

WALL-CROSSING FORMULA FOR BPS STATES & SOME APPLICATIONS

TRIESTE SPRING SCHOOL LECTURE IV
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BASED ON WORK DONE WITH
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1. INTRODUCTION

THE "SPACE OF BPS STATES" HAS BEEN A CENTRAL CONCEPT IN SUSY GAUGE THEORY & STRING THEORY FOR ALMOST 30 YEARS.

TODAY I'LL FOCUS ON RECENT PROGRESS IN UNDERSTANDING PHENOMENA ASSOCIATED TO MARGINAL STABILITY.

1. INTRODUCTION

2. WALL-CROSSING FORMULAE:

3. PHYSICAL DERIVATION

4. D6-D2-D0 SYSTEM

5. D4D2D0 SYSTEM: MODULAR GEN. FUNCTIONS

6. ROUTE TO OSV: ENTROPY ENIGMA &

DEGENERACY DICHOTOMY

7. KONTSEVICH-SOIBELMAN FORMULA

8. OPEN PROBLEMS

A. DEFINING THE "SPACE OF BPS STATES"

FOR DEFINITENESS, WE FOCUS ON THEORIES WITH $d=4$, $\mathcal{N}=2$ SUSY IN (ASYMPTOTIC) MINKOWSKI SPACE $\mathcal{M}_4$

HILBERT SPACE OF ONE-PARTICLE STATES, $\mathcal{H}$, IS A REP. OF THE $d=4$, $\mathcal{N}=2$ ALGEBRA.

$\hat{Z}$: CENTRAL CHARGE OPERATOR

$$\{\hat{Q}_{i\alpha}, \hat{Q}_{j\beta}\} = \delta_{ij} (C\Gamma^\mu)_{\alpha\beta} \hat{P}_\mu + \epsilon_{ij} C_{\alpha\beta} \hat{Z}$$

DECOMPOSE $\mathcal{H} = \bigoplus_{z \in \mathbb{C}} \mathcal{H}_{\hat{Z}=z}$

LEMMA: $E \geq |Z|$ ON $\mathcal{H}_Z$

PROOF: $N = 2 \Rightarrow$

$$\{Q_{i\alpha}, Q_{j\beta}\} = \delta_{ij} (C\Gamma^\mu)_{\alpha\beta} P_\mu + \epsilon_{ij} C_{\alpha\beta} Z$$

THIS IS A 6D SUSY ALGEBRA Q_A ,

$$\{Q_A, Q_B\} = (C\Gamma^M)_{AB} P_M$$

WITH $P_4 + iP_5 = Z$. BUT

$$M^2 = E^2 - \vec{P}^2 - |Z|^2 \geq 0. \quad \blacksquare$$

DEF'N: $\mathcal{H}_{\text{BPS}}$ IS THE SUBSPACE OF $\mathcal{H}$ WHERE $E = |Z|$.

NOW - SPECIALIZE TO **TYPE II**
STRING THEORY ON $M_4 \times X$.

- M_4 IS NONCOMPACT $\Rightarrow$ TO DEFINE THE HILBERT SPACE AS A REP. OF $W=2$ WE MUST SPECIFY BOUNDARY COND'S FOR THE MASSLESS FIELDS:

$$\lim_{\vec{x} \rightarrow \infty} (g_{\mu\nu}, \phi, B_{\mu\nu}, RR) := \underline{\Phi}_\infty \in \tilde{\mathcal{M}}$$

$\mathcal{H}_{\underline{\Phi}_\infty}$: 1-PARTICLE HILBERT SPACE
DEPENDS ON $\underline{\Phi}_\infty$

- GENERALIZED MAXWELL THEORY $\Rightarrow$
 $\mathcal{H}_{\underline{\Phi}_\infty}$ IS GRADED BY ELECTRIC/MAGNETIC
CHARGE SECTORS:

$$\mathcal{H}_{\underline{\Phi}_\infty} = \bigoplus_{\Gamma} \mathcal{H}_{\underline{\Phi}_\infty}^{\Gamma}$$

$$\Gamma \in (\text{TWISTED}) \text{ K-THEORY}(X)$$

K-THEORY TO COHOMOLOGY

PHYSICISTS USUALLY WORK
WITH COHOMOLOGY

$$E \in K^0(X) \longrightarrow \text{ch}(E)\sqrt{\hat{A}} \in H^{\text{ev}}(X, \mathbb{Q})$$

D-BRANES ARE SOURCES:

D6	D4	D2	D0
p^0	$\underline{P}$	Q	q_0
H_6	H_4	H_2	H_0
H^0	H^2	H^4	H^6

Often identify $H^6(X, \mathbb{Z}) \cong \mathbb{Z}$

$$K^0(X) / \text{TORSION} = \text{LATTICE } \Lambda$$

$$\text{Ch}(\mathcal{E}) \sqrt{\hat{A}} \Rightarrow \text{CORRESPONDING LATTICE IN } H^{\text{ev}}(X, \mathbb{Q})$$

Λ HAS A $\natural$ SYMPLECTIC FORM

$$\begin{aligned} \langle \mathcal{E}_1, \mathcal{E}_2 \rangle &= \text{Index } \not{D}_{\mathcal{E}_1 \otimes \overline{\mathcal{E}}_2} \\ &= \int (\text{Ch } \mathcal{E}_1 \sqrt{\hat{A}}) \wedge (\text{Ch } \overline{\mathcal{E}}_2 \sqrt{\hat{A}}) \end{aligned}$$

IN TERMS OF COHOMOLOGY

$$\langle \Gamma, \Gamma' \rangle = \int -p^0 q'_0 + p q' - q p' + q_0 p^0$$

PHYSICALLY: DIRAC-SCHWINGER-ZWANNZ.

DUALITY INVT. PRODUCT OF
ELECTRIC AND MAGNETIC
CHARGES.

NOW WE PUT THESE THINGS TOGETHER:

CONSIDER IIA STRINGS WITH

1. X = STATIC, COMPACT, CY 3-FOLD
2. FLAT B-FIELD: $B \in H^2(X, \mathbb{R})$
3. FLAT RR FIELDS

$\Rightarrow \mathcal{N}=2, d=4$ SUGRA

• EACH $\mathcal{H}_{\Xi_\infty}^\Gamma$ IS A REP OF $\mathcal{N}=2$

• CENTRAL CHARGE $\boxed{Z = Z(\Gamma; \Xi_\infty)}$

SO, WE STUDY THE BPS SPECTRUM

$$\mathcal{H}_{\text{BPS}} = \bigoplus_{\Gamma \in K^0(X)} \mathcal{H}_{\Xi_\infty, \text{BPS}}^\Gamma$$

$\uparrow$
FINITE DIMENSIONAL

B. DEPENDENCE ON MODULI

THE SPACES $\mathcal{H}_{\mathbb{E}_\infty, \text{BPS}}^\Gamma$ ARE
LOCALLY CONSTANT BUT NOT GLOBALLY
CONSTANT AS FUNCTIONS OF $\mathbb{E}_\infty$

MODULI SPACE $\tilde{\mathcal{M}}$ IS A PRODUCT:

HYPERMULTIPLETS $\times$ VECTORMULTIPLETS
[CPLX STR., ϕ , RR FIELDS] [COMPLEXIFIED KÄHLER]

WE WORK AT A GENERIC HYPERMULTIPLY.

RECENT PROGRESS HAS BEEN
CONCERNED WITH THE DEPENDENCE
ON VECTORMULTIPLETS, IN THIS TALK,

$$t = B + iJ$$

• THE JUMPING LOCUS IS REAL
CODIMENSION ONE

DEFINE AN INDEX

$$\Omega(\Gamma; \Xi_\infty) = -\frac{1}{2} \text{Tr}_{\mathcal{H}_{\Xi_\infty, \text{BPS}}^\Gamma} (2J_3)^2 (-1)^{2J_3}$$

(COMPARE A. SEN'S TALK: HE HAD 6TH HELICITY SUPERTRACE.)

● TECHNICAL POINT:

$$\mathcal{H}_{\Xi_\infty, \text{BPS}}^\Gamma = \underbrace{\mathcal{H}_{\frac{1}{2}\text{HM}}}_{\substack{1/2 \text{ hyper} \\ \text{spin rep}^h}} \otimes \mathcal{H}(\Gamma, t_\infty)$$

$2(0) + (\frac{1}{2})$ as

$$\Omega(\Gamma; t_\infty) = \text{Tr}_{\mathcal{H}(\Gamma, t_\infty)} (-1)^F$$

HENCEFORTH FOCUS ON $\mathcal{H}(\Gamma; t_\infty)$

● KEY POINT: Ω CHANGES ACROSS WALLS OF MARGINAL STABILITY

C. WHY DO WE CARE?

PHYSICS MOTIVATION

1. THE MAIN MOTIVATION FOR RECENT WORK IS THE PROGRAM, INITIATED BY STROMINGER-Vafa (1995) OF ACCOUNTING FOR BH ENTROPY VIA MICROSTATE COUNTING. THAT GOAL IS STILL NOT FULLY ACCOMPLISHED.

WE DON'T KNOW BPS DEGENERACY FOR CERTAIN NATURAL CHARGE REGIMES, FOR EXAMPLE:

$$\Gamma \rightarrow \lambda \Gamma \quad \lambda \rightarrow \infty$$

2. OSV CONJECTURE:

RELATION BETWEEN

$$\Omega(\Gamma) \quad \Big| \quad \text{GW/DT/GV INVARIANTS}$$

$\Rightarrow$ NONPTVE TOPOLOGICAL STRING?

MATH MOTIVATION

1. PHYSICAL STABILITY OF BPS STATES IS RELATED TO MATH. STABILITY IN THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES ON A C.Y.: KONTSEVICH, DOUGLAS, BRIDGELAND, THOMAS, PANDHARIPANDE

PHYSICS $\Rightarrow$ PREDICTIONS/CONSTRAINTS ON WHAT WE EXPECT SHOULD BE TRUE.

2. MANY INTERESTING CONNECTIONS TO AUTOMORPHIC FORMS AND ANALYTIC NUMBER THEORY; SOME RELATIONS TO ARITHMETIC CY'S.

3. THERE ARE SEVERAL OTHER MORE SPECULATIVE APPLICATIONS, E.G. BPS ALGEBRAS: GENERALIZING NAKAJIMA'S WORK AND SUGGESTED BY TYPE II/HET DUALITY SHOULD BE CLOSELY RELATED.

2. WALL-CROSSING FORMULAE: STATEMENT

$N=2, d=4$ Algebra $\Rightarrow$

- MODULI OF VACUA $\widetilde{\mathcal{M}}$
- LATTICE OF ELECTRIC/MAGNETIC CHARGES Λ
- CENTRAL CHARGE: $Z: \Lambda \times \widetilde{\mathcal{M}} \longrightarrow \mathbb{C}$

WALLS WHERE $\mathcal{L}_{\text{BPS}}$ MIGHT JUMP

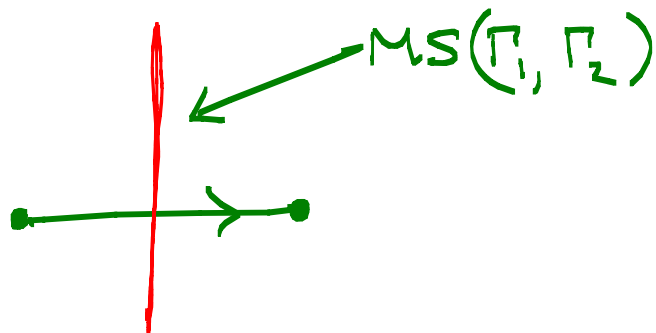
$$MS(\Gamma_1, \Gamma_2) := \{t \mid Z(\Gamma_1, t) = \lambda Z(\Gamma_2, t), \lambda \in \mathbb{R}_+\}$$

$$|Z_1 + Z_2| = |Z_1| + |Z_2|$$

CECOTT, INTRILIGATOR, VAFA ; SEIBERG & WITTEN:

A BOUNDSTATE OF PARTICLES WITH CHARGES

Γ_1, Γ_2 CAN DECAY

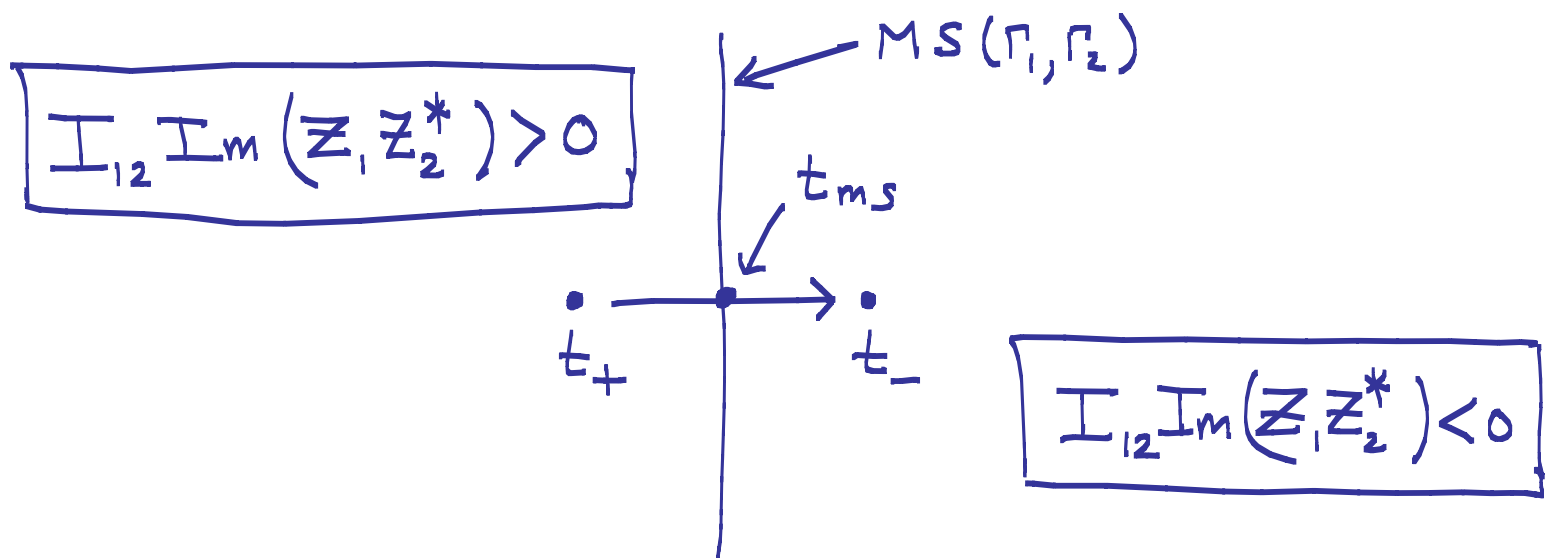


WE WANT TO SAY HOW MANY STATES DECAY.

PRIMITIVE WALL-CROSSING FORMULA:

Λ HAS SYMPLECTIC FORM $\langle \cdot, \cdot \rangle$

LET $I_{12} := \langle \Gamma_1, \Gamma_2 \rangle$



Γ_1, Γ_2 PRIMITIVE, t_{ms} GENERIC $\Rightarrow$

$$\mathcal{H}_+ - \mathcal{H}_- = (J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})$$

$$J_{12} = \frac{1}{2} (|I_{12}| - 1)$$

$$\Delta \Omega = (-1)^{|I_{12}|-1} |I_{12}| \Omega(\Gamma_1, t_{ms}) \Omega(\Gamma_2, t_{ms})$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA

IN ADDITION TO $\Gamma_1 + \Gamma_2$ BOUNDSTATES

WE CAN ALSO FORM $N_1 \Gamma_1 + N_2 \Gamma_2$ BOUNDSTATES

$$MS(\Gamma_1, \Gamma_2) = MS(N_1 \Gamma_1, N_2 \Gamma_2) \quad N_1, N_2 \in \mathbb{Z}_+$$

CONSIDER $N_1 = 1, N_2 \geq 1$:

$$\bigoplus_{N_2} u^{N_2} \Delta \mathcal{H} / \Gamma \rightarrow \Gamma_1 + N_2 \Gamma_2$$

CLAIM: THIS IS A $\mathbb{Z}_2$ -GRADED FOCK SPACE

$$\mathcal{H}(\Gamma_1; t_{ms}) \bigotimes_{k=1}^{\infty} \underbrace{\mathcal{F} \left(u^k \left(\mathcal{J}_{\Gamma_1, k\Gamma_2} \right) \otimes \mathcal{H}(k\Gamma_2; t_{ms}) \right)}_{\text{GRADED SPACE OF OSCILLATORS}}$$

IN PARTICULAR:

$$\begin{aligned} \Omega_1 + \sum_{N \geq 0} u^N \Delta \Omega(\Gamma_1 + N\Gamma_2) &= \\ &= \Omega(\Gamma_1) \prod_{k \geq 0} \left(1 - (-1)^{\langle \Gamma_1, k\Gamma_2 \rangle} u^k \right)^{|\langle \Gamma_1, k\Gamma_2 \rangle|} \Omega(k\Gamma_2) \end{aligned}$$

3. PHYSICAL DERIVATION OF WCF

A. SUPERGRAVITY TOOLS

D-BRANES ARE OBJECTS IN A CATEGORY

IN TYPE IIA/CY, THE SUBCATEGORY OF SUSY BRANES IS PROBABLY THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES.

BUT WE WANT TO DESCRIBE THE (PHYSICALLY) STABLE OBJECTS.

AT WEAK STRING COUPLING, AND $J \rightarrow \infty$
 $\exists$ A BEAUTIFUL DESCRIPTION OF STABLE BPS STATES USING SUGRA.

IN THE SEMICLASSICAL LIMIT

$\psi \in \mathcal{H}_{\text{BPS}}$  BPS SOLUTION OF SUGRA EQUATIONS

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.

ATTRACTOR MECHANISM: (F.K.S. ; STROMINGER)

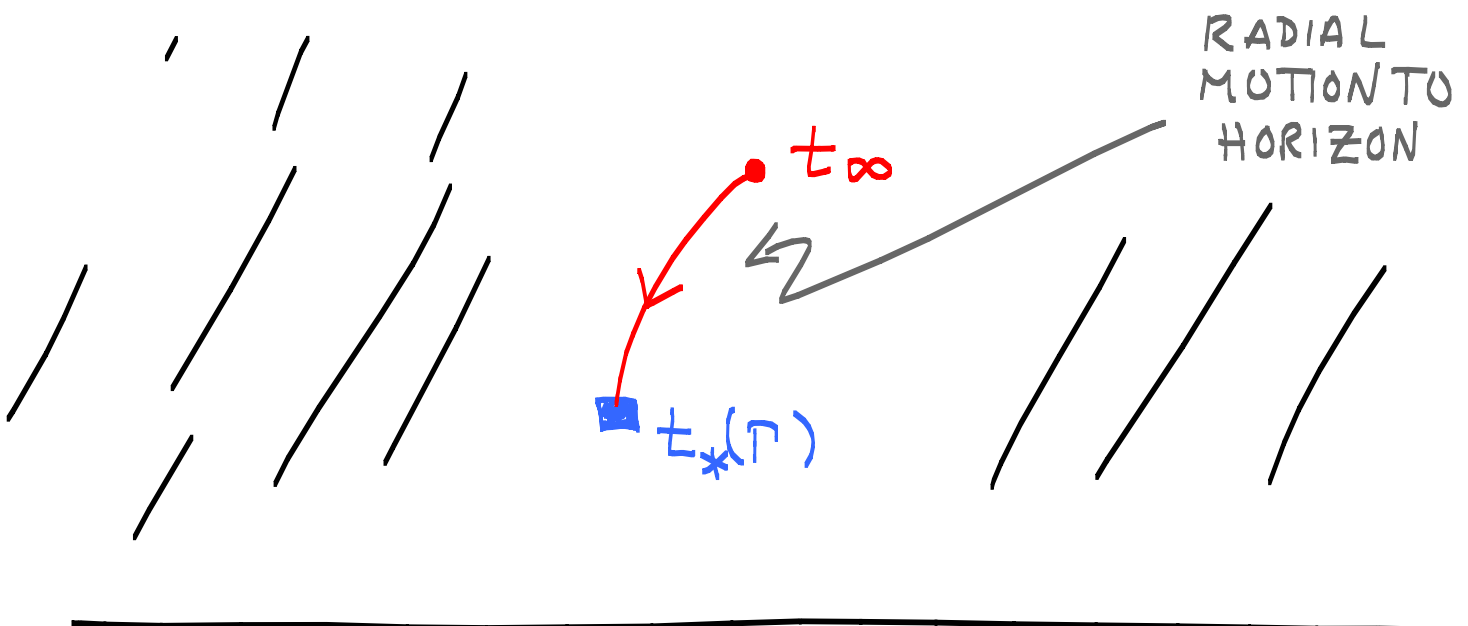
$\Gamma, t_\infty \in$ SPHERICAL SYMMETRY

$\Rightarrow \exists$ AT MOST ONE BPS
SOLUTION OF SUGRA.

IF IT EXISTS

SCALAR FIELDS $t = t(r)$, AND
EVOLUTION FROM $r = \infty$ TO $r = 0$
APPROACHES AN ATTRACTIVE
FIXED POINT $t_*(\Gamma)$:

$\widetilde{\mathcal{M}}_{VM}$



ATTRACTOR FLOW = GRADIENT FLOW FOR

$$\log |Z(\Gamma; t)|^2$$

$$Z = \frac{\langle \Gamma, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}}$$

$$\langle \Gamma, \Gamma' \rangle = \int -p^0 q'_0 + p q' - q p' + q_0 p'_0$$

ω = PERIOD VECTOR

IN LARGE RADIUS APPROXIMATION:

$$\omega = -e^t = -e^{B+itJ}$$

$$Z \approx \frac{\frac{1}{6} p^0 t^3 - \frac{1}{2} p t^2 + Q t - q_0}{\sqrt{(Im t)^3}}$$

BASIC TRICHOTOMY

1. $t_*(\Gamma) \in \text{Interior}(\tilde{\mathcal{M}})$
and $\mathcal{Z}(\Gamma; t_*(\Gamma)) \neq \emptyset$

"REGULAR ATTRACTOR POINT"

2. $\exists$ NONEMPTY SUBVARIETY $\subset \tilde{\mathcal{M}}$
 $\mathcal{Z}(\Gamma; t) = \emptyset$

3. $t_*(\Gamma) \in \partial \tilde{\mathcal{M}}$

(1.) $\exists$ SPHERICALLY SYMMETRIC BPS
BLACK HOLES IN $\mathcal{H}_{\text{BPS}}(\Gamma; t)$ FOR ALL t

(2.) $\mathcal{H}_{\text{BPS}}(\Gamma; t) = \emptyset$ IN AN OPEN
REGION OF THE ZERO LOCUS.

$\mathcal{H}_{\text{BPS}}$ MIGHT BE NONEMPTY FURTHER AWAY

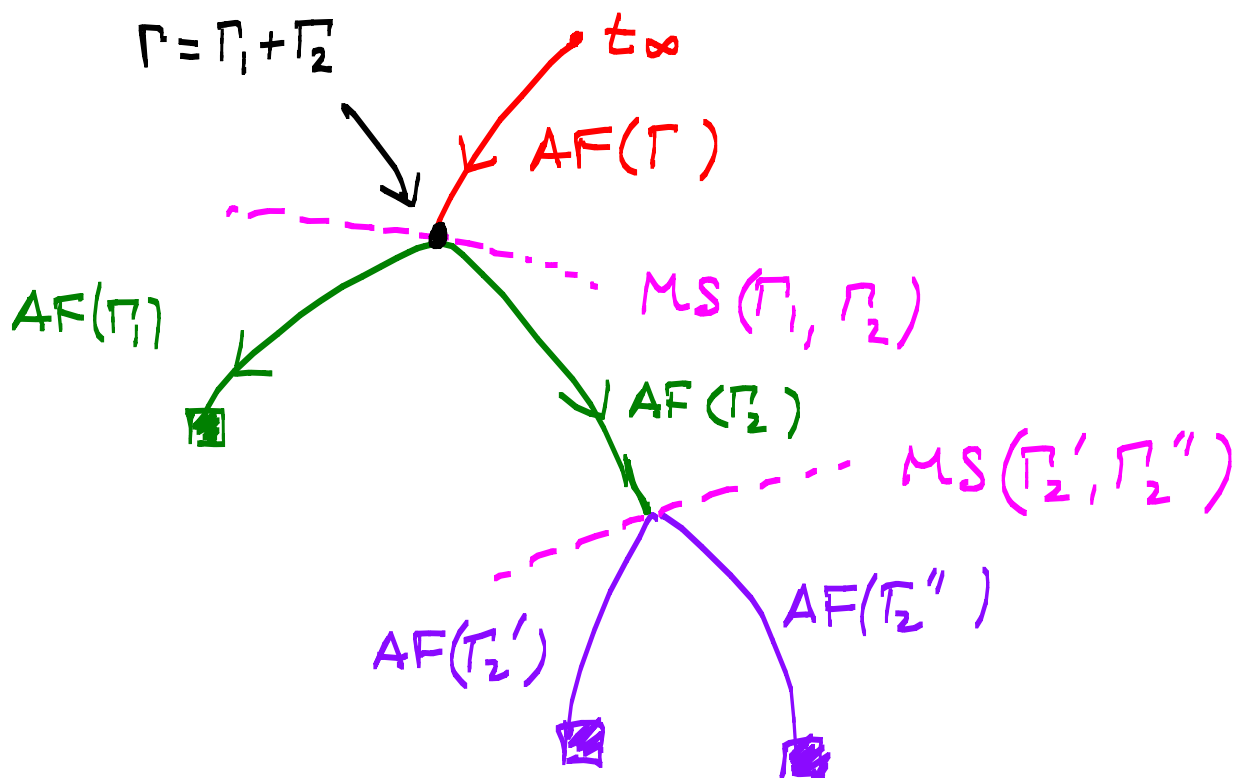
(3.) CANNOT USE SUGRA TO ESTABLISH
EXISTENCE: MUST USE MICROSCOPIC
ARGUMENTS.

B. SPLIT ATTRACTOR FLOWS

IF $\mathbb{Z}(\Gamma; t) = 0$ HAS SOLUTIONS IN THE INTERIOR OF MODULI SPACE THEN USE:

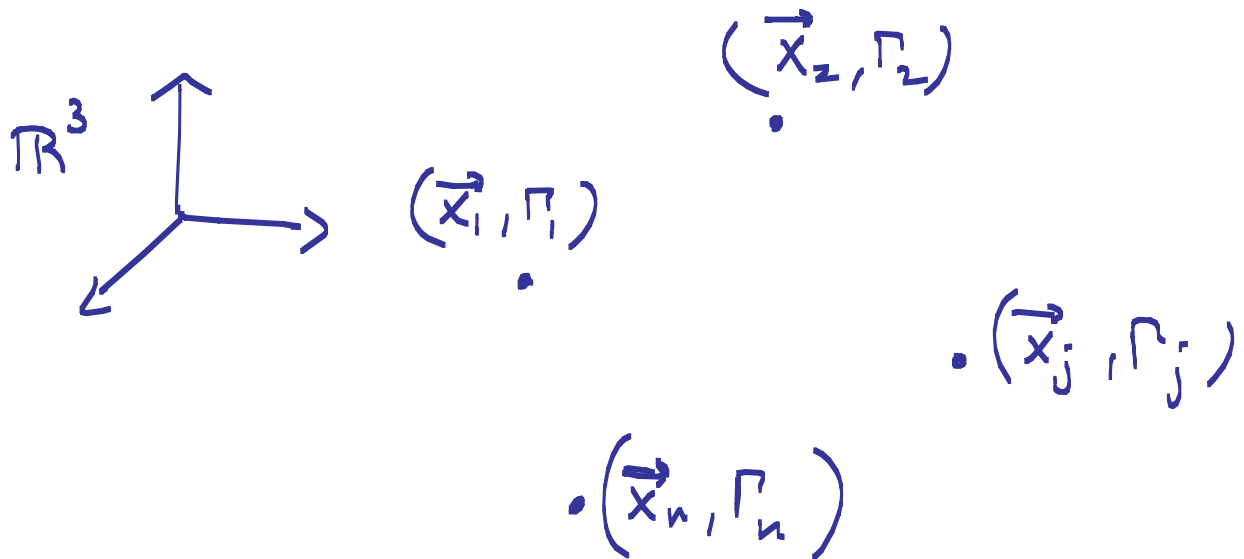
DENEFF'S RULE: $\mathbb{Z}(\Gamma; t) \neq 0 \iff$
 $\exists$ A SPLIT ATTRACTOR FLOW (S.A.F.)

S.A.F.: A PIECEWISE ATTRACTOR FLOW,
JOINED ALONG WALLS OF M.S.,
CONSERVING CHARGE AT THE
VERTICES, TERMINATING ON R.A.P.'S :



- IF SUCH ATTRACTOR FLOW TREES EXIST WE CAN CONSTRUCT A CORRESPONDING SOLUTION OF SUGRA.

- SPACETIME PICTURE:



- NEAR EACH POINT $\vec{x}_i$ THE SOLUTION LOOKS LIKE THE SINGLE-CENTERED SOLUTION: "BLACK-HOLE MOLECULES"

MULTICENTERED SOLUTIONS:

GENERAL BPS EQUATIONS

$$(1.) \quad ds^2 = -e^{2U} (dt + \Theta)^2 + e^{-2U} d\vec{x}^2$$

$$U = U(\vec{x}), \quad \vec{x} \in \mathbb{R}^3$$

(2.) CHOOSE A HARMONIC MAP

$$H: \mathbb{R}^3 \longrightarrow H^{ev}(X, \mathbb{R})$$

$$H(\vec{x}) = \sum_j \frac{\Gamma_j}{|\vec{x} - \vec{x}_j|} + H_\infty$$

$$\boxed{2e^U \operatorname{Im}(e^{-i\alpha} \omega) = -H(\vec{x})} \Rightarrow$$

(a.) $U(\vec{x})$ completely fixed,

$$(b.) \quad e^{-2U(\vec{x})} = S(H(\vec{x}))$$

$$(3.) \quad *_3 d\mathbb{H} = \langle dH, H \rangle$$

$\Rightarrow$ INTEGRABILITY CONDITION:

$$\sum_{\substack{j \\ j \neq i}} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2 \operatorname{Im} \left(e^{-i\alpha} Z(\Gamma_i) \right)_{\infty}$$

SUGRA SOLUTION EXISTS $\iff$

$\forall \vec{x} \in \mathbb{R}^3:$

$t(\vec{x}) \in \mathcal{M}_{\text{VM}} \quad \begin{matrix} | \\ \varepsilon \\ | \end{matrix}$

$$\pi e^{-2U(\vec{x})} = S(H(\vec{x})) \geq 0$$

(A VERY NONTRIVIAL CONDITION
TO CHECK ...)

SPLIT ATTRACTOR CONJECTURE (DENEFF)

- (a.) (COMPONENTS OF MODULI OF) MULTICENTERED SOLUTIONS ARE IN $1 \leftrightarrow 1$ CORRESPONDENCE WITH S.A.F.'S.
- (b.) FOR A FIXED (t_∞, Γ) THERE ARE A FINITE NUMBER OF S.A.F.'S

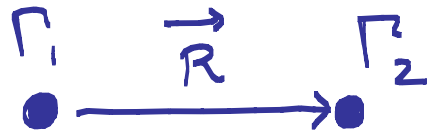
- USEFUL BECAUSE CHECKING $S(H(\vec{x})) > 0$ IS DIFFICULT
- $\mathcal{H}_{BPS}$ IS PARTITIONED BY SPLIT ATTRACTOR FLOWS
- $\exists$ SOME INTERESTING OPEN

PROBLEMS HERE

- * QUANTUM MIXING BETWEEN DIFFERENT TREES
- * USEFUL EXISTENCE CRITERION FOR SCALING SOLUTIONS.

C. DERIVATION OF PRIMITIVE WCF:

CONSIDER BOUNDSTATE OF TWO
PRIMITIVE CHARGES:

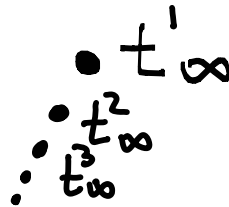
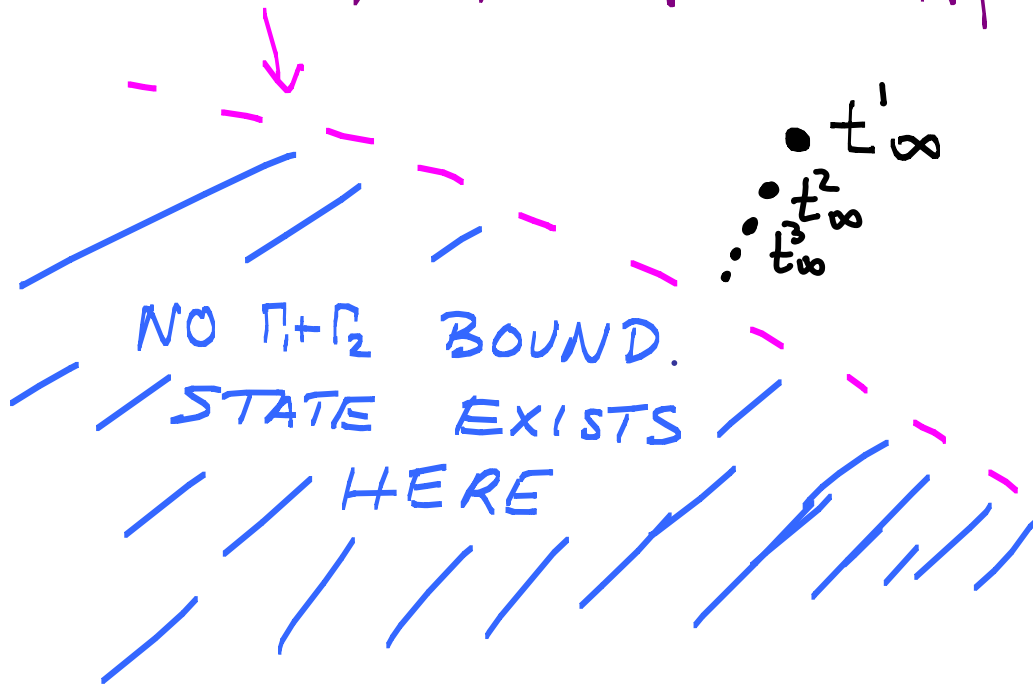


$$R = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|z_1 + z_2|_\infty}{\text{Im}(z_1, \bar{z}_2)_\infty}$$

- NOTE: $\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(z_1, \bar{z}_2)_\infty > 0$
- NOTE THAT BY CHANGING t_∞
WE CAN MAKE $\text{Im}(z_1, \bar{z}_2)|_{t_\infty} \rightarrow 0$
WHILE $|z_1 + z_2|_{t_\infty} \neq 0$

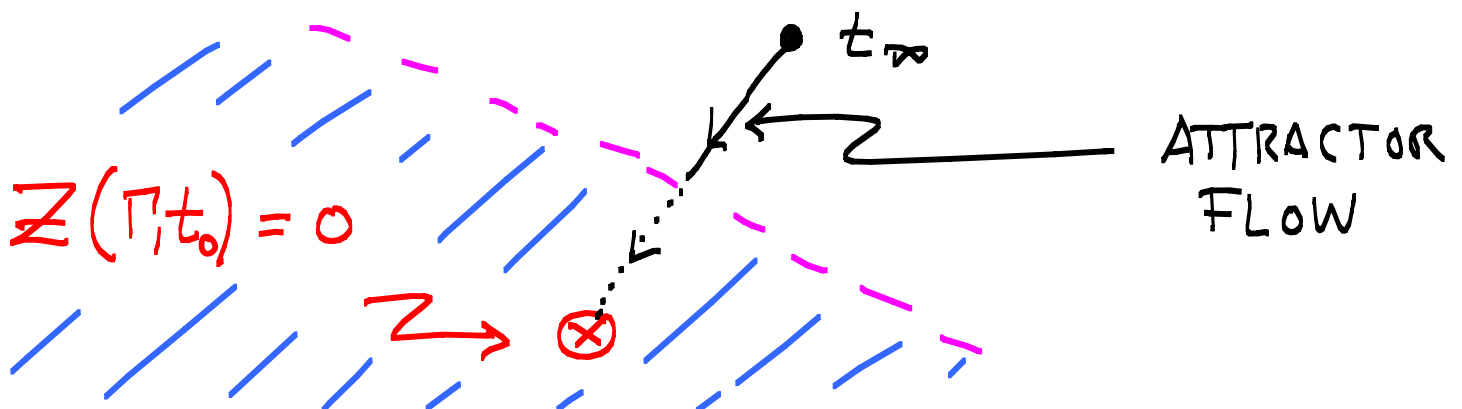
ILLUSTRATES THE KEY POINT
OF MARGINAL STABILITY:

$$MS(\Gamma_1, \Gamma_2) := \{t \in \mathcal{M}_{VM} \mid \frac{z_1}{z_2} \in \mathbb{R}_+\}$$



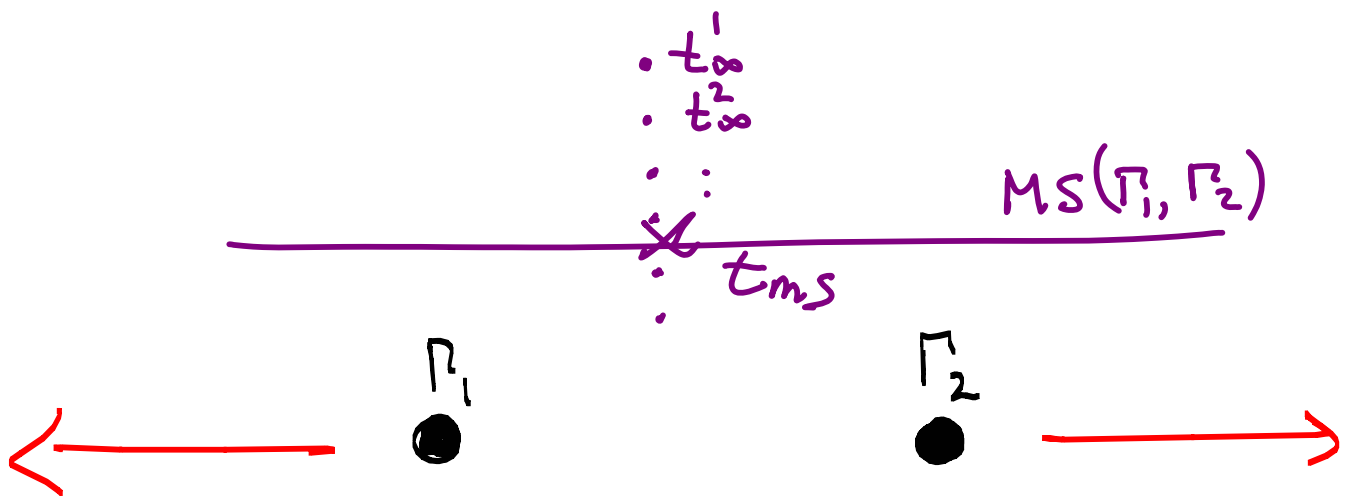
CHANGE BC'S
③ $r = \infty \Rightarrow$
 $R_{1,2} \rightarrow \infty$

IF $Z(\Gamma; t)$ HAS A ZERO THEN
THERE IS NO BOUNDSTATE OF TYPE $\Gamma_1 + \Gamma_2$
IN THE BLUE REGION.



MACROSCOPIC ARGUMENT FOR WCF:

$$R_{12} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\text{Im}(Z_1 \bar{Z}_2)_\infty}$$



ELECTROMAGNETIC FIELD OF TWO DYONS
HAS SPIN:

$$J_{12} = \frac{1}{2} (K_{\Gamma_1, \Gamma_2} - 1) \quad \text{quantum correction}$$

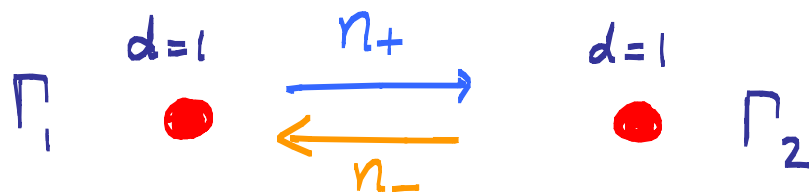
LOCALITY $\Rightarrow$ FOR Γ_1, Γ_2 PRIMITIVE:

STATES LOST FROM $\mathcal{H}(\Gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})$$

MICROSCOPIC ARGUMENT FOR WCF:

WHEN $\vartheta = \arg z_2/z_1 \rightarrow 0$, MODEL
LIGHT D.O.F BY A QUIVER GAUGE THRT:



TRANSLATION TO SUPERGRAVITY:

STABILITY DATA: $(\vartheta, -\vartheta)$

$$n_+ - n_- = \mathbb{I}_{12}$$

GENERICALLY $n_+ = 0$ or $n_- = 0$.

SUPPOSE $n_- = 0$:

$$\vartheta > 0 \quad \mathcal{M} = \mathbb{CP}^{n_+-1}$$

$$\vartheta < 0 \quad \mathcal{M} = \emptyset$$

$$\Delta \mathcal{H} = H^*(\mathbb{CP}^{n_+-1})$$

$$\text{spin}(3) \approx \text{Lefschetz}$$

QUIVER QUANTUM MECHANICS

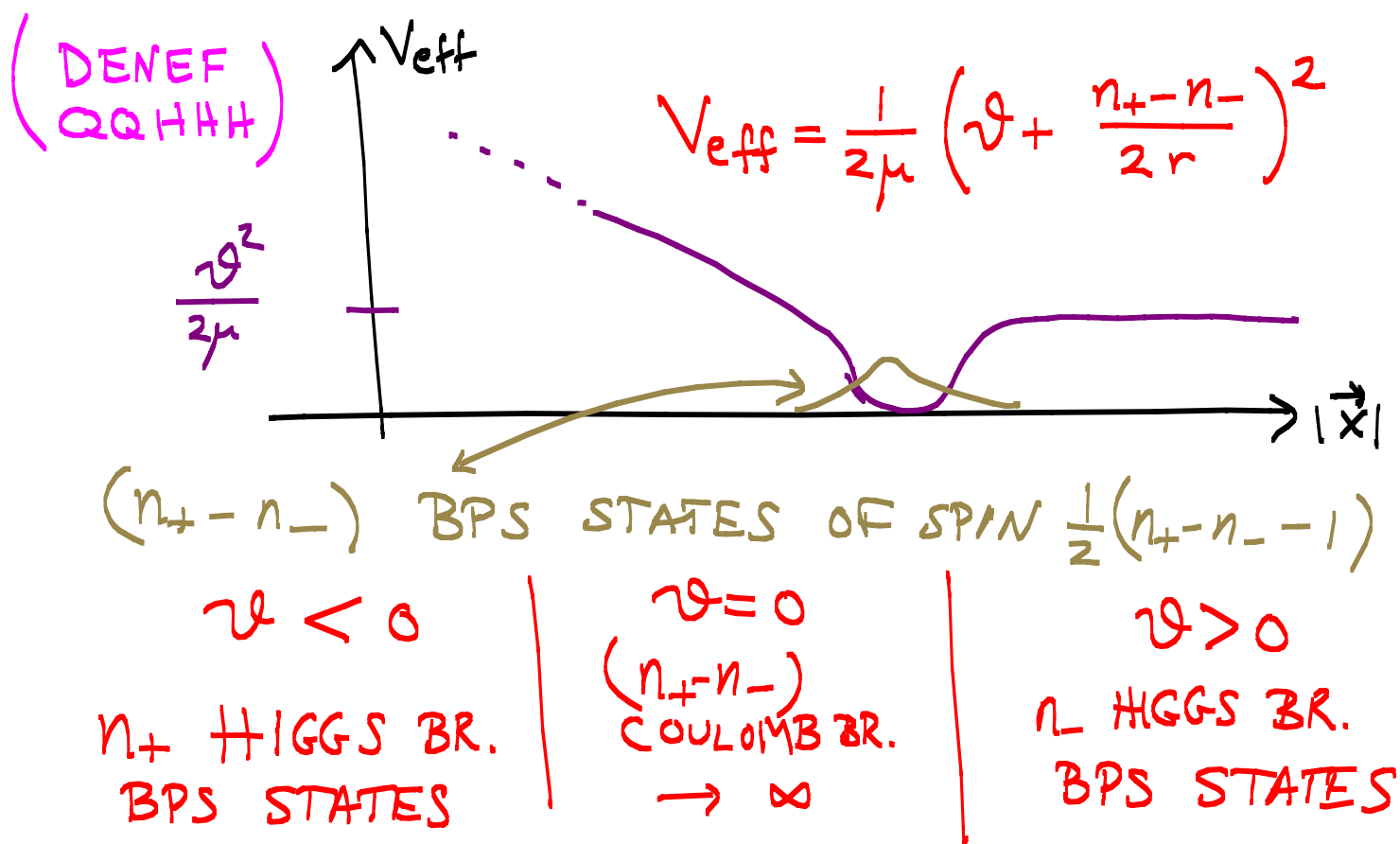
0+1 SUSY QED WITH

1 VM $(A_0, \vec{x}, \lambda)$

$n_{\pm}$ CM's $(\vec{\Phi}_{\pm}, \vec{\Psi}_{\pm})$ CHARGE ± 1

SMALL $|\langle \vec{x} \rangle| \Rightarrow$ HIGGS BRANCH = MODULI OF STABLE QUIVER REPS

LARGE $|\langle \vec{x} \rangle| \Rightarrow$ INTEGRATE OUT $\vec{\Phi}_{\pm} \Rightarrow$



D. DERIVATION OF SEMI-PRIMITIVE WCF

HALO STATES

SUPPOSE $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$,

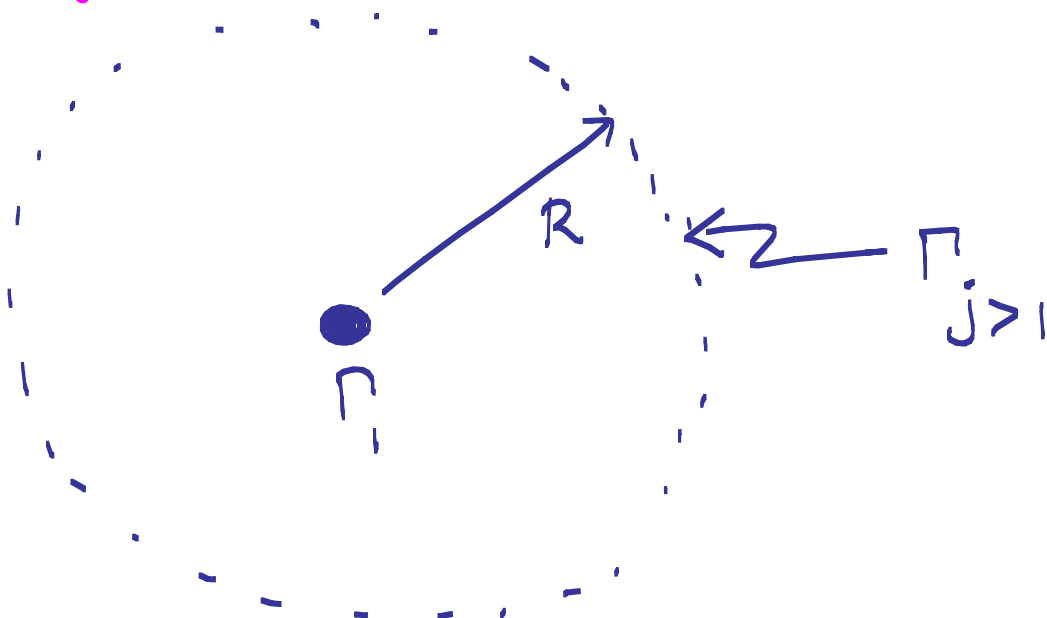
$$\Gamma_j = \lambda_j \Gamma_2 \quad \lambda_j > 0, j=2, \dots, N$$

ARE ALL MUTUALLY LOCAL

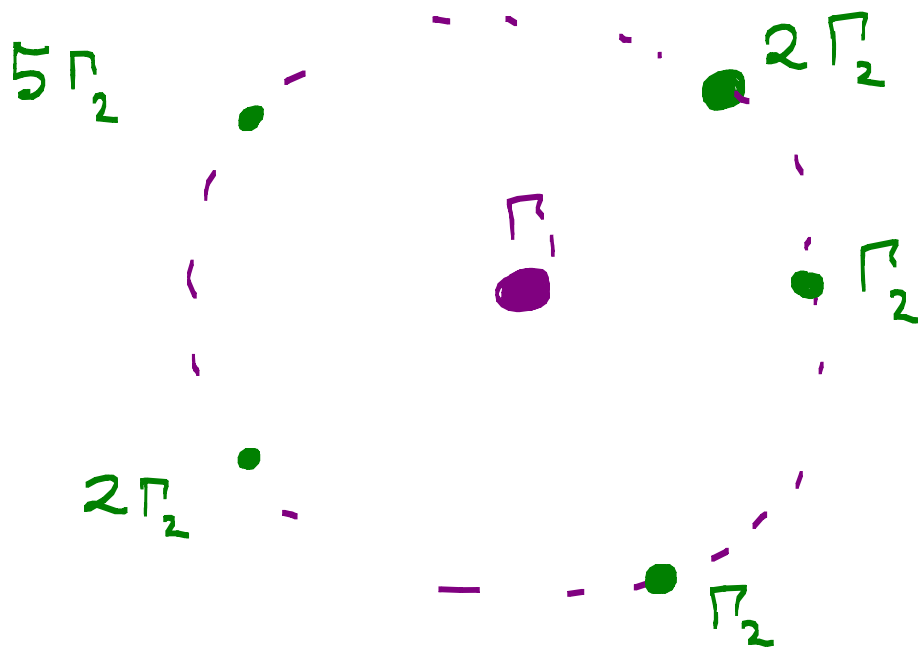
INTEGRABILITY CONDITIONS SAY

$$j \geq 2: \quad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|\vec{x}_j - \vec{x}_1|} = 2 \frac{\text{Im}(z(\Gamma_j) \overline{z(\Gamma_1)})}{|z(\Gamma_1)|}$$

$\Rightarrow$ ALL $|\vec{x}_j - \vec{x}_1|$ ARE EQUAL



CROSS $MS(\Gamma_1, \Gamma_2)$: HALO RADIUS $\nearrow \infty$



THE PARTICLES IN THE HALO
GENERATE A FOCK SPACE WITH

$(J_{\Gamma_1, k\Gamma_2}) \otimes \mathcal{H}(k\Gamma_2; t_{ms})$ CREATION
OPERATORS OF
CHARGE $k\Gamma_2$

ALL WALLS $W(\Gamma_1, N\Gamma_2)$ COINCIDE $\Rightarrow$
CROSSING A WALL WE LOSE ENTIRE
FOCK SPACE:

$$\Omega(\Gamma_1) + \sum_{N \geq 1} \Delta\Omega(\Gamma_1 \rightarrow \Gamma_1 + N\Gamma_2) u^N$$

$$= \Omega(\Gamma_1) \prod_{k > 0} \left(1 - (-1)^{k \langle \Gamma_1, \Gamma_2 \rangle} u^k \right)^{|\langle \Gamma_1, k\Gamma_2 \rangle|} \Omega(k\Gamma_2)$$

4. D6D2D0 SYSTEM

AN IMPORTANT AND USEFUL EXAMPLE IS THE SYSTEM OF 1 D6 BRANE WRAPPING X , BOUND TO D2 & D0 BRANES IN X .

$$\underset{\text{D6}}{H^0} \oplus \underset{\text{D4}}{H^2} \oplus \underset{\text{D2}}{H^4} \oplus \underset{\text{D0}}{H^6} \ni \Gamma = (p^0, P, Q, q_0)$$

CONSIDER: $\Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n)$

$\beta = \text{P.D.}[\sigma] \quad \sigma \subset X \quad \text{HOLOMORPHIC CURVE}$

CHARGE OF (THE DUAL OF) AN IDEAL SHEAF:

$$\text{ch } \mathcal{G} \sqrt{\hat{A}} = 1 - \beta + ndV$$

CONSIDER BINDING THESE

TO D2D0 PARTICLES WITH CHARGE:

$$\Gamma_h = (0, 0, -\beta_h, n_h)$$

PLOT MARGINAL STABILITY CURVE

$$\mathbb{Z}(\Gamma(\beta, n); t) = \lambda \mathbb{Z}(\Gamma_n; t) \quad \lambda \in \mathbb{R}_+$$

$$\mathbb{Z}(\Gamma, t) = \frac{\langle \Gamma, \omega \rangle}{\sqrt{\langle \omega, \omega^* \rangle}}$$

$$\text{SUGRA REGIME: } \Omega = -e^t$$

$$t = B + iJ$$

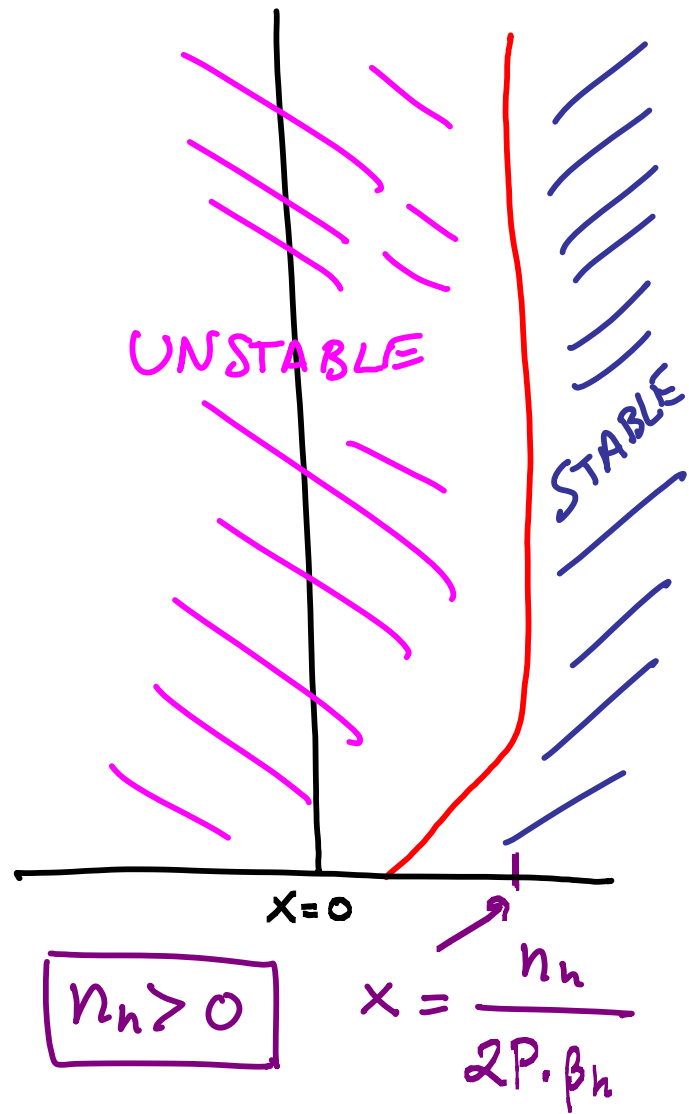
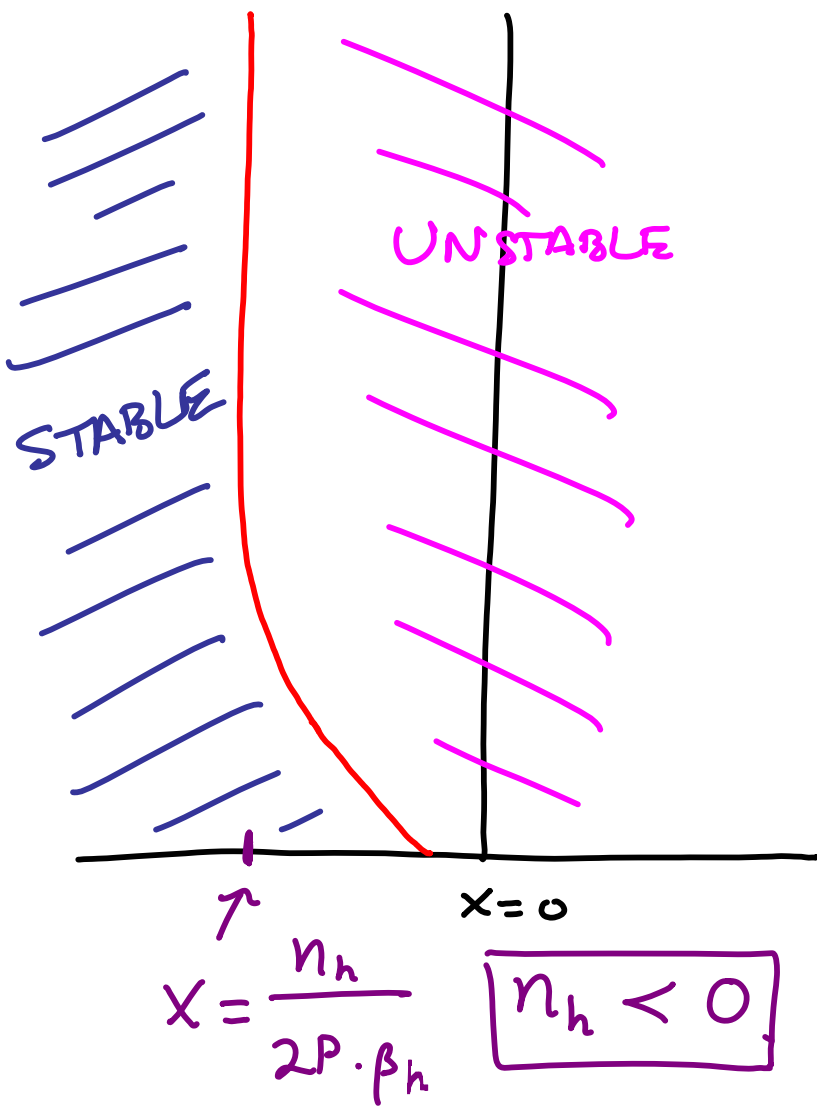
$$\boxed{\frac{t^3}{6} - \beta \cdot t - n = \lambda (-\beta_n \cdot t - n_n)} \quad \lambda \in \mathbb{R}_+$$

THESE WALLS EXTEND TO ∞ IN
THE KÄHLER CONE!

SET $t = z \cdot \bar{z}$

$\bar{z} \in \mathbb{R}$

$z = x + iy$



CONSIDER THE HALO BOUNDSTATES
WITH CENTRAL PARTICLE $\Gamma(\beta, n)$ AS
WE INCREASE THE B-FIELD

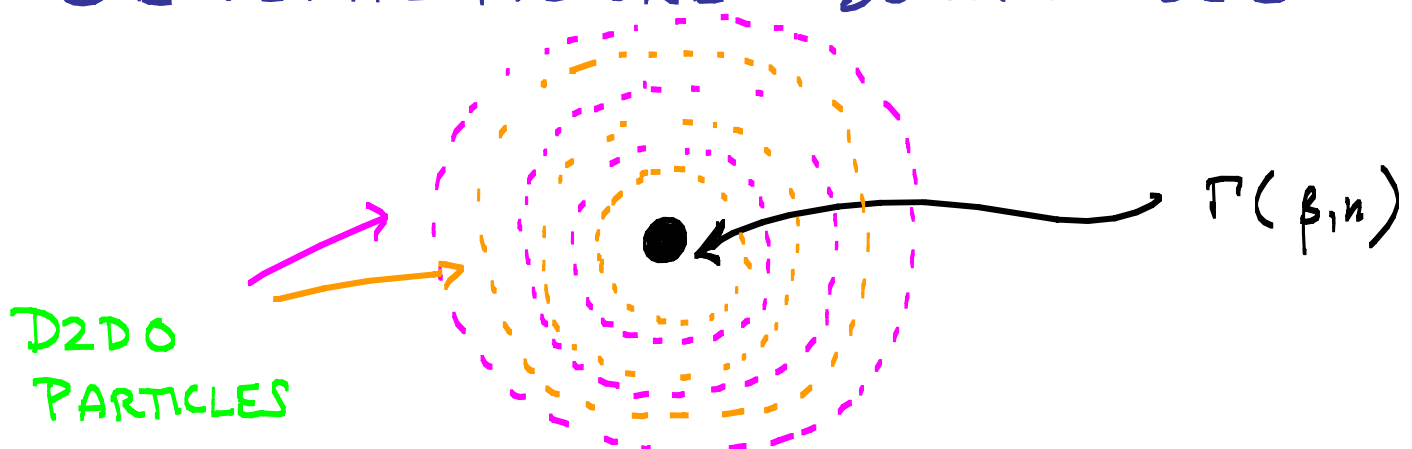
$$B = x P \quad x \text{ INCREASES}$$

HALOS OF D2DO PARTICLES $(0, 0, -\beta_n, n_h)$.
APPEAR & DISAPPEAR.

FOR $x > 0$

ALL $n_h < 0$ STATES HAVE DECAYED.
AS $x \rightarrow +\infty$ WE MOVE INTO THE STABLE
REGION FOR ALL $n_h > 0$, AND EVER
LARGER "ATOMS" BECOME STABLE

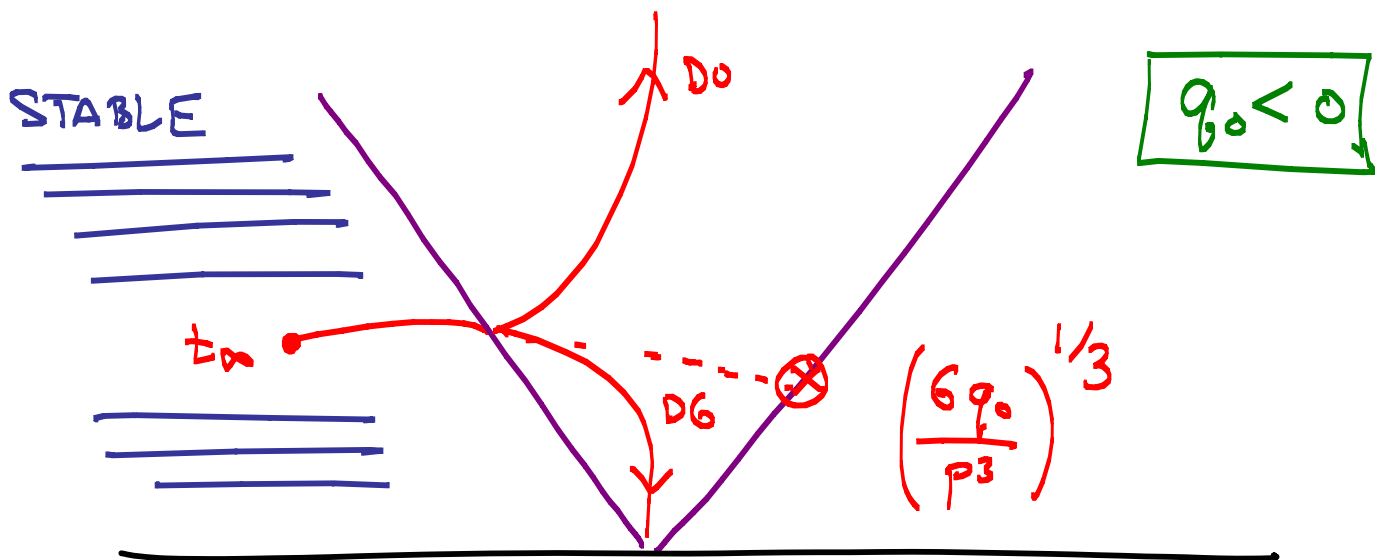
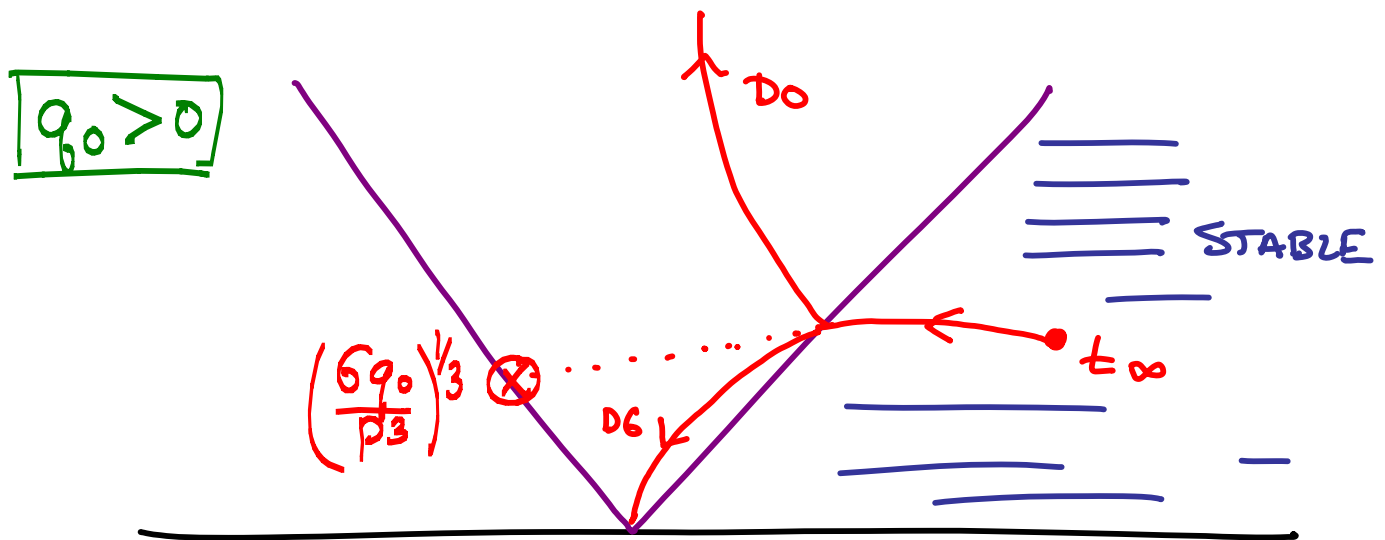
GENERAL PICTURE: BOHR MODEL



WHEN $\beta_h = 0$ WALLS LOOK DIFFERENT

$$\Gamma = \underbrace{1}_{\Gamma_1} + \underbrace{q_0 dV}_{\Gamma_2} \quad Z = \frac{t^3}{6} - q_0$$

SET $t = (x+iy)P \Rightarrow \text{ZERO @ } z = \left(\frac{6q_0}{p^3}\right)^{1/3} P$



INTRODUCE GENERATING FUNCTION

$$Z_{\text{D6D2D0}}(u, v; t) := \sum_{n, \beta} \Omega(\Gamma(\beta, n); t) u^n v^\beta$$

SEMI-PRIMITIVE WALL-CROSSING FORMULA:

CONTRIBUTION OF FOCK SPACE GENERATED
BY $\Gamma_h = -\beta_h + n_h dV$ CROSSING INTO
STABLE REGION:

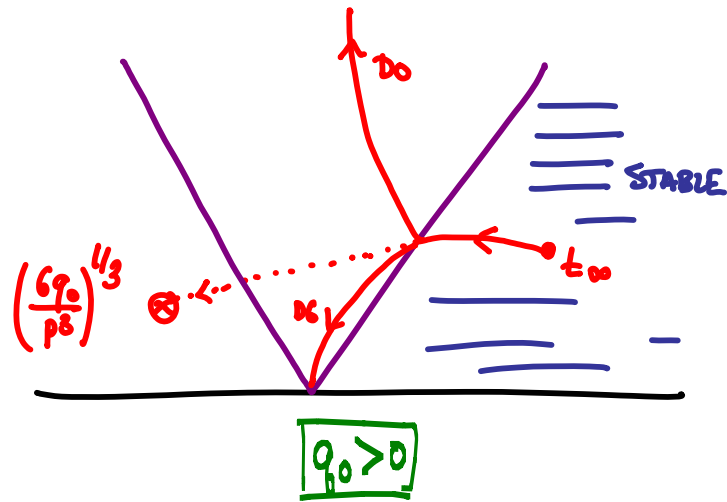
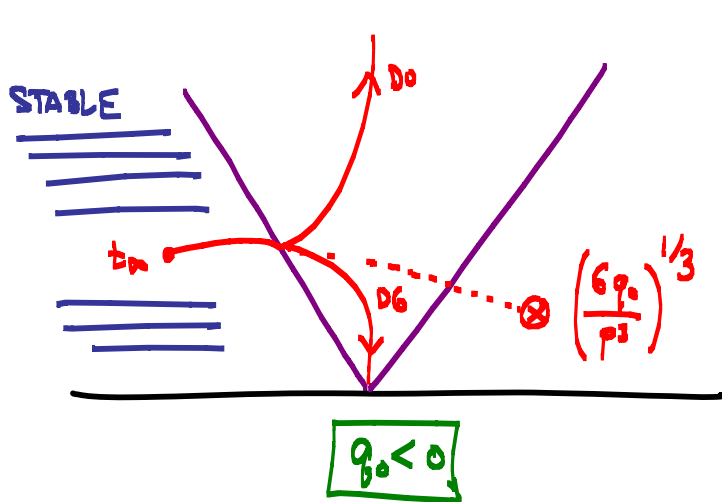
$$Z_{\text{D6D2D0}} \rightarrow \left(1 - (-u)^{n_h} v^{\beta_h}\right)^{|n_h| n_{\beta_h}^0} Z_{\text{D6D2D0}}$$

$$\begin{aligned} \Omega(-\beta_h + n_h dV) &= \sum_{m_L, m_R} (-1)^{2m_L + 2m_R} N_{\beta_h}^{m_L m_R} \\ &= n_{\beta_h}^0 \end{aligned}$$

"SPIN ZERO GV INVARIANT" ($\beta_h \neq 0$)

EXAMPLE: D6D0

$$Z_{D6D0}(u) = \sum \Omega(1 + q_0 dV; t) u^{q_0}$$



$$\Omega(q_0 dV) = -\chi(x)$$

$$Z_{D6D0}(u) = \begin{cases} (M(-u))^{\chi(x)} & \arg z < \frac{\pi}{3} \\ 1 & \frac{\pi}{3} < \arg z < \frac{2\pi}{3} \\ (M(-\bar{u}^{-1}))^{\chi(x)} & \frac{2\pi}{3} < \arg z \end{cases}$$

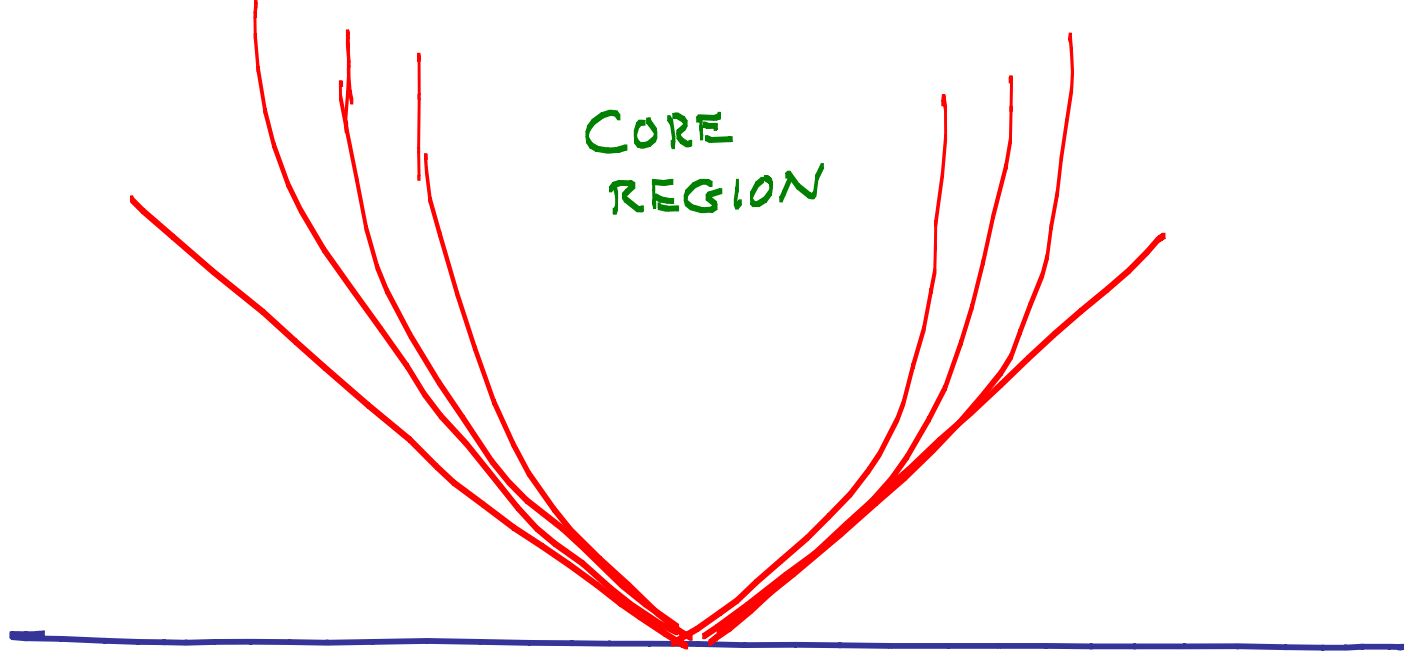
$$M(u) := \prod_{k \geq 1} (1 - u^k)^{-k}$$

SIMILARLY, WALL-CROSSINGS FOR
 THE FULL Z_{D6D2D0} AS $x \rightarrow \infty$
 BUILD UP AN INFINITE PRODUCT
SIMILAR TO THE INFINITE
 PRODUCT FORM OF $Z_{DT}(u, v)$

ON THE OTHER HAND, AN ARGUMENT
 FROM M-THEORY [Dijkgraaf, Verlinde, Vafa; Denef, Moore]
 IMPLIES:

$$\lim_{x \rightarrow +\infty} Z_{D6D2D0}(u, v; z^P) = Z_{DT}(u, v)$$

$$\lim_{x \rightarrow -\infty} Z_{D6D2D0}(u, v; z^P) = Z_{DT}(\bar{u}', v)$$



- STATES IN CORE REGION ARE COMPLICATED BOUND STATES
- PRODUCT OF WALL-CROSSINGS $\Rightarrow$

$$Z_{DT}^{l, r=0}(u, v) = \prod_{\beta > 0, k > 0} \left(1 - (-u)^k v^\beta \right)^{k n_\beta^0}$$

- LIMIT FOR $x \rightarrow +\infty$:

$$Z_{DT}'(u, v) = \underbrace{Z_{DT}^{l, r=0}(u, v)}_{\text{HALOS}} \underbrace{Z_{DT}^{l, r>0}(u, v)}_{\text{CORES}}$$

$$Z_{DT}^{l, r>0}(u, v) = \prod_{\substack{\beta > 0, k > 0 \\ r > 0}} \prod_{l=0}^{2r-2} \left(1 - (-u)^{r-l-1} v^\beta \right)^{(-1)^{r+l} \binom{2r-2}{l} n_\beta^r}$$

5. THE D4-D2-D0 SYSTEM: MODULARITY

NOW CONSIDER $p^0 = 0$

$$\Gamma = P + Q + q_0 dV$$

REGULAR ATTRACTOR POINT:

P IN KÄHLER CONE $\wedge \hat{q}_0 < 0$

$$\hat{q}_0 := q_0 - \frac{1}{2} (D_{ABC} P^C)^{-1} Q_A Q_B$$

THESE ARE BLACK HOLES:

$$\text{HORIZON AREA} = 4 S(\Gamma) = 4\pi |Z_*(\Gamma)|^2$$

$$S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0 \chi(P)}$$

$$\chi(P) := P^3 + c_2 \cdot P > 0 \text{ FOR } P \in \text{KÄHLER CONE}$$

EXPECT: $\log \Omega(\Gamma; t) \sim S(\Gamma)$
FOR "LARGE" Γ AND "LARGE" $\text{Im} t$

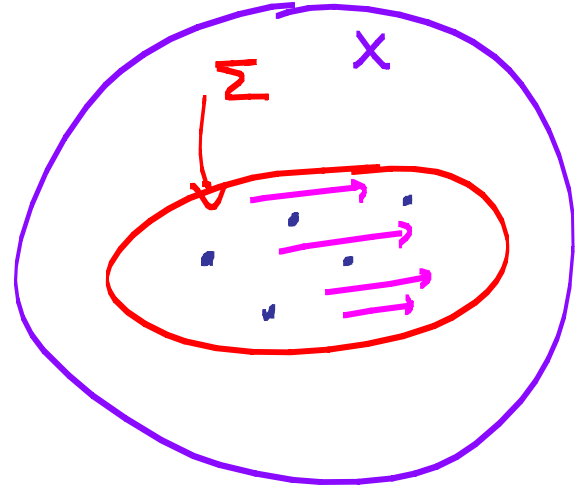
A. ROUGH MICROSCOPIC DESCRIPTION

FOR LARGE J : SINGLE D4 WRAPS $\Sigma \in |P|$

$$\chi(P) = P^3 + c_2 \cdot P = \text{EULER CHARACTER OF } \Sigma$$

FLUX $F \in H^2(\Sigma, \mathbb{Z})$

AND $N \overline{D0}$'s



COMPUTE INDUCED RR CHARGES:

D2: $Q = (2\Sigma)_*(F)$

Do: $\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$

SUSY $\Rightarrow N \geq 0, F^{2,0} = 0 \Rightarrow (F^-)^2 \leq 0 \Rightarrow$

$$\hat{q}_0 \leq (\hat{q}_0)_{\max} = \frac{\chi(P)}{24}$$

$\mathcal{M}(P, F, N) :=$ MODULI OF SUCH $D4$ 'S

$$\mathrm{Hilb}^N(\Sigma) \hookrightarrow \mathcal{M}(P, F, N)$$

ROUGHLY:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Sigma & \hookrightarrow & \{\Sigma \in |\mathcal{P}| \mid F \in H^{1,1}(\Sigma)\} \\ \text{Smooth} & & \end{array}$$

//
MODULI OF STABLE OBJECTS E
IN THE DERIVED CATEGORY
WITH SPECIFIED CHERN CLASSES //

$$\mathrm{Ch} E \sqrt{\hat{A}} = P + Q + q_0 \quad (*)$$

$$= \bigcup_{\substack{F, N \\ \text{s.t.} \\ (*)}} \mathcal{M}(P, F, N)$$

B. INDEX OF BPS STATES

$$"\Omega(\Gamma)_{\infty} := \lim_{J \rightarrow \infty} \Omega(\Gamma; B+iJ)"$$

$$d(F, N) := (-1)^{\dim \mathcal{M}} \chi(\mathcal{M}(P, F, N))$$

$$\Omega(\Gamma)_{\infty} = \text{FINITE SUM OF } d(F, N)$$

SURPRISE: WHEN $h''(x) > 1$ THERE ARE SPLITTINGS @ ∞ :

$$\Gamma = P + Q + q_0 dV$$

$$= (P' + Q' + q'_0 dV) + (P'' + Q'' + q''_0 dV)$$

$$\text{WITH: } \sqrt{-\hat{q}_0'' (P'')^3} > \sqrt{-\hat{q}_0 P^3}$$

$\Rightarrow$ EVEN THE LEADING ORDER

ENTROPY IS CHAMBER DEPENDENT

[E. ANDRIYASH + G.M.]

• FOR $\Gamma = P + Q + q_\bullet dV$,

$P \in \text{KÄHLER CONE}$, $\exists$ DISTINGUISHED
CHAMBER:

$$\Omega(\Gamma)_\infty := \lim_{\lambda \rightarrow \infty} \Omega(\Gamma; B + i\lambda P)$$

CLAIM: LIMIT EXISTS $\frac{1}{\epsilon}$ IS
B-INDEPENDENT

(FINITENESS OF ATTRACTOR FLOW TREES)

HENCEFORTH WORK IN THIS
CHAMBER.

C. MODULARITY

$$\tau \in \mathcal{H} \quad \& \quad C \in \mathcal{L}_{\Sigma}^*(H^2(X, \mathbb{C}))$$

$$\mathcal{Z}(\tau, \bar{\tau}, C) :=$$

$$\sum_{F, N} d(F, N) \exp \left\{ -2\pi i \tau \hat{q}_0 - 2\pi i \bar{\tau} \frac{1}{2}(F^+)^2 - 2\pi i F \cdot (C + \frac{P}{2}) \right\}$$

SUSY PARTITION FUNCTION OF D3 INSTANTON

U-DUALITY $\Rightarrow$

$\mathcal{Z}(\tau, \bar{\tau}, C)$ is a JACOBI FORM $\Rightarrow$

$$\mathcal{Z}(\tau, \bar{\tau}, C) = \sum_{\mu \in L^*/L} H_{\mu}(\tau) \underbrace{\oplus_{\mu, L}(\tau, \bar{\tau}, C)}_{\text{SIEGEL-NARAIN}}$$

$$L := \mathcal{L}_{\Sigma}^*(H^2(X, \mathbb{Z})) \subset \underbrace{H^2(\Sigma; \mathbb{Z})}_{\text{SELF-DUAL}}$$

$\ell \in L$ IS ALWAYS IN $H^{1,1}(\Sigma) \Rightarrow$

$$d(F + \ell, N) = d(F, N) \quad \forall \ell \in L$$

- $H_\mu(\tau)$ IS A VECTOR-VALUED NEARLY HOLO.

MODULAR FORM OF WEIGHT $W = -1 - \frac{h'(x)}{2}$

AND MULTIPLIER SYSTEM M^* DUAL TO THAT OF $\oplus_{\mu \in L}$

- $W < 0 \Rightarrow H_\mu$ IS DETERMINED BY ITS POLAR TERMS.

SUPPRESS μ -INDEX FOR SIMPLICITY:

$$H(\tau) = \sum_{\hat{q}_0} \Omega(\Gamma)_\infty e^{-2\pi i \hat{q}_0 \tau}$$

$$= \underbrace{\sum_{0 < \hat{q}_0 \leq \frac{\chi(p)}{24}} (\dots)}_{\text{POLAR}} + \underbrace{\sum_{-\infty < \hat{q}_0 \leq 0} (\dots)}_{\text{NONPOLAR}}$$

D. MACROSCOPIC POLAR STATES

$$\text{IF } \Gamma = (0, P, Q, q_0) = P + Q + q_0 dV$$

$$\text{IS POLAR: } 0 < \hat{q}_0 \leq (\hat{q}_0)_{\max}$$

THEN $Z(\Gamma; t)$ HAS A ZERO.

$$\text{INDEED } S(\Gamma) = \frac{2\pi}{\sqrt{6}} \cdot \sqrt{-\hat{q}_0 \chi(P)}$$

SO NO SINGLE-CENTERED SOLUTION

BUT $H(\tau)$ HAS $W < 0 \Rightarrow$ SOME
POLAR DEGENERACIES ARE NONZERO

$\Rightarrow$ THESE MUST BE REALIZED AS
SPLIT ATTRACTOR STATES.

SIMPLE EXAMPLE

$$\text{PURE D4: } \Gamma = P + q_0 dV$$

$$\text{WITH } q_0 = \hat{q}_0 = (\hat{q}_0)_{\max} = \frac{\chi(P)}{24}$$

FIND ONLY ONE SPLITTING

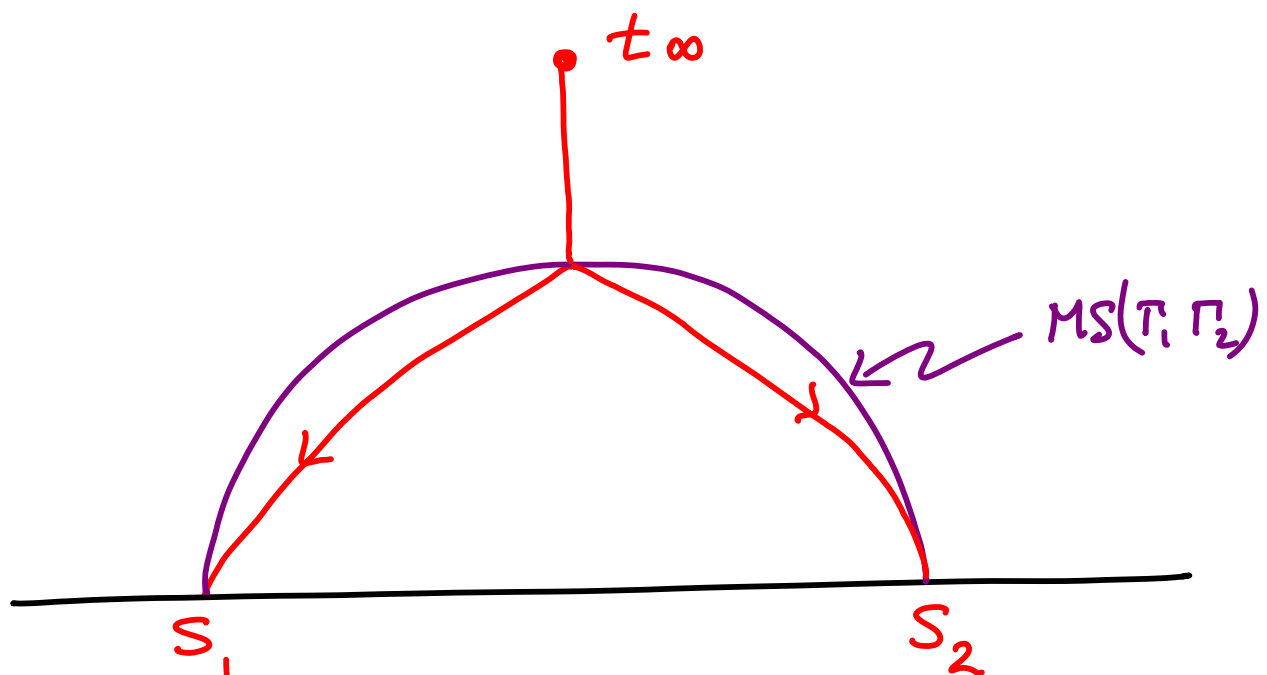
$$\Gamma = P + q_0 dV = \Gamma_1 + \Gamma_2$$

$$= e^{S_1} \left(1 + \frac{C_2(x)}{24} \right) - e^{S_2} \left(1 + \frac{C_2(x)}{24} \right)$$

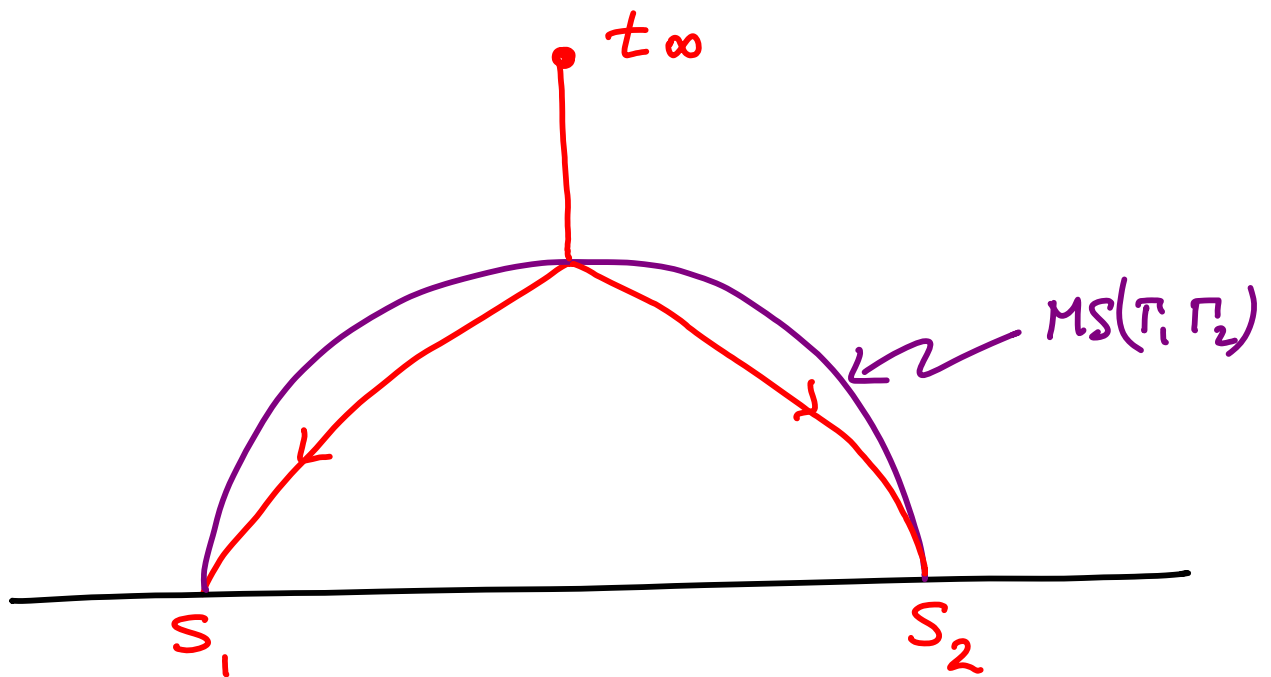
1 D6 WITH FLUX S_1

1 $\overline{\text{D6}}$ w/ FLX S_2

$$S_1 - S_2 = P$$



MOREOVER - YOU CAN COMPUTE THE
POLAR DEGENERACY:



$$\Omega(\Gamma, t_\infty) = (-1)^{I_{12}-1} |\mathcal{I}_{12}| \Omega(\Gamma_1) \Omega(\Gamma_2) = (-1)^{I_{12}-1} |\mathcal{I}_{12}|$$

$$I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{P^3}{6} + \frac{C_2(X) \cdot P}{12}$$

INDEED = THE CORRECT ANSWER FOR
 $\chi(\text{MODULI OF PURE D4}) = \chi(|P|)$

DESCRIBING THE SPLIT ATTRACTOR
FLOWS FOR $0 < \hat{q}_0 < \frac{\chi(p)}{24}$
IS MUCH MORE COMPLICATED...

IN GENERAL, POLAR STATES CAN
BE VERY COMPLICATED SPLIT
ATTRACTORS, REALIZED IN MANY
DIFFERENT WAYS....

BUT IN THE LIMIT $p \rightarrow \infty$ WE CAN
SAY SOMETHING

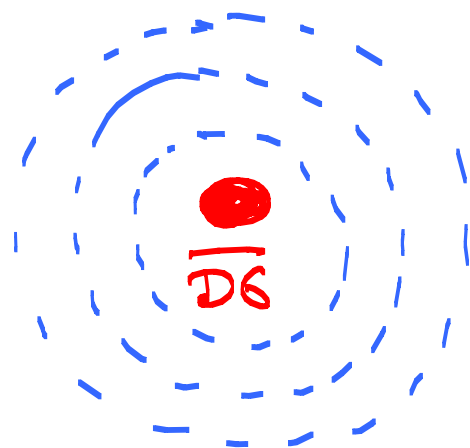
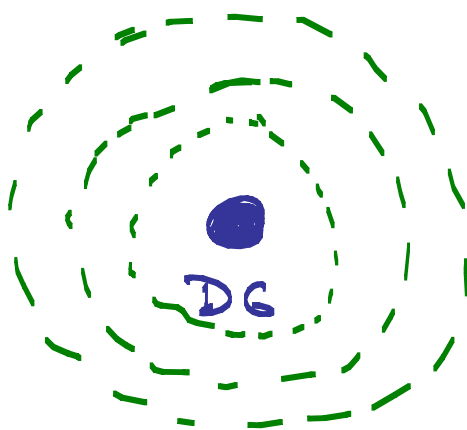
EXTREME POLAR STATES

$$H^{\text{POLAR}}(\tau) = \underbrace{|I_p| e^{-2\pi i \tau \frac{\chi(p)}{24}} + \dots}_{\text{"EXTREME POLAR"}} + \underbrace{O\left(e^{\frac{-2\pi i \tau}{|p|}}\right)}_{\text{"BARELY POLAR"}}$$

E.P.S. CONJECTURE: $\exists \epsilon < 1$ SO THAT

$$\frac{\hat{q}_0^{\max} - \hat{q}_0}{\hat{q}_0^{\max}} < \epsilon \Rightarrow$$

POLAR STATES SPLIT AS $D\overline{6}D\overline{6} + \text{HALOS}$:



$$\Gamma_1 = e^{S_1} (1 - \beta_1 + n_1 dV)$$

$$\Gamma_2 = -e^{S_2} (1 - \beta_2 + n_2 dV)$$

6. ROUTE TO THE OSV CONJECTURE

A. BY THE W.C.F. THE (EXTREME)
POLAR DEGENERACIES GO LIKE

$$\Omega(D6-D2-D0) \times \Omega(\overline{D6-D2-D0})$$

B. BUT BPS INVARIANTS OF
THE D6-D2-D0 SYSTEM ARE
RELATED TO GROMOV-WITTEN
INVARIANTS COUNTING WORLD SHEET
INSTANTONS IN X

SO, BY THE W.C.F. TOGETHER
WITH RESULTS ON $Z_{D_6 D_2 D_0}$
THE EXTREME POLAR
DEGENERACIES ARE RELATED

$$|Z_{\text{TOP}}|^2$$

SUGGESTING A RELATION LIKE
THE GSV CONJECTURE

$$\Omega(\Gamma)_\infty = \int d\phi \, |Z_{\text{top}}(g_{\text{top}}, t)|^2 e^{-2\pi g \phi}.$$

- $\exists$ STRONG ARGUMENTS FOR $|\hat{q}_0| \gg p^3$
- $\exists$ POTENTIAL COUNTEREXAMPLES FOR $|\hat{q}_0| \lesssim p^3$: "ENTROPY ENIGMA"

IN THE CHARGE REGIME

$$g_{\text{top}} \sim \sqrt{\frac{-\hat{g}_0}{p^3}} \lesssim \mathcal{O}(1)$$

THE DERIVATION IN DENEF-MOORE
BREAKS DOWN.

- BARELY POLAR DEGENERACIES
BECOME LARGE

- CORRECTIONS TO THE CARDY
FORMULA BECOME LARGE.

THERE IS A GOOD PHYSICAL
REASON THE DERIVATION BREAKS
DOWN ...

ENTROPY ENIGMA

NOW CHOOSE $q_0 < 0$, P AMPLE SO

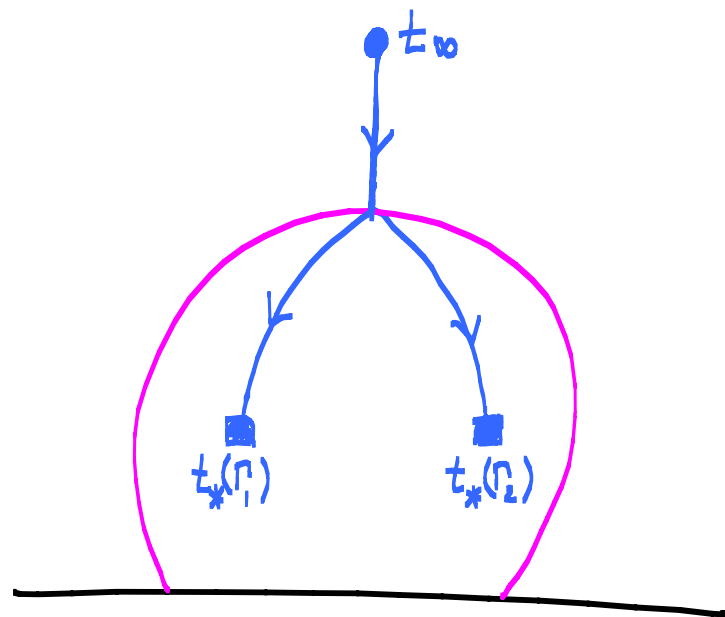
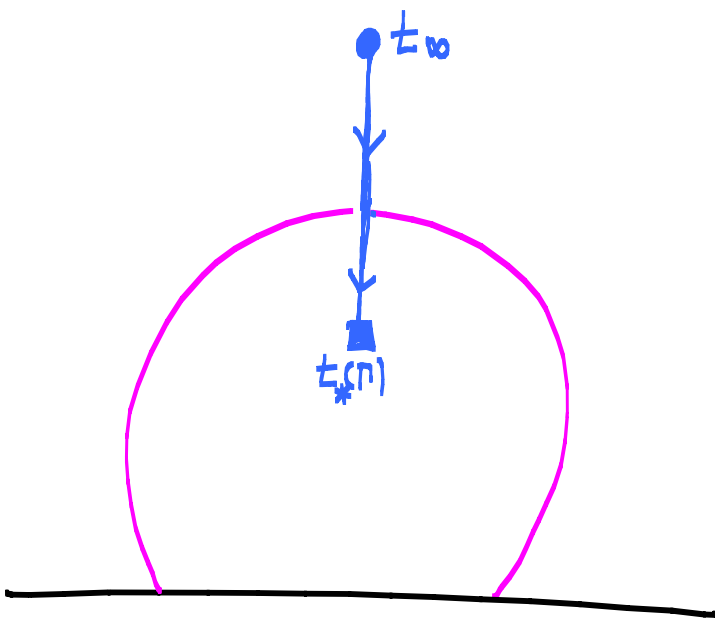
$$\Gamma = (0, P, 0, q_0)$$

HAS A REGULAR ATTRACTOR POINT

NEVERTHELESS! WE CAN CHOOSE

q_0, Q_A SO THAT $\exists$ A TWO-CENTERED
SOLUTION WITH $\Gamma = \Gamma_1 + \Gamma_2$

$$\Gamma_1 = (r, \frac{1}{2}P, Q, \frac{1}{2}q_0) \quad \Gamma_2 = (-r, \frac{1}{2}P, -Q, \frac{1}{2}q_0)$$



BOTH SOLUTIONS EXIST

SO... COMPARE ENTROPIES

$$S(\Gamma) \quad \text{vs.} \quad S(\Gamma_1) + S(\Gamma_2)$$

IN FACT,

$\exists$ FAMILY OF CHARGES

$$\lambda \Gamma = \lambda(0, P, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda$$

$$\Gamma_1^\lambda = (r, \frac{\lambda}{2} P, \lambda^2 Q, \frac{\lambda}{2} q_0) \quad \Gamma_2^\lambda = (-r, \frac{\lambda}{2} P, -\lambda^2 Q, \frac{\lambda}{2} q_0)$$

SCALING OF ENTROPIES:

$$S(\lambda \Gamma) = \lambda^2 S(\Gamma)$$

BUT!

$$S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda P)^3}{r} \sim \lambda^3$$

$\Rightarrow$ MANY IMPLICATIONS FOR PHYSICS & MATHEMATICS

SOME TECHNICAL DETAILS

1. CONSTRUCT A FAMILY OF 2-CENTERED

$$\tilde{\Gamma}_1^\lambda = (r, \frac{p}{2}, Q, \lambda^{-2} \frac{q_0}{2})$$

$$\tilde{\Gamma}_2^\lambda = (-r, \frac{p}{2}, -Q, \lambda^{-2} \frac{q_0}{2})$$

$\tilde{\Gamma}_i^\lambda$ CAN BE 1-CENTERED BH'S OR
CAN THEMSELVES BE POLAR

2. ATTRACTOR FORMALISM HAS A
SCALING SYMMETRY UNDER

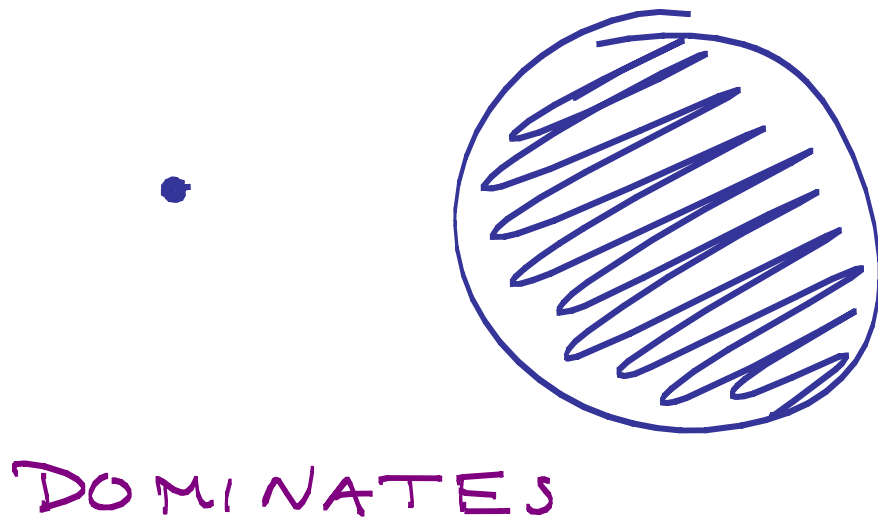
$$T_\lambda (p^0, p, Q, q_0) = (p^0, \lambda p, \lambda^2 Q, \lambda^3 q_0)$$

$$S(T_\lambda \Gamma) = \lambda^3 S(\Gamma)$$

3. APPLY TO $T_\lambda \tilde{\Gamma}_1^\lambda + T_\lambda \tilde{\Gamma}_2^\lambda = \lambda \Gamma$

RECENTLY, DEBOER ET. AL.

SHOWED THAT IF WE SPLIT
THE D2-D0 CHARGE ASYMMETRICALLY
BETWEEN THE TWO CENTERS
THEN THE COEFFICIENT OF THE
 λ^3 GROWTH CAN BE INCREASED:



BUT BOTH CONTRIBUTIONS SCALE
LIKE λ^3 .

DEGENERACY DICHOTOMY

- WE HAVE FOUND CONTRIBUTIONS TO $\Omega(\lambda\Gamma)_\infty$ GROWING LIKE e^{λ^3}

- IF INDEED $\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^3}$ THEN WEAK COUPLING OSV IS WRONG, SINCE OSV $\Rightarrow \Omega(\lambda\Gamma)_\infty \sim e^{\lambda^2}$.

- BUT $\Omega(\lambda\Gamma)_\infty$ IS AN INDEX. IT IS POSSIBLE THAT

$$\Omega(\lambda\Gamma)_\infty = \sum \pm e^{\lambda^3} \sim e^{\lambda^2}$$

- WE ARGUE THAT THIS IS UNLIKELY, BUT IT IS NOT EXCLUDED.

SUPPOSE THAT THERE ARE
"MAGICAL CANCELLATIONS" AND

$$\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^2}$$

• THIS RAISES THE QUESTION
OF $\dim \mathcal{H}(\Gamma; t)$ vs. $\Omega(\Gamma; t)$

• PHYSICALLY: THE DIMENSION IS RELEVANT

• BUT ALL TESTS OF THE STROMINGER-
Vafa PROGRAM USE THE INDEX
(WITH ONE EXCEPTION).

• IT IS ∇ TO SUPPOSE THAT IN
THE EXACT THEORY, NONPTVE
STRINGY EFFECTS GIVE:

$$\dim \mathcal{H}(\Gamma; t) = \Omega(\Gamma; t)$$

IF WE GRANT THIS POINT,
AND IF, MOREOVER, THERE ARE
"MAGICAL CANCELLATIONS" SO THAT
 $\log \Omega(0, \lambda p, 0, \lambda q_0) \sim \lambda^2$

THEN THE SPECTRUM OF
NEAR-BPS STATES TAKES A
REMARKABLE FORM:

$$E - |Z| = 0 \quad \sim e^{\lambda^2} \text{ states}$$

$$E - |Z| \sim e^{-1/g_s} \quad \sim e^{\lambda^3} \text{ states}$$

7. KONTSEVICH-SOIBELMAN FORMULA

THE KS FORMULA IS A RELATION BETWEEN $\Omega(\Gamma; t_{\pm})$ ACROSS MS WALLS WITH NO RESTRICTION ON PRIMITIVITY OF CONSTITUENTS.

- NO PHYSICAL DERIVATION YET

EVIDENCE THAT KES $\Omega(\Gamma; t)$'s

ARE THE SAME AS PHYSICAL $\Omega(\Gamma; t)$'s.

- CAN RECOVER PRIMITIVE WCF
- CAN RECOVER SEMI-PRIMITIVE WCF
- NONTRIVIAL CHECKS FOR $SU(2)$ SEIBERG-WITTEN WITH $N_f = 0, 1, 2, 3$ HYPERMULTIPLETS

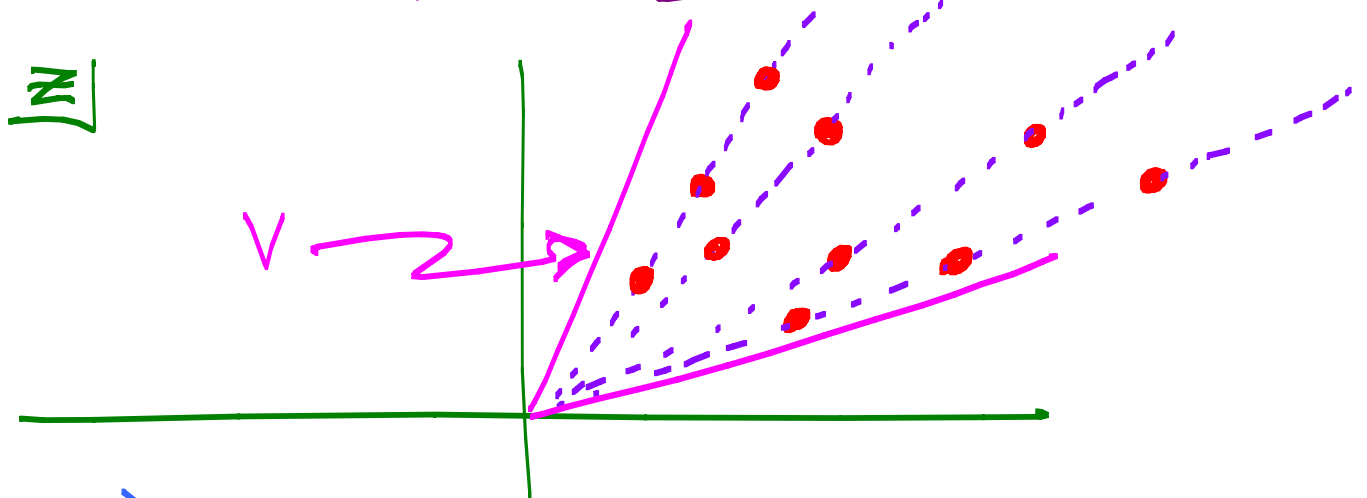
(LAST TWO ARE RESULTS W/ Wu-yen Chuang)

THE KONTSEVICH-SOIBELMAN FORMULA

FOR THE LATTICE Λ OF CHARGES
INTRODUCE A LIE ALGEBRA $\mathbb{Z}[\Lambda]$
WITH ONE GENERATOR FOR EACH
 $\gamma \in \Lambda$:

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

① FIXED $\pm$, $Z: \Lambda \rightarrow \mathbb{C}$,
CHOOSE ANY CONVEX ANGULAR SECTOR V



$$\prod_{\gamma \in \bar{Z}^+(V) \cap \Lambda} \left(\exp \sum_{n=1}^{\infty} \frac{e_n \gamma}{n^2} \right) \Omega^-(\gamma) \quad \text{INCREASING SLOPE}$$

$$\prod_{\gamma \in \bar{Z}^-(V) \cap \Lambda} \left(\exp \sum_{n=1}^{\infty} \frac{e_n \gamma}{n^2} \right) \Omega^+(\gamma) \quad \text{DECREASING SLOPE}$$

AT A GENERIC POINT $t \in \text{MS}(\Gamma_1, \Gamma_2)$

$$\mathbb{Z}(\Gamma; t) \parallel \mathbb{Z}_1, \mathbb{Z}_2 \implies$$

$$\Gamma = \Gamma_{a,b} = a\Gamma_1 + b\Gamma_2$$

$(\Gamma_1, \Gamma_2 \text{ primitive})$

FOR SMALL CONE ANGLE ONLY THE
LIE SUBALGEBRA $\mathbb{Z}\Gamma_1 + \mathbb{Z}\Gamma_2$
CONTRIBUTES:

$$[e_{a,b}, e_{c,d}] = (-1)^{(ad-bc)\mathbb{I}_{12}} (ad-bc)\mathbb{I}_{12} e_{a+c, b+d}$$

DEFINE:

$$U_{a,b} := \exp\left(\sum_{m=1}^{\infty} \frac{e_{ma, mb}}{m^2}\right)$$

$$\prod_{\substack{a/b \uparrow \\ a \geq 0}} U_{a,b}^{\bar{\Omega}(\Gamma_{a,b})} = \prod_{\substack{a/b \downarrow \\ a \geq 0}} U_{a,b}^{\Omega^+(\Gamma_{a,b})}$$

LIE ALGEBRA IS FILTERED $\Rightarrow$
CAN RESTRICT TO

Heisenberg
Algebra $\left\{ \begin{array}{l} [e_{0,1}, e_{1,0}] = (-1)^{I_{12}-1} I_{12} e_{1,1} \\ e_{1,1} \text{ CENTRAL} \end{array} \right.$

$$U_{0,1}^{\Omega^-(\Gamma_1)} U_{1,1}^{\Omega^-(\Gamma_1+\Gamma_2)} U_{1,0}^{\Omega^-(\Gamma_2)}$$

$$= U_{1,0}^{\Omega^+(\Gamma_2)} U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2)} U_{0,1}^{\Omega^+(\Gamma_1)}$$

$$\boxed{U_{0,1} U_{1,0} = U_{1,1}^{\pm I_{12}} \cdot U_{1,0} U_{0,1}} \Rightarrow$$

$$U_{1,1}^{\Omega^+(\Gamma_1+\Gamma_2) - \Omega^-(\Gamma_1+\Gamma_2)} = U_{0,1}^{\Omega(\Gamma_1)} U_{1,0}^{\Omega(\Gamma_2)} U_{0,1}^{-\Omega(\Gamma_1)} U_{1,0}^{-\Omega(\Gamma_2)}$$

$$= U_{1,1}^{I_{12}} \Omega(\Gamma_1) \Omega(\Gamma_2)$$

PRIMITIVE W.C. FORMULA!

SU(2) SEIBERG-WITTEN THEORY

Γ_1 = MONOPOLE

Γ_2 = DYON



$$[e_{a,b}, e_{c,d}] = 2(bc - ad) e_{a+c, b+d}$$

STRONG : $\pm(1,0), \pm(0,1)$ $\Omega = +1$ HM

WEAK : $\pm(1,1)$ $\Omega = -2$ VM

$\pm(n, n+1), \pm(n+1, n)$ $\Omega = +1$ HM

STRONG $U_{1,0} \cdot U_{0,1}$

WEAK :

$$(U_{0,1} U_{1,2} U_{2,3} \dots) U_{1,1}^{-2} (\dots U_{3,2} U_{2,1} U_{1,0})$$

EQUALITY APPEARS TO BE TRUE !

$\Rightarrow$ NEW IDENTITIES FOR $N_f = 1, 2, 3$

8. SOME OPEN PROBLEMS

- a.) PHYSICAL DERIVATION OF THE KS FORMULA
- b.) HOW TO COMPUTE POLAR DEGENERACIES EFFECTIVELY?
- c.) RESOLVE THE QUESTION OF THE ENTROPY ENIGMA: ARE THERE CANCELLATIONS BRINGING $e^{\lambda^3} \rightarrow e^{\lambda^2}$?
- d.) IS THERE AN OSV-LIKE RELATION FOR $\Omega(\Gamma, t_*(\Gamma))$? DO THESE ENJOY AUTOMORPHY PROPERTIES?