE GROUP, RANK $= r$

$\Longrightarrow$ $D=4, N=2$ FIELD THEORY

(CAN ALSO INCLUDE HM's)

$$\mathcal{M}_V = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r \quad \left(u_2 = \mathrm{Tr}\,\Phi^2,\ u_3 = \mathrm{Tr}\,\Phi^3, \cdots \right)$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

IN TERMS OF

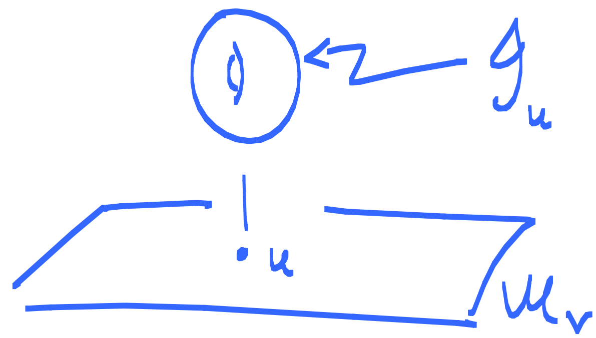
<u>SPECIAL KÄHLER GEOMETRY</u>

REVIEW SPECIAL KÄHLER GEOM:

c.f. D. FREED, hep-th/9712042

VIEW Γ AS A LOCAL SYSTEM OVER $\mathcal{M}_v$

$$\begin{tikzcd} \mathcal{J}_u \arrow[r] \arrow[d] & \mathcal{J} \arrow[d] \\ u \arrow[r, hook] & \mathcal{M}_v \end{tikzcd}$$



$$\mathcal{J}_u = \Gamma_u^* \otimes_{\mathbb{Z}} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

FIBERS = ABELIAN VARIETIES

IN REGIONS OF $\mathcal{M}_v$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag} , \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \mathrm{Span}\{\alpha_I\} \oplus \mathrm{Span}\{\beta^I\}$$

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0 \qquad \langle \alpha_I, \beta^J \rangle = \delta_I{}^J$$

CHOOSING A DUALITY FRAME,

Σ_u HAS PERIOD MATRIX τ_{IJ}

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1. LOW ENERGY LAGRANGIAN:

$$\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im}\tau_{IJ}\left(da^I * d\bar{a}^J + F^I * F^J\right)$$

$$+ \frac{1}{4\pi} \operatorname{Re}\tau_{IJ}\; F^I \wedge F^J$$

$$a^I = Z_{\alpha_I}(u) \qquad I = 1, \ldots, r$$

LOCAL COORD'S ON $\mathcal{M}_V$

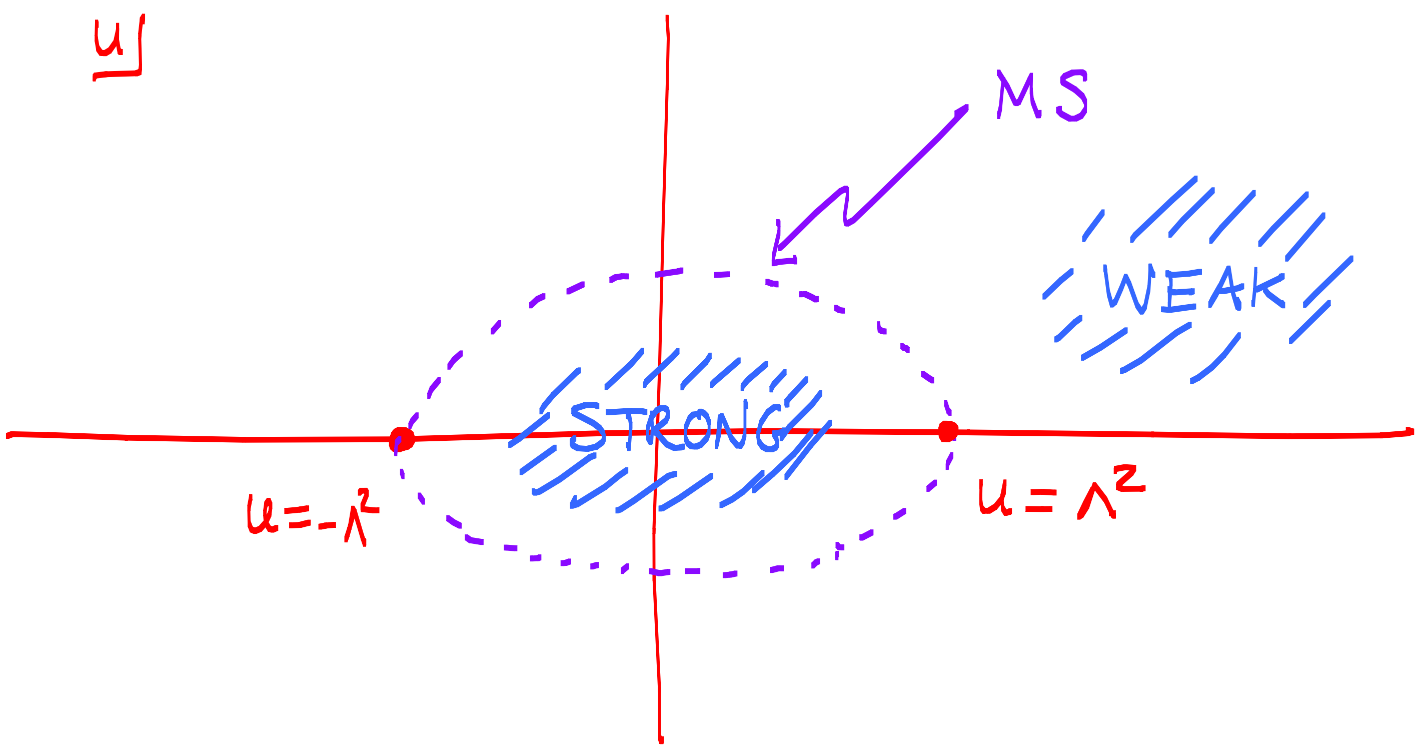
2. CENTRAL CHARGE FUNCTION

$$\boxed{\begin{aligned} Z_\gamma(u) &= a\cdot\gamma_{el} + a_D\cdot\gamma_{mg} \\ \tau_{IJ} &= \frac{\partial a_{DI}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \end{aligned}}$$

S & W IDENTIFY $\mathcal{J}_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u: \quad y + \frac{\Lambda^4}{y} = x^2 - 2u$$



$$a = \oint_\alpha x \frac{dy}{y} \qquad a_D = \oint_\beta x \frac{dy}{y}$$

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0)$$

$$\oplus \text{ CONJUGATE}$$

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1)$$

$$\oplus \text{ CONJUGATE}$$

[Bilal & Ferrari]

KS IDENTITY:

$$K_{2,-1} K_{0,1} = K_{0,1} K_{2,1} K_{4,1} \cdots K_{2,0}^{-2} \cdots K_{6,-1} K_{4,-1} K_{2,-1}$$

IT IS TRUE !!!

B. COMPACTIFY ON A CIRCLE.

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.

- LOW ENERGY THEORY IS A 3D σ-MODEL : $\mathbb{R}^3 \longrightarrow \mathcal{M}$

$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_4^I \, dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,4})_I \, dx^4 \end{aligned}\right\} \underline{\text{PERIODIC}}\,!$$

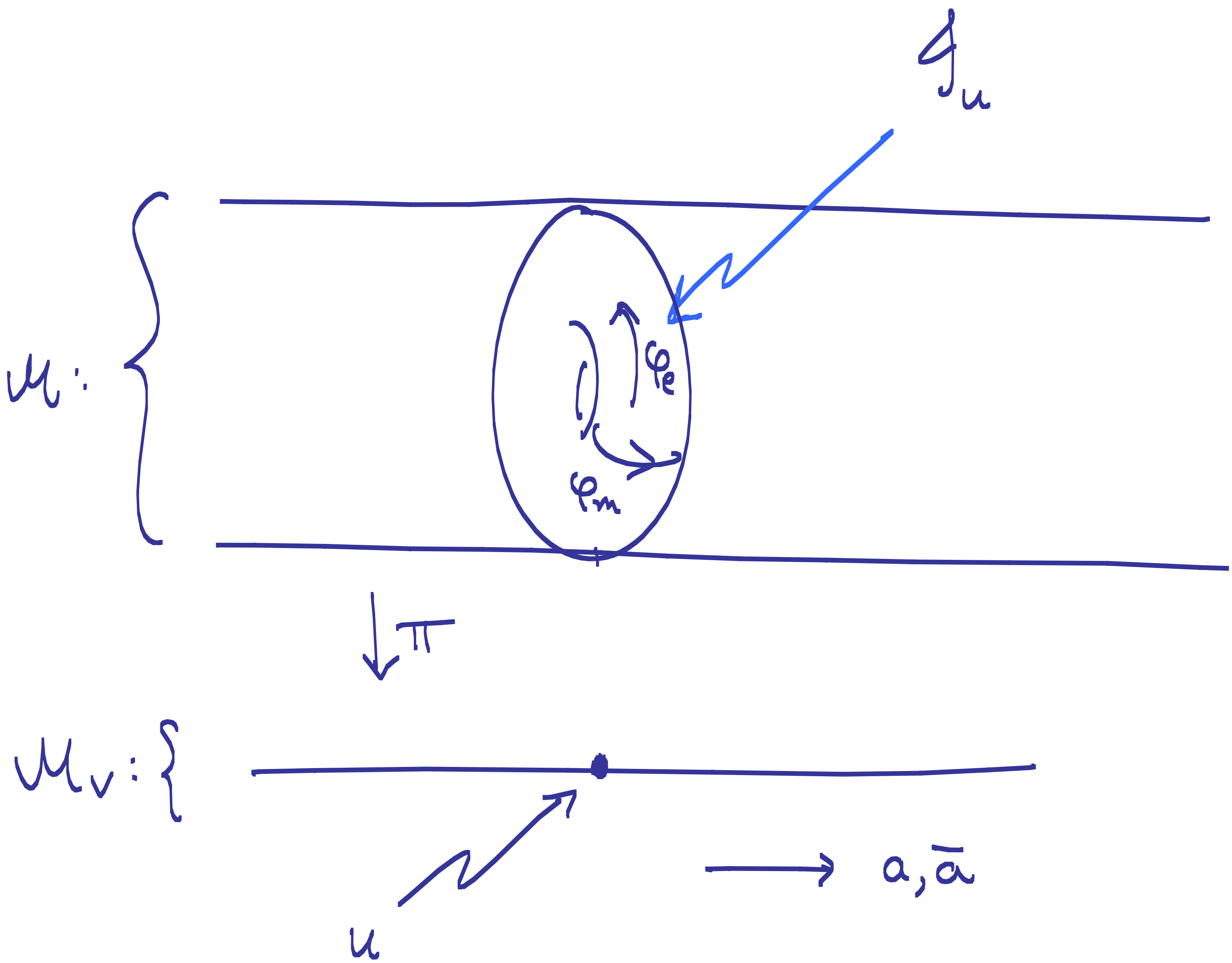
- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

TOPOLOGICALLY $\mathcal{M}$ IS A TORUS FIBRATION OVER $\mathcal{M}_v$:

IT IS EXACTLY $\mathcal{M} = \mathcal{J} = \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$ THAT APPEARED ABOVE:



THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:
USE DIMENSIONAL REDUCTION
+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J$$

$$- \frac{1}{8\pi^2 R} (\operatorname{Im}\tau)^{-1, IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R (\operatorname{Im}\tau) |da|^2 + \frac{1}{4\pi^2 R} (\operatorname{Im}\tau)^{-1} |dz|^2$$

C. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{M}_V$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.

5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN'S THEOREM: KNOWING (M, g) IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := M \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

<u>THEOREM</u>: IF (M, g) IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p: Z \longrightarrow \mathbb{CP}^1$$

$\mathcal{M}_\zeta = p^{-1}(\zeta) = \mathcal{M}$ IN COMPLEX STRUCTURE ζ

2. $\exists$ HOLOMORPHIC SECTION

ϖ OF $\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$

$\varpi_\zeta := \varpi|_{\mathcal{M}_\zeta}$ = HOLOMORPHIC SYMPLECTIC FORM ON $\mathcal{M}_\zeta$

3. $\forall x \in \mathcal{M}$, $\exists$ HOLOMORPHIC SECTION

$s_x : \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

4. $\exists$ ANTI-HOLOMORPHIC $\sigma : Z \to Z$ COVERING $\zeta \to -1/\bar{\zeta}$

CONVERSELY,

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

FOR $\zeta \in \mathbb{C}^*$:

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

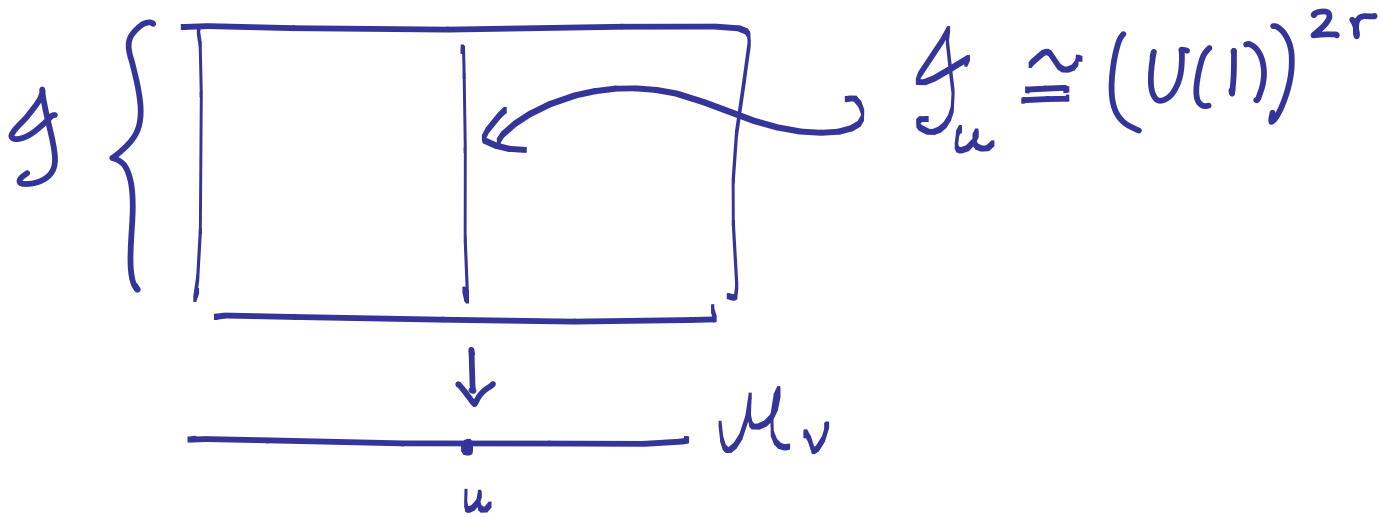
KÄHLER FORM

$$\omega_+ = \omega_1 + i\omega_2$$

OUR STRATEGY IS TO CONSTRUCT ϖ_ζ EXPLICITLY FOR $\mathcal{I}_\zeta$ USING A "NICE" SET OF HOLOMORPHIC FUNCTIONS ON TWISTOR SPACE:

$$\mathcal{X}_\gamma \quad , \quad \gamma \in \Gamma$$

USE THE TORUS FIBRATION OF $\mathcal{J}$:



FOR $\zeta \neq 0, \infty$ $\mathcal{J}_u$ IS NOT HOLOMORPHIC

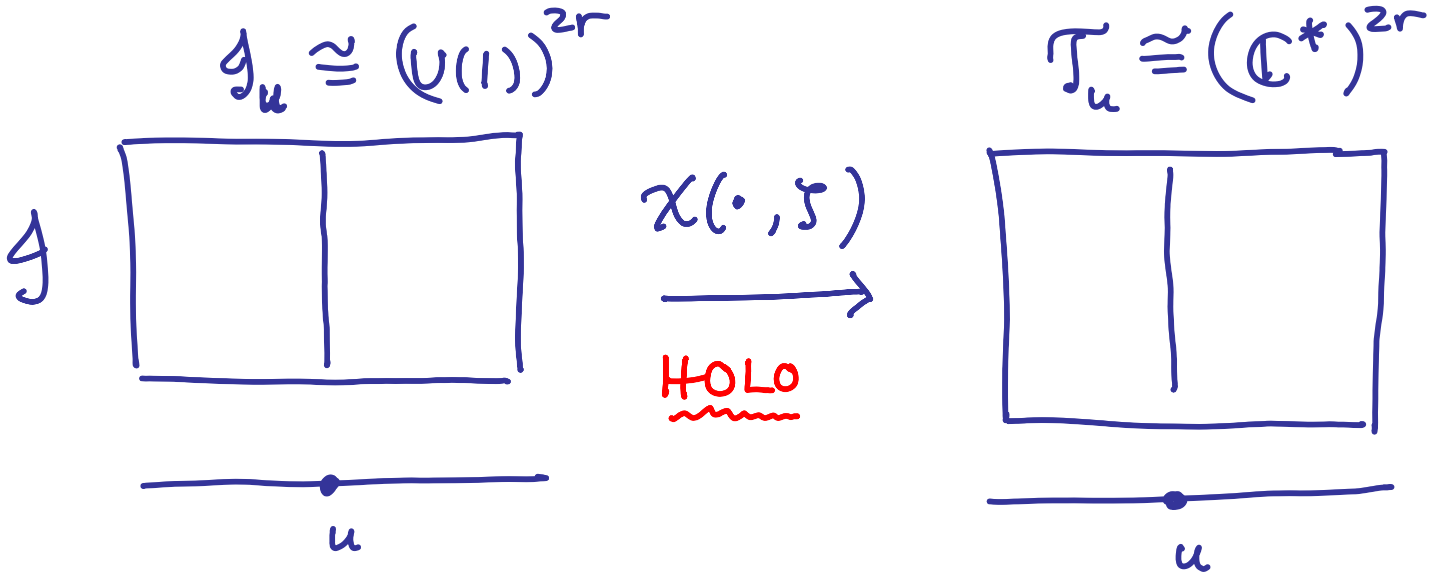
BUT CONSIDER $\mathcal{T} := \Gamma^* \otimes \mathbb{C}^*$

- $\mathcal{T}$ HAS A FIXED COMPLEX STRUCTURE WITH HOLOMORPHIC FIBERS $\cong (\mathbb{C}^*)^{2r}$
- $\mathcal{T}$ HAS HOLOMORPHIC FUNCTIONS X_γ
- $\mathcal{T}$ HAS A FIBERWISE HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^{\mathcal{T}} := \frac{1}{2} \epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}}$$

WE SEARCH FOR A HOLOMORPHIC MAP

$$\mathcal{X} : \mathcal{Z} \longrightarrow \mathcal{T} = \Gamma^* \otimes \mathbb{C}^*$$



SO THAT: $\varpi_\zeta = \mathcal{X}^*(\varpi^{\mathcal{T}})$

IF WE DEFINE $\mathcal{X}_\gamma := \mathcal{X}^*(X_\gamma)$

$$\Rightarrow \quad \varpi_\zeta = \frac{1}{2} \epsilon^{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

WE CAN VIEW

$$\varpi_\zeta = \frac{1}{2}\epsilon^{ij}\,\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

IN TWO WAYS:

- KNOW THE METRIC $\Longrightarrow$ CONSTRUCT $\mathcal{X}_\gamma$

– DO THIS FOR THE SEMIFLAT METRIC AND FIRST QUANTUM CORRECTION

- ULTIMATELY, WE DEFINE THE $\mathcal{X}_\gamma$ AND USE THEM TO DEFINE ϖ_ζ (AND HENCE THE HK METRIC)

EXAMPLE: SEMI-FLAT LIMIT

WE KNOW g^{sf} $\Longrightarrow$ COMPUTE

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

$$= \frac{i}{8\pi\zeta}\, da^I \wedge dz_I + \cdots + \frac{i\zeta}{8\pi}\, \overline{da^I \wedge dz_I}$$

FIND $\mathcal{X}_{\gamma_i}$ SO THAT:

$$\varpi_\zeta = \frac{1}{8\pi^2 R}\,\epsilon^{ij}\,\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

SOLUTION: DEFINE $\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$

$$\boxed{\mathcal{X}^{sf}_\gamma = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\,\theta_\gamma + \pi R \zeta\, \bar{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

- LEADING APPXT. TO $\mathcal{X}_\gamma$ FOR $R \to \infty$
- NO Q.C.'s FROM BPS STATES.

6. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.
- CONSIDER A POINT $u_* \in \mathcal{M}_V$

WHERE A SINGLE HM HAS $M \to 0$

$\Rightarrow$ DOMINANT CONTRIBUTION NEAR u_*.

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$

KK REDUCTION $\Rightarrow$ TARGET SPACE METRIC IS A GIBBONS-HAWKING ANSATZ: [Seiberg Witten; Ooguri Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left(\frac{d\varphi_m}{2\pi} + A \right)^2 + V(\vec{x})\, d\vec{x}^2$$

$$F = * dV \qquad \vec{x} \in \mathbb{R}^3$$

INTEGRATE OUT KK TOWER:

$$V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{g^2R^2|a|^2 + \left(g\frac{\varphi_e}{2\pi} + n\right)^2}}$$

$$a = x^1 + i x^2$$

$$\varphi_e = 2\pi R\, x^3 \qquad \text{PERIODIC}$$

$$V(\vec{x}) = V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi} \sum_{n\neq 0} e^{i n g \varphi_e} K_0\left(2\pi R |n g a|\right)$$

$\sim e^{-2\pi R |n g a|}$: INSTANTON CONTRIBUTION

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\mathcal{X}_\gamma\}$ ON TWISTOR SPACE IS GENERATED BY:

$$\mathcal{X}_e := \mathcal{X}_{(1,0)} = \exp\left\{ i\varphi_e + \cdots \right\}$$

$$\mathcal{X}_m := \mathcal{X}_{(0,1)} = \exp\left\{ i\varphi_m + \cdots \right\}$$

$$\mathcal{X}_{(k_1,k_2)} = (\mathcal{X}_e)^{k_1}(\mathcal{X}_m)^{k_2}$$

$$(k_1,k_2) \in \mathbb{Z}^2 \cong \Gamma_u$$

DETERMINE χ_e AND χ_m

FROM A DIFFERENTIAL EQUATION

HK STRUCTURE: $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A \right) + \frac{1}{2} V \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Rightarrow$ COMPUTE $\varpi_5 = -\frac{i}{2J}\omega_+ + \omega_3 - \frac{i}{2}J\omega_-$

$$\boxed{\varpi_5 = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}}$$

WE FIND:

$$\mathcal{X}_e = \mathcal{X}_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i\varphi_e + \pi R \zeta \bar{a} \right]$$

BUT

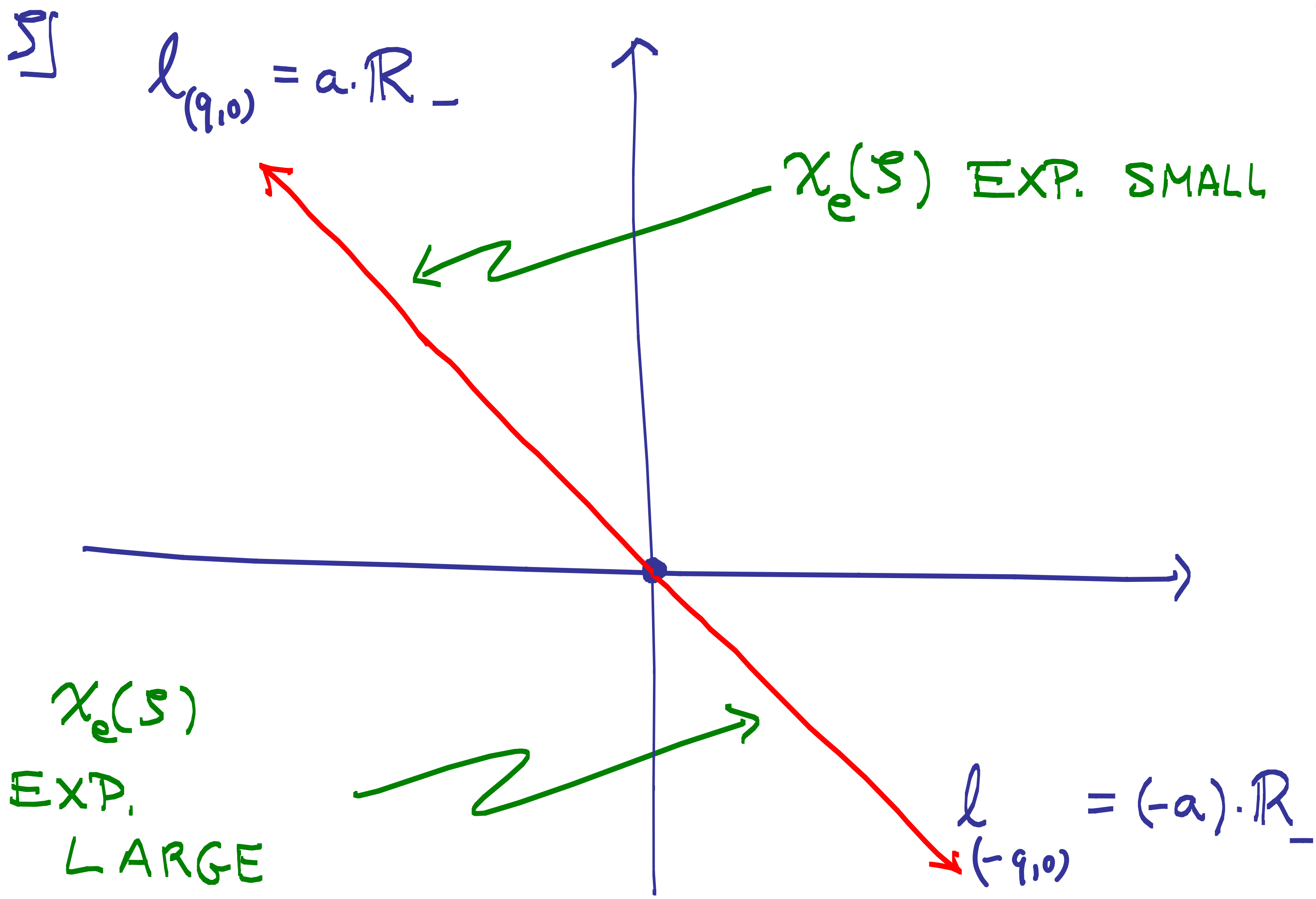
$$\mathcal{X}_m = \mathcal{X}_m^{s.f.} \cdot \mathcal{X}_m^{inst.}$$

$$\mathcal{X}_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i\varphi_m + \pi R \zeta \bar{a}_D \right]$$

$$a_D = \frac{q^2}{2\pi i}\left(a \log \frac{a}{e\Lambda} \right)$$

$$\mathcal{X}_m^{inst} = \text{INSTANTON CONTRIBUTION}$$

$$\boxed{\mathcal{X}_m^{inst}(\zeta) = \exp\left\{ \frac{iq}{4\pi}\int_{\ell_{(q,0)}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{q}\right) - \frac{iq}{4\pi}\int_{\ell_{(-q,0)}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{-q}\right)\right\}}$$



EMERGENCE OF THE KS TRANSFORMATION

AS A FUNCTION OF ζ, $\mathcal{X}_m$ IS DISCONTINUOUS ACROSS THE BPS RAYS OF THE HYPERMULTIPLET OF CHARGE $(\pm q, 0)$

$$\ell_\gamma := \left\{ \zeta \;\middle|\; \frac{Z_\gamma}{\zeta} \in \mathbb{R}_- \right\}$$

ACROSS THESE RAYS:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = K_{(0,\pm q)}(\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= \left(\mathcal{X}_e, \mathcal{X}_m (1 - \mathcal{X}_e^{\pm q})^{\mp q}\right)^{ccw}$$

KEY FEATURES OF $\mathcal{X}_\gamma$:

1. $\mathcal{X}_\gamma$ ARE HOLOMORPHIC ON $\mathcal{Z}$

2. $\mathcal{X}_\gamma \cdot \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$

3. $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$

4. $\mathcal{X}_\gamma \sim \mathcal{X}_\gamma^{s.f.}$ FOR $R \to \infty$

5. $$\left.\begin{array}{l} \lim\limits_{\zeta \to 0} \mathcal{X}_\gamma \exp\left(-\frac{\pi R}{\zeta} Z_\gamma(u)\right) \\ \lim\limits_{\zeta \to \infty} \mathcal{X}_\gamma \exp\left(-\pi R \zeta \overline{Z_\gamma(u)}\right) \end{array}\right\} \text{FINITE}$$

6. $\mathcal{X}_{\gamma'}(\zeta)$ TRANSFORMS BY $K_\gamma^{\Omega(\gamma;u)}$ ACROSS THE BPS RAY ℓ_γ.

7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: PROPERTIES 1–6

HOLD FOR THE <u>EXACT</u> FUNCTIONS $\mathcal{X}_{\gamma_g}$ USING ALL THE BPS RAYS ℓ_γ WITH DISCONTINUITY $K_\gamma^{\Omega(\gamma;u)}$

THIS WILL DETERMINE THEM UNIQUELY

OBSERVATION: $\mathcal{X}_\gamma$ ARE THE SOLUTION OF A RIEMANN-HILBERT PROBLEM.

(R-H PROBLEM:

FIND A PIECEWISE HOLO. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.)

SUMMARIZING THE $\mathcal{X}_\gamma$ BY A SINGLE MAP $\mathcal{X}$ (RECALL $\mathcal{X}_\gamma = \mathcal{X}^*(X_\gamma)$)

$\Rightarrow$ A RIEMANN-HILBERT PROBLEM IN THE ζ-PLANE FOR THE MAP

$$\mathcal{X}(\zeta): \mathcal{J} \longrightarrow T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ :

$$\mathcal{X}^{cw} = S_\gamma(\mathcal{X}^{ccw})$$

[RECALL: $S_\gamma = \prod_{\ell_{\gamma'}=\ell_\gamma} K_{\gamma'}^{\Omega(\gamma';u)}$]

2.) $\mathcal{X}(\zeta)$ HAS ASYMPOTICS FOR $\zeta \longrightarrow 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $\mathcal{O}(1)$ CORRECTIONS

$$Y := (\mathcal{X}^{sf})^{-1}\mathcal{X} : \mathfrak{g} \to \mathfrak{g}$$

i.e.

$$Y_0 = \lim_{\zeta\to 0} Y(\zeta) \;\&\; Y_\infty = \lim_{\zeta\to\infty} Y(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{sf}_\gamma(\zeta) \cdot \exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_{\gamma'}(\zeta')\right] \right\}$$

ITERATING THIS EQUATION

(AS A SUM OVER TREES...)

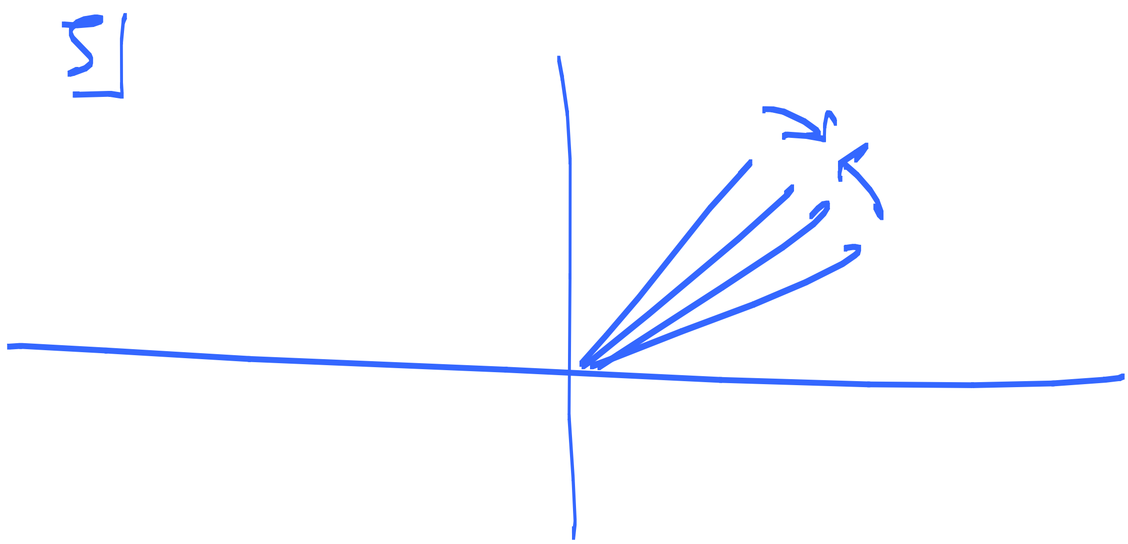
GIVES THE FULL INSTANTON EXPANSION!

$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S

- WE RECONSTRUCT THE METRIC FROM

$$\varpi = \frac{1}{4\pi^2 R}\, \mathcal{X}^*\left(\varpi^T\right)$$

- AS u CROSSES A WALL OF MS BPS RAYS PILE UP



BUT THE JUMP OF $\mathcal{X}$ IN THE RH PROBLEM IS CONTINUOUS AS A FUNCTION OF u: THAT IS THE KS FORMULA

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE HK. METRIC ACROSS WALLS OF M.S. !

THE RESULTING METRIC PASSES A NUMBER OF CONSISTENCY TESTS.

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM?

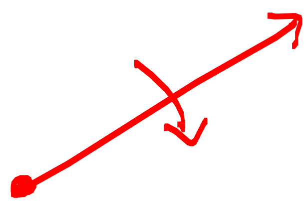
8. PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$A_\zeta = \mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X}$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$\mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X} \to (S\mathcal{X})^{-1} \zeta \partial_\zeta (S\mathcal{X})$$

$$= \mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X}$$

$\Rightarrow A_\zeta$ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Rightarrow \quad \zeta \partial_\zeta X = A_\zeta X \qquad 2$$

STRUCTURE GROUP: SYMPL(Γ)

ASYMPTOTICS $\Rightarrow$

$$A_\zeta = \zeta^{-1} A_\zeta^{(-1)} + A_\zeta^{(0)} + \zeta A_\zeta^{(+1)}$$

SINCE S_γ IS INDPT. OF R, u, Λ ....

SAME ARGUMENT $\Rightarrow$ X SATISFIES A SET OF DIFFERENTIAL EQUATIONS:

$$\frac{\partial}{\partial u}\chi = \mathcal{A}_u \cdot \chi$$

$$\frac{\partial}{\partial \bar{u}}\chi = \mathcal{A}_{\bar{u}} \cdot \chi$$

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \mathcal{A}_\Lambda \cdot \chi$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \mathcal{A}_{\bar{\Lambda}} \cdot \chi$$

$$R \frac{\partial}{\partial R}\chi = \mathcal{A}_R \cdot \chi$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \mathcal{A}_\zeta \cdot \chi$$

$$\mathcal{A}_i = \zeta^{-1} \mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta \mathcal{A}_i^{(+1)}$$

KEY POINT: <u>THESE EQUATIONS ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!</u>

$$\left.\begin{aligned} \frac{\partial}{\partial u}\chi &= \mathcal{A}_u \cdot \chi \\ \frac{\partial}{\partial \bar{u}}\chi &= \mathcal{A}_{\bar{u}} \cdot \chi \end{aligned}\right\}$$

HOLOMORPHY ON $\mathcal{M}_g$

$$\left.\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda}\chi &= \mathcal{A}_\Lambda \cdot \chi \\ \bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi &= \mathcal{A}_{\bar{\Lambda}} \cdot \chi \end{aligned}\right\}$$

ALSO HOLOMORPHY... VIEW Λ AS BACKGROUND VEV OF A VM.

$$\left.\begin{aligned} R \frac{\partial}{\partial R}\chi &= \mathcal{A}_R \cdot \chi \\ \zeta \frac{\partial}{\partial \zeta}\chi &= \mathcal{A}_\zeta \cdot \chi \end{aligned}\right\}$$

ANOMALOUS SCALE AND R-SYMMETRY

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.

$\mathcal{A}_\gamma^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS ℓ_γ

DENOTE STOKES FACTORS BY $\mathcal{S}_\gamma$

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\mathcal{S}_\gamma$ ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1- INSTANTON APPROXIMATION:

$$\mathcal{S}_\gamma = S_\gamma^{K.S.}$$

9. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE QUANTUM-CORRECTED METRIC IS EQUIVALENT TO THE KS WCF.

4. USE TWISTOR TRANSFORM AND WRITE HOLOMORPHIC FUNCTIONS AS AN EXPLICIT SUM OVER BPS INSTANTONS.

10. CONCLUSION

— OTHER THINGS WE HAVE DONE —

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAPA.

- THE FUNCTIONS $\mathcal{X}_\gamma$ ARE 'tHOOFT-WILSON-MALDACENA LOOP OPERATOR VEV'S; MOREOVER THERE IS A NICE INTERPRETATION IN TERMS OF A 3D TFT [TO APPEAR]

- THE MODULI SPACE $(\mathcal{M}, g)$ IS A MODULI SPACE OF A HITCHIN SYSTEM [S. CHERKIS & A. KAPUSTIN].

FOR HITCHIN SYSTEMS WITH GAUGE GROUP $SU(2)$ WITH SINGULARITIES ON $\mathbb{P}^1$ WE HAVE CONSTRUCTED THE FUNCTIONS $\mathcal{X}_\gamma$.

THE KS TRANSFORMATIONS AND $\zeta \to 0, \infty$ ASYMPTOTICS EMERGE VERY NATURALLY...

— TO DO —

- SINGULARITIES AT SUPERCONFORMAL POINTS REMAIN TO BE UNDERSTOOD

- RELATIONS TO INTEGRABLE SYSTEMS AND THE "MOTIVIC W.C.F."

- RELATION TO THE WORK OF JOYCE & BRIDGELAND / TOLEDANO LAREDO

- SOME OF THE DISCUSSION GENERALIZES NICELY TO SUGRA, BUT PUZZLES REMAIN

- MIGHT GIVE EXPLICIT FORMULATION OF Q.C.'s TO HYPERMULTIPLET MODULI SPACES.