

LECTURE III: EXTREMAL $N=2$ CFT'S

1. SUMMARY OF KEY POINTS FROM LECTURES I + II.
2. QUANTUM GRAVITY IN $2+1$ DIMENSIONS.
3. DEFINING EXTREMAL $N=2$ CFT
4. COUNTING POLAR POLYNOMIALS
5. COUNTING WEAK JACOBI FORMS
6. SEARCH FOR THE EXTREMAL E.G.
7. NEAR-EXTREMAL $N=2$ CFT
8. CONCLUDE: TWO OPEN PROBLEMS

SECTIONS 3-7 ARE UNPUBLISHED RESULTS WITH

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1. SUMMARY OF KEY POINTS FROM LECTURES I & II.

A. A VECTOR-VALUED NEARLY HOLOMORPHIC MODULAR FORM OF WEIGHT $W < 0$ IS DETERMINED BY ITS POLAR PART

B. THE ELLIPTIC GENUS OF AN $W=2$ THEORY WITH INTEGRAL $U(1)$ CHARGES AND $C=6m$ IS A WEAK JACOBI FORM $\chi(\tau, z; \ell) \in \tilde{J}_{0,m}$

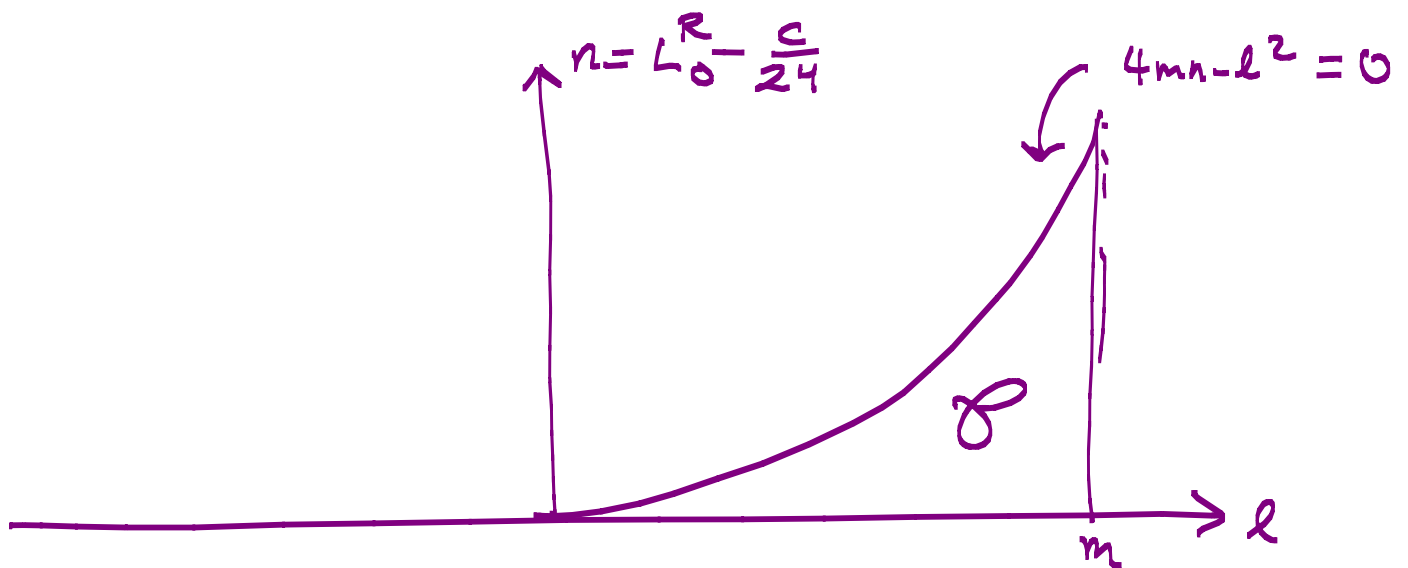
C. JACOBI FORMS ARE EQUIVALENT TO V-V-N-H MOD. FORMS

$$\begin{aligned}\chi(\tau, z) &= \sum_{n, \ell} c(n, \ell) q^n y^\ell \\ &= \sum_{\mu \bmod 2m} h_\mu(\tau) \bigoplus_{\mu, m} (\tau, z)\end{aligned}$$

D. THE POLAR TERMS IN $h_\mu(\tau)$
ARE THE TERMS $C(n, l)$ WITH

$$\mathcal{P} = 4mn - l^2 < 0$$

AFTER SPECTRAL FLOW & CHARGE
CONJUGATION THE INDEPENDENT
 $C(n, l)$ HAVE $(l, n) \in \mathcal{P}$



E. IF $\mathcal{C}$ HAS AdS_3 DUAL,
BTZ BLACK HOLES ONLY CONTRIBUTE
TO THE NON-POLAR PART OF $\mathcal{X}$.

2. QUANTUM GRAVITY IN 2+1 DIM'S

RECENTLY, WITTEN REVIVED AN OLD (1986) PROPOSAL THAT QUANTUM GRAVITY IN 2+1 DIM'S IS EXACTLY SOLUBLE.

FOCUS ON THE CASE $\Lambda < 0$

WHAT IS THE HOLOGRAPHIC DUAL OF 2+1 Q.G. ?

TO MOTIVATE WITTEN'S ANSWER

LET US RECALL WHY PEOPLE THINK THAT 2+1 Q.G. IS EXACTLY SOLUBLE:

THE ACTION

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{l^2} \right) + \frac{k}{4\pi} \int \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right)$$

l - AdS LENGTH

ω - SPIN CONNECTION

k - QUANTIZED

IS CLASSICALLY EQUIV. TO: 2

$$S = \frac{k_+}{4\pi} \int \text{Tr} \left(A_+ dA_+ + \frac{2}{3} A_+^3 \right) - \frac{k_-}{4\pi} \int \text{Tr} \left(A_- dA_- + \frac{2}{3} A_-^3 \right)$$

$$k_{\pm} = \frac{l}{16G} \pm \frac{k}{2}, \quad A_{\pm} = \omega \mp *e/l$$

WITTEN SUGGESTS THAT "THEREFORE"

2 THE HOLOGRAPHICALLY DUAL
PARTITION FUNCTION IS FACTORIZED:

LET'S JUST ACCEPT IT.

$$Z(\tau, \bar{\tau}) = Z_{k_+}(\tau) \overline{Z_{k_-}(\tau)}$$

ON A COMPACT SPACE C.S. HAS
NO LOCAL DEGREES OF FREEDOM.

NEITHER DOES 2+1 GRAVITY:
NO GRAV. WAVES. LOCALLY SOLUTIONS
ARE JUST AdS

BROWN+HENNEAUX: EDGE STATES
= VIRASORO ALGEBRA DESCENDENTS

$$C_L + C_R = \frac{3\ell}{G} = 24(k_+ + k_-)$$
$$C_L - C_R = 24(k_+ - k_-) \quad [\text{LORENTZ ANOMALY}]$$

$$C_L = 24k_+, \quad C_R = 24k_-$$

$$Z_k(\tau) \stackrel{?}{=} \chi_{\text{vac}} = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}$$

(NOTE THAT $L_{-1}|0\rangle$ IS A NULL STATE)

$$\chi_{\text{vac}}^{(\tau)} = q^{-k+\frac{1}{12}} (1-q) \frac{1}{\eta(\tau)}$$

EVIDENTLY, THIS IS NOT MODULAR

BUT WE EXPECT MODULARITY
IN A DIFF - INV T THEORY.

WHAT TO DO?

WITTEN PROPOSES THAT THE
P.F. SHOULD BE AS CLOSE TO
THE VIRASORO CHARACTER AS
POSSIBLE:

$$Z_k(\tau) = \left[q^{-k \frac{\infty}{12}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q \rightarrow 0} + O(q)$$

AS I EXPLAINED IN LECTURE 1
THIS MEANS:

$$Z_k(\tau) = a_k j^k + \dots + a_0$$

THAT IS - BY ADDING NONPOLAR TERMS
WE CAN RENDER $Z(\tau)$ MODULAR

WITTEN INTERPRETS THESE TERMS
AS THE CONTRIBUTION OF BTZ
BLACK HOLES.

THIS FITS IN PERFECTLY WITH
THE FAREYTAIL STORY OF
LECTURE II.

OBVIOUSLY THE REASONING IS
FAR FROM AIR-TIGHT. WHETHER
YOU LIKE IT OR NOT, WITTEN
MAKES A SHARP PROPOSAL:

LET $Z_k(\tau)$ BE THE
UNIQUE MOD. INVT. FUNCTION
SUCH THAT:

$$Z_k(\tau) = \left[q^{-k \frac{\infty}{1}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \right]_{q \rightarrow 0} + O(q)$$

DEF. AN EXTREMAL CFT OF
LEVEL k IS A CFT WITH
PARTITION FUNCTION $Z_k(\tau)$

WITTEN'S PROPOSAL: THE HOLOGRAPHIC
DUAL OF PURE 2+1 GRAVITY
IS $\mathcal{C}_k \otimes \tilde{\mathcal{C}}_k$ WHERE $\mathcal{C}_k$
IS AN ECFT $_k$.

- BUT DO ECFT'S EXIST?
- YES FOR $k=1$
- CONTROVERSIAL FOR $k>1$

THIS LEADS TO A ∇ QUESTION:
MAYBE EXTREMAL SUPERCONFORMAL
THEORIES ARE EASIER TO FIND....

THE STORY FOR $N=1$ IS DISCUSSED
IN WITTEN'S PAPER AND IS SIMILAR TO $N=0$.

3. DEFINING EXTREMAL $N=2$ CFT

I WILL NOW DESCRIBE SOME WORK
IN PROGRESS WITH

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SOMETHING QUALITATIVELY NEW
HAPPENS IN THE $N=2$ CASE.

WE WILL USE MODULARITY OF
THE ELLIPTIC GENUS TO PUT CONSTRAINTS
ON THE SPECTRUM OF $N=2$ PRIMARY
FIELDS.

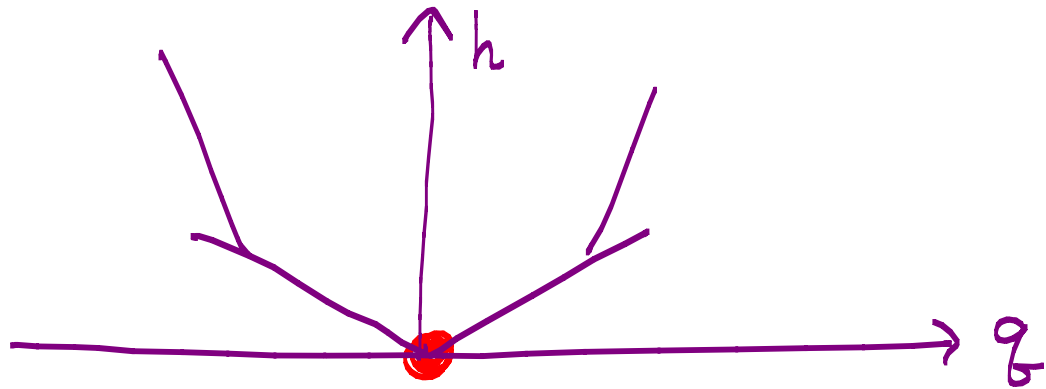
HOW SHOULD WE DEFINE AN
"EXTREMAL $N=2$ CFT" ?

IT'S UP TO US.

LET'S FOLLOW WITTEN'S LEAD

AND TAKE IT TO BE SUCH THAT
THE PARTITION FUNCTION

$Z(z, \tau; \bar{z}, \bar{\tau})$ DEFINED LAST
TIME IS "AS CLOSE AS POSSIBLE"
TO THE VACUUM CHARACTER.



VACUUM CHARACTER FOR $|h, q\rangle = |0, 0\rangle$

AND $c = 6m$:

$$\chi_{\text{vac}}(\tau, z) = q^{-m/4} \frac{\prod_{j=1}^{\infty} (1 + y q^{j+1/2}) (1 + \bar{y} q^{j+1/2})}{\prod_{j=1}^{\infty} (1 - q^j) \prod_{j=2}^{\infty} (1 - q^j)}$$

$\begin{array}{ccc} G_{-j-1/2}^{+} & & G_{-j-1/2}^{-} \\ \downarrow & & \downarrow \\ \uparrow & & \uparrow \\ J_{-j} & & L_{-j} \end{array}$

THIS IS NOT SPECTRAL FLOW
 INVT & DOES NOT HAVE GOOD
 MODULAR PROPERTIES - SO WE
 FORCE IT TO HAVE THESE PROP'S:

DEF: AN $N=(2,2)$ EXTREMAL CFT
 IS A HYPOTHETICAL THEORY WITH

$$Z_{NSNS} = \left| \sum_{\Theta \in \mathbb{Z}} SF_{\Theta} \chi_{vac} \right|^2 + \sum_{\substack{p > 0 \\ \text{OR} \\ \tilde{p} > 0}} c(n, l; \tilde{n}, \tilde{l}) q^n y^l \bar{q}^{\tilde{n}} \bar{y}^{\tilde{l}}$$

WE COULD HAVE FORMULATED THIS
 IN THE RAMOND OR NS SECTOR,
 BUT THE DEF. SEEMS BEST
 MOTIVATED IN THE NS-SECTOR.

THIS DEFINITION IMPLIES THAT
AN $N=2$ ECFT MUST HAVE
ELLIPTIC GENUS

$$\chi_{\text{EXT}}(\tau, z) = 2(-1)^m \sum_{\theta \in \mathbb{Z} + 1/2} S F_{\theta} \chi_v + \text{NONPOLAR}$$

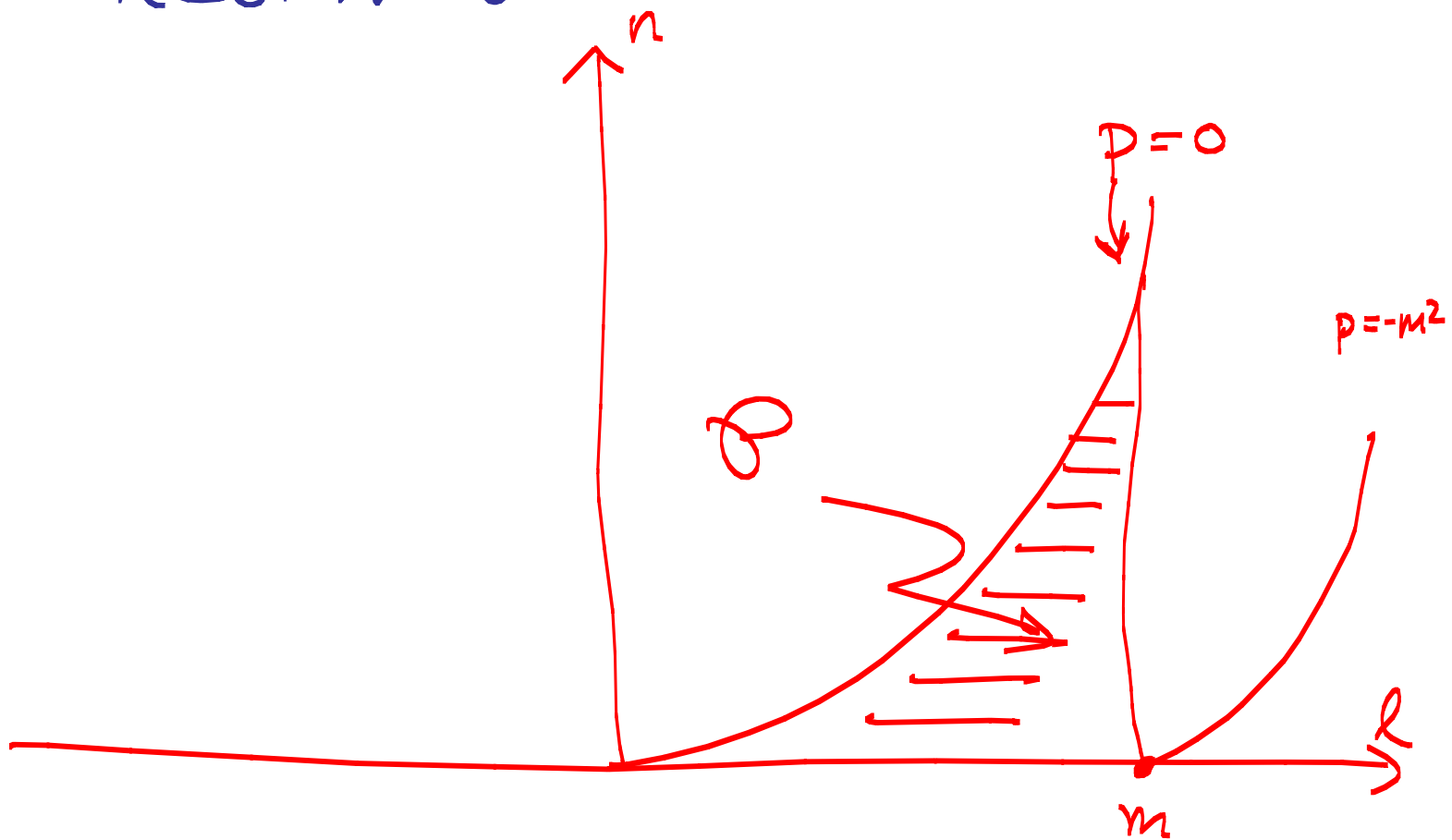
$$= 2(-1)^m \left\{ (1-q)y^m \prod_{m=1}^{\infty} \frac{(1-yq^{m+1})(1-\bar{y}^{-1}q^m)}{(1-q^m)^2} \right. \\ \left. + (y \rightarrow y^{-1}) + \text{NONPOLAR} \right\}$$

- IS THIS COMPATIBLE WITH MODULAR INVARIANCE?
- DOES THERE EXIST SUCH A

$$\chi_{\text{EXT}} \in \hat{J}_{0,m} \quad ?$$

4. COUNTING POLAR POLYNOMIALS

AS I EXPLAINED LAST TIME
THE ELLIPTIC GENUS IS COMPLETELY
DETERMINED BY ITS FOURIER
COEFFICIENTS IN THE POLAR
REGION $\mathcal{P}$



$$p = 4mn - \ell^2$$

Def: IF $\phi = \sum \hat{\phi}(n, l) q^n y^l$ IS
ANY FOURIER SERIES ITS POLAR
POLYNOMIAL IS:

$$\text{Pol}(\phi) := \sum_{(l, n) \in P} \hat{\phi}(n, l) q^n y^l$$

LET US COUNT THE DIMENSION OF

$P_m :=$ VECTOR SPACE OF POLAR POLY'S.

$$p(m) = \dim P_m = \sum_{l=1}^m \left\lfloor \frac{l^2}{4m} \right\rfloor$$

THIS IS THE NUMBER OF LATTICE
POINTS IN THE SHADED REGION P .

IT IS NOT ENTIRELY TRIVIAL
TO EVALUATE, AND WE EXPLAIN IN
THE NOTES THAT

$$P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m)$$

↑
AREA

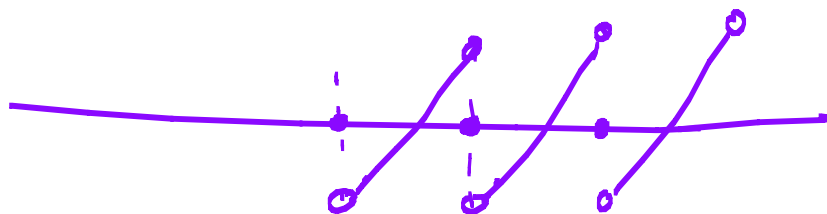
↖
LATTICE CORRECTION

$A(m)$ = # - THEORETIC EXPRESSION
INVOLVING CLASS NUMBERS,
MAIN POINT FOR US IS THAT
FOR LARGE m IT GROWS ROUGHLY
LIKE $O(m^{1/2})$

Proof (time permitting)

Introduce sawtooth function

$$((x)) = x - \frac{1}{2} (\lfloor x \rfloor + \lceil x \rceil) = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha & x = n + \alpha \\ & 0 < \alpha < 1 \end{cases}$$



Write

$$\sum \left\lfloor \frac{r^2}{4m} \right\rfloor = \underbrace{\sum \frac{r^2}{4m}}_{(A)} - \underbrace{\sum ((\frac{r^2}{4m}))}_{(B)} + \frac{1}{2} \underbrace{\sum \left(\left\lceil \frac{r^2}{4m} \right\rceil - \left\lfloor \frac{r^2}{4m} \right\rfloor \right)}_{(C)}$$

$$(A) = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24}$$

$$(C) = \frac{m}{2} + \text{correction when } \frac{r^2}{4m} \in \mathbb{Z} \text{ happens } O(m^{1/2})$$

$$= \frac{m}{2} - \left\lfloor \frac{b}{2} \right\rfloor \quad b = \text{largest integer } b^2 \mid m$$

(B) Roughly, a random walk between $-1/2$ and $+1/2$. So $O(m^{1/2})$. Can express exactly in terms of class numbers.

5. COUNTING WEAK JACOBI FORMS

NOW THE ELLIPTIC GENUS MUST BE A WEAK JACOBI FORM OF WEIGHT ZERO AND INDEX m .

WHAT IS THE DIMENSION OF $\hat{J}_{0,m}$?

THM [EICHLER-ZAGIER] THE BIGRADED RING $\hat{J}_{*,*}$ OF WEAK JACOBI FORMS IS A POLYNOMIAL RING:

$$\hat{J}_{*,*} = \mathbb{C}[E_4, E_6, \phi_{0,1}, \phi_{-2,1}]$$

NOTE THE SIMILARITY TO THE THEOREM WE PROVED ABOUT M_* IN LECTURE I. THE PROOF IS NOT HARD - BUT SKIP IT.

WE THEREFORE HAVE A BASIS FOR $\hat{J}_{0,m}$

$$(\phi_{-2,1})^a (\phi_{0,1})^b \Xi_4^c \Xi_6^d$$

$$* \begin{cases} a+b=m \\ -2a+4c+6d=0 \end{cases} \quad a,b,c,d \in \mathbb{Z}_+$$

$j(m) := \dim \widetilde{J}_{0,m}$ IS THE # OF
SOLUTIONS TO (*)

A STRAIGHTFORWARD COMPUTATION
SHOWS THAT

$$j(m) = \frac{m^2}{12} + \frac{m}{2} + \widetilde{A}(m)$$

$$\widetilde{A}(m) = \left(\delta_{s,0} + \frac{s}{2} - \frac{s^2}{12} \right) ; \quad \begin{matrix} m=s \bmod 6 \\ 0 \leq s \leq 5 \end{matrix}$$

$$\text{BUT } P(m) = \frac{m^2}{12} + \frac{5m}{8} + O(m^{1/2}) !$$

$$P(m) > j(m) \quad \text{FOR } m \geq 5$$

$$P(m) - j(m) = \frac{m}{8} + O(m^{1/2})$$

RECALL THAT WHEN WE
DISCUSSED THE RECONSTRUCTION
FORMULA IN LECTURE II

$$f(\tau) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (j(\gamma\tau))^{-w} f(\gamma\tau) + \text{REG}$$

I SAID THAT IN GENERAL YOU
CANNOT TAKE AN ARBITRARY f^-
AND GET A MODULAR FORM.

THE OBSTRUCTION IS MEASURED
BY A SPACE OF CUSP FORMS

$$0 \rightarrow \tilde{J}_{0,m} \xrightarrow{\text{Pol}} P_m \rightarrow S_{5/2}(\Gamma, M)$$

AS DESCRIBED IN PAPER WITH MANSCHOT.

RECENTLY J. MANSCHOT COMPUTED
THE DIMENSION OF THE OBSTRUCTION
SPACE AND REPRODUCES THE FORMULA
FOR $P(m) - j(m)$. [TO APPEAR.]

6. SEARCH FOR THE EXTREMAL ELLIPTIC GENUS

NOW RETURN TO OUR HYPOTHETICAL EXTREMAL $\mathcal{N}=2$ CFT WITH

$$\chi_{\text{EXT}} = SF_{1/2} \chi_V + S\bar{F}_{-1/2} \chi_V + NP.$$

THERE IS NO GUARANTEE THAT

$$P_{\text{EXT}}^m := \text{Pol}(\chi_{\text{EXT}}) \in \mathcal{I}_m(\widetilde{\mathcal{J}}_{0,m})$$

BUT MAYBE THERE IS MAGIC...

CHOOSE A BASIS ϕ_i FOR $\widetilde{\mathcal{J}}_{0,m}$.

IF THE EXTREMAL $\mathcal{N}=2$ CFT EXISTS THEN

$$\exists x_i \quad \sum_{i=1}^{j(m)} x_i \text{Pol}(\phi_i) = p_{\text{EXT}}^m \quad (*)$$

EVEN IF WE FIND SOLUTIONS
THERE IS A FURTHER TEST SINCE

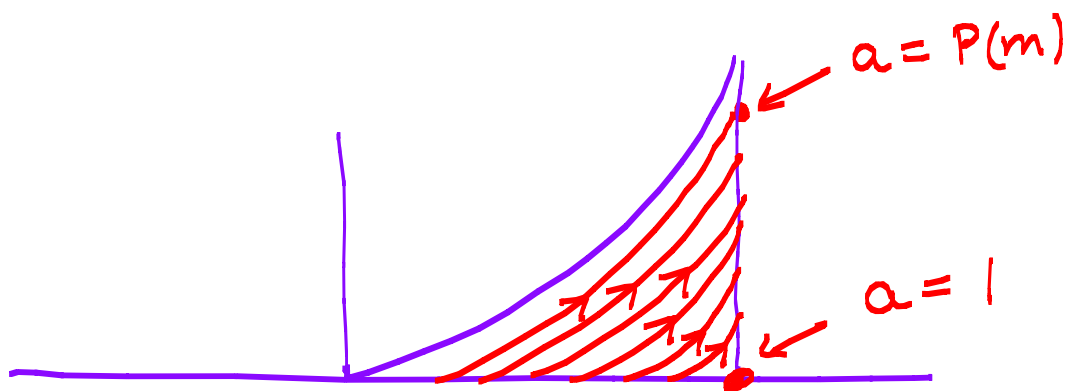
$$\sum_i x_i \phi_i = \sum c(n, l) q^n y^l$$

MUST HAVE $c(n, l) \in \mathbb{Z}$

OF COURSE, THESE ARE NECESSARY,
NOT SUFFICIENT CONDITIONS FOR
THE EXISTENCE OF THE EXTREMAL
THEORY.

TO ANALYZE EQUATION (*)
INTRODUCE A POLARITY-ORDERED
BASIS FOR P_m ,

$$q^{n(a)} y^{l(a)} \quad a = 1, \dots, P_m$$



POLARITY $\nearrow$ AS $a \nearrow$

FOR $a=1$ y^m HAS $p=-m^2$,

DEFINE:

$$\text{Pol } \phi_i := \sum_{a=1}^{P(m)} N_{ia} g^{n(a)} y^{l(a)}$$

$$P_{\text{EXT}}^m := \sum_{a=1}^{P(m)} d_a g^{n(a)} y^{l(a)}$$

EQUATION $\otimes$ IS:

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \quad a=1, \dots, P(m)$$

RECALL: $P(m) > j(m) \quad m \geq 5$

COMPUTER: $1 \leq m \leq 36$

SOLN'S X_i EXIST FOR: $1 \leq m \leq 5, 7, 8, 11, 13$

$\nexists X_i$ FOR $m = 6, 9, 10, 12, 14 \leq m \leq 36$

MOREOVER: IN THE CASES
WHERE SOLUTIONS EXIST THE
 $C(n, l)$ SEEM TO BE INTEGRAL.

(THIS IS NONTRIVIAL. E.G. $m=2$

$$\frac{1}{6} \phi_{0,1}^2 + \frac{5}{6} \phi_{-2,1}^2 \in \mathbb{Z} \quad .)$$

RECENTLY, WE FOUND AN ANALYTIC
ARGUMENT:

FOR m SUFF. LARGE, SOLUTIONS
 X_i DO NOT EXIST

THUS, EXTREMAL $N=2$ THEORIES
AT BEST EXIST FOR A FINITE
"SPORADIC" SET OF m .

7. NEAR EXTREMAL $W=2$ CFT

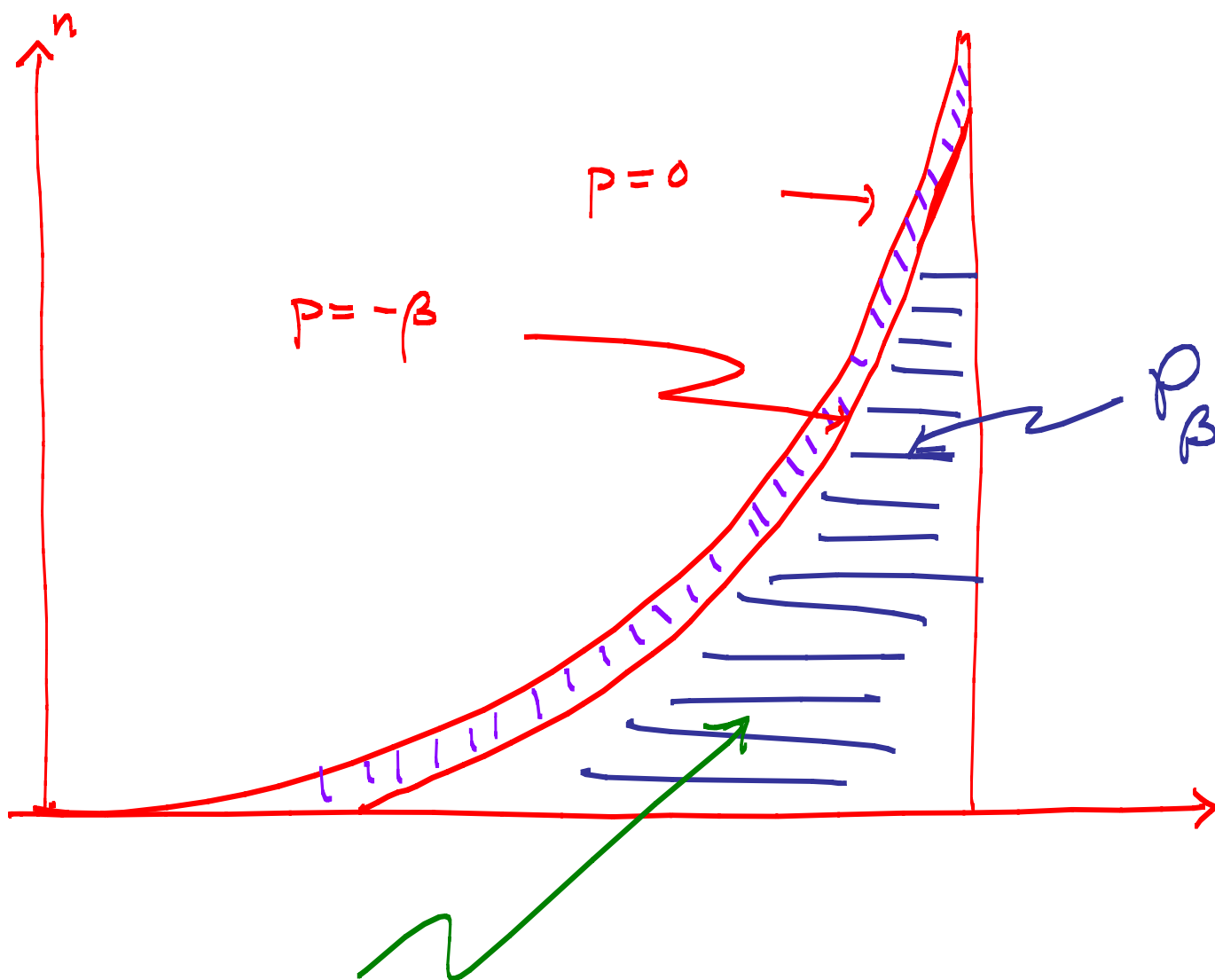
PERHAPS OUR DEFINITION WAS
TOO RESTRICTIVE...

MAYBE THERE ARE QUANTUM
CORRECTIONS TO THE COSMIC
CENSORSHIP BOUND...

LET US DEFINE A

β -EXTREMAL $W=2$ CFT

IF WE ONLY DEMAND AGREEMENT
WITH THE VACUUM CHARACTER
UP TO POLARITY $-\beta$



ONLY DESCENDENTS OF THE VACUUM

NOW WE ONLY TRY TO MATCH
THE POLAR DEGENERACIES IN p_β .

SO WE INCREASE β UNTIL
WE ARE SOLVING

$$\sum_{i=1}^{j(m)} x_i N_{ia} = d_a \quad a=1, \dots, j(m)$$

THIS HAPPENS FOR $\beta = \frac{m}{2} + O(m^{1/2})$

COMPUTER:

- SOLUTIONS x_i DO EXIST!

$$1 \leq m \leq 36$$

- AND

$$\sum_i x_i \phi_i = \sum c(n,l) q^n y^l$$

DO HAVE $c(n,l) \in \mathbb{Z} \dots$

EXCEPT FOR $m=17$!

QUI IN ITALIA IL DICIASSETTE
PORTA SFIGA!

FOR $m=17$, WE MUST LOWER THE
POLARITY CUTOFF STILL FURTHER....

ON THE OTHER HAND,

THERE IS SOME $\beta_*(m)$ SO THAT
WE CAN MATCH X_{vac} FOR ALL
STATES OF POLARITY LESS
THAN $-\beta_*(m)$

THE COMPUTER EVIDENCE SUGGESTS

$$\beta_*(m) \geq \frac{m}{2} + O(m^{1/2})$$

FOR ALL m .

IF TRUE, THEN FOR ANY
 $W=(2,2)$ CFT THERE MUST BE
AN $W=2$ PRIMARY WITH

$$4m\left(h^R - \frac{c}{24}\right) - \ell^2 > -\beta_*(m)$$

THUS WE FORMULATE THE CONJECTURE:

ANY $N=(2,2)$ CFT WITH INTEGRAL
U(1) CHARGES MUST HAVE A STATE
 $\psi_L \otimes \widetilde{\text{BRS}}$ WHERE ψ_L IS AN
 $N=2$ PRIMARY WITH:

$$h^{NS} > \frac{m}{4} + \frac{(J_0^m)^2}{4m} - \frac{1}{8} + O(m^{-1/2})$$

ON THE OTHER HAND - USING
A DIFFERENT METHOD - IT IS
POSSIBLE TO SHOW HOW TO CONSTRUCT
A χ WITH ONLY CONTRIBUTIONS
FROM DESCENDENTS WITH

$$h^{NS} \leq \frac{5m}{16}$$

NOW - IT IS ABOUT TIME I
REVEAL MY REAL MOTIVATION
FOR PURSUING THIS PROBLEM:

IN RECENT YEARS THERE HAS
BEEN MUCH ACTIVITY IN FLUX
COMPACTIFICATION AND MODULI
STABILIZATION. (C.F. L. McAllister
TALKS.) SHAMIT KACHRU
HAS BEEN A DRIVING FORCE IN
THIS DEVELOPMENT.

WITH MANY CHOICES FOR FLUX
THERE ARE, FAMOUSLY, MANY
POSSIBLE COMPACTIFICATIONS

WE EXPECT THERE ARE SOME WITH

- $AdS_3 \times R$

$\Rightarrow \exists$ HOLOGRAPHIC DUAL CFT_2

- $0 < -\frac{\Lambda}{M_{Pl}^2} = + \left(\frac{1}{2M_{Pl}} \right)^2 \ll 1$

$\Rightarrow c = \frac{3\ell}{2G} \gg 1$

- KK LENGTHSCALE OF R IS ORDER c IN ADS UNITS

$\Rightarrow$ "MOST" PRIMARIES HAVE A LARGE GAP FROM THE VACUUM.

THE SPECTRUM RESEMBLES AN EXTREMAL CFT !

8. CONCLUSION

I WILL LEAVE YOU WITH
TWO OBVIOUS OPEN PROBLEMS

- PROVE THE BOUND ON h^N 'S
- DOES THIS BOUND PUT ANY INTERESTING CONSTRAINTS ON AdS_3 FLUX COMPACTIFICATIONS?