

Monopolia

Gregory Moore

Nambu Memorial Symposium

University of Chicago

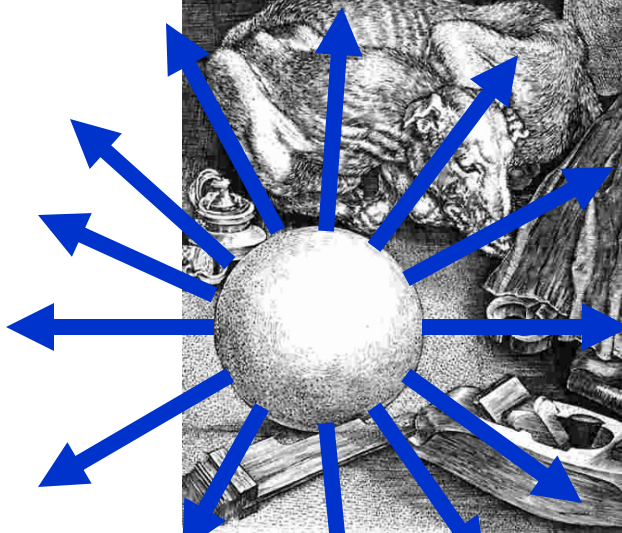
March 13, 2016



MONOPOLIA

a_1	a_2	c_3	f_4
a_2	$a_2 \times a_2$	a_5	e_6
c_3	a_5	d_6	e_7
f_4	e_6	e_7	e_8

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$



Robbert Dijkgraaf's Thesis Frontispiece



“Making the World
a Stabler Place”

The BPS Times

Late Edition

Today, BPS degeneracies,
wall-crossing formulae.
Tonight, Sleep. Tomorrow, K3
metrics, BPS algebras, p.B6

Est. 1975 www.bpstimes.com

SEOUL, FRIDAY, JUNE 28, 2013

₩ 2743.75

INVESTIGATORS SEE NO EXOTICS IN PURE SU(N) GAUGE THEORY

Use of Motives Cited

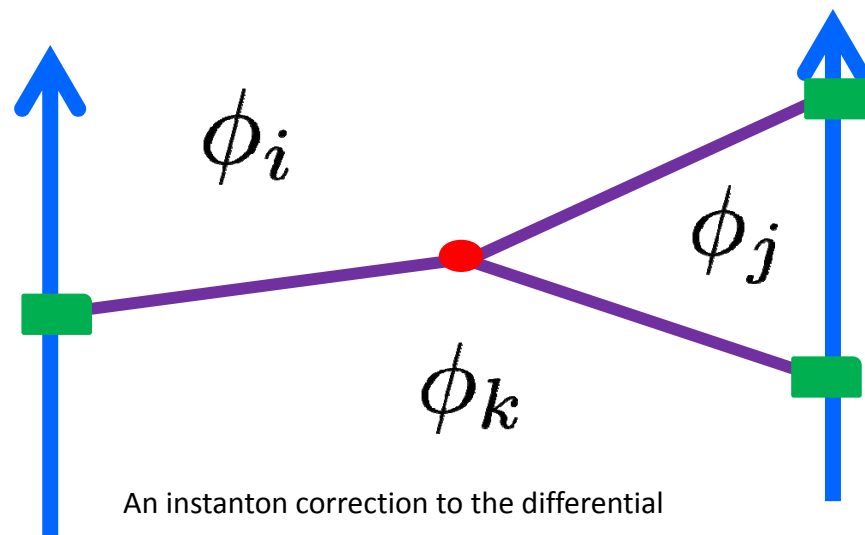
By E. Diaconescu, et. al.

RUTGERS – An application of
results on the motivic
structure of quiver moduli
spaces has led to a proof of
a conjecture of GMN. p.A12

Semiclassical, but Framed, BPS States

By G. Moore, A. Royston, and
D. Van den Bleeken

RUTGERS – Semiclassical
framed BPS states have
been constructed as



An instanton correction to the differential

Operadic Structures Found in Infrared Limit of 2D LG Models

NOVEL CONSTRUCTION OF d ON INTERVAL

Hope Expressed for Categorical WCF

By D. Gaiotto, G. Moore, and E. Witten

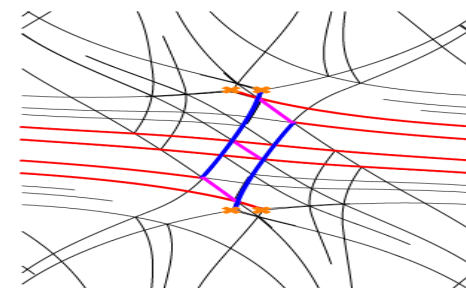
PRINCETON – A Morse-theoretic formulation of LG models has revealed ∞ -
structures familiar from String Field Theory. LG models are nearly trivial in

WILD WALL CROSSING IN SU(3)

EXPONENTIAL
GROWTH OF Ω

By D. Galakhov, P. Longhi, T. Mainiero,
G. Moore, and A. Neitzke

AUSTIN – Some strong coupling
regions exhibit wild wall crossing.
“I didn’t think this could happen,”
declared Prof. Nathan Seiberg of the
Institute for Advanced Study in
Princeton. *Continued on p.A4*





[Semiclassical framed BPS states](#)

[arXiv:1512.08924](#)

[L²-Kernels Of Dirac-Type Operators On Monopole Moduli Spaces](#)

[arXiv:1512.08923](#)

[Brane bending and monopole moduli](#)

[arXiv:1404.7158](#)

[Parameter counting for singular monopoles on \$\mathbb{R}^3\$](#)

[arXiv:1404.5616](#)

Goal Of Our Project

Recently there has been some nice progress in understanding BPS states in $d=4$, $N=2$ supersymmetric field theory:

No Exotics Theorem, Wall-Crossing Formulae,
Exact Results For Line Defect Vev's

What can we learn about the differential geometry of monopole moduli spaces from these results?

1 Introduction

2 Monopoles & Monopole Moduli Space

3 Singular Monopoles

4 Singular Monopole Moduli: Dimension & Existence

5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates

6 Semiclassical (Framed) BPS States

7 Application 1: No Exotics & Generalized Sen Conjecture

8 Application 2: Wall-crossing & Fredholm Property

9 Future Directions

Lie Algebra Review: 1/4

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$.

$X \in \mathfrak{g}$ is regular if $Z(X)$ has minimal dimension.

Then $Z(X) = \mathfrak{t}$ is a Cartan subalgebra.

$T = \exp[2\pi\mathfrak{t}]$ is a Cartan subgroup.

$\Lambda_G^\vee := \text{Hom}(T, U(1))$ character lattice

$\Lambda_G := \text{Hom}(U(1), T)$ $\exp(2\pi X) = 1$

$$\Lambda_{rt} \subset \Lambda_G^\vee \subset \Lambda_{wt} \subset \mathfrak{t}^\vee$$

$$\Lambda_{cr} \subset \Lambda_G \subset \Lambda_{mw} \subset \mathfrak{t}$$

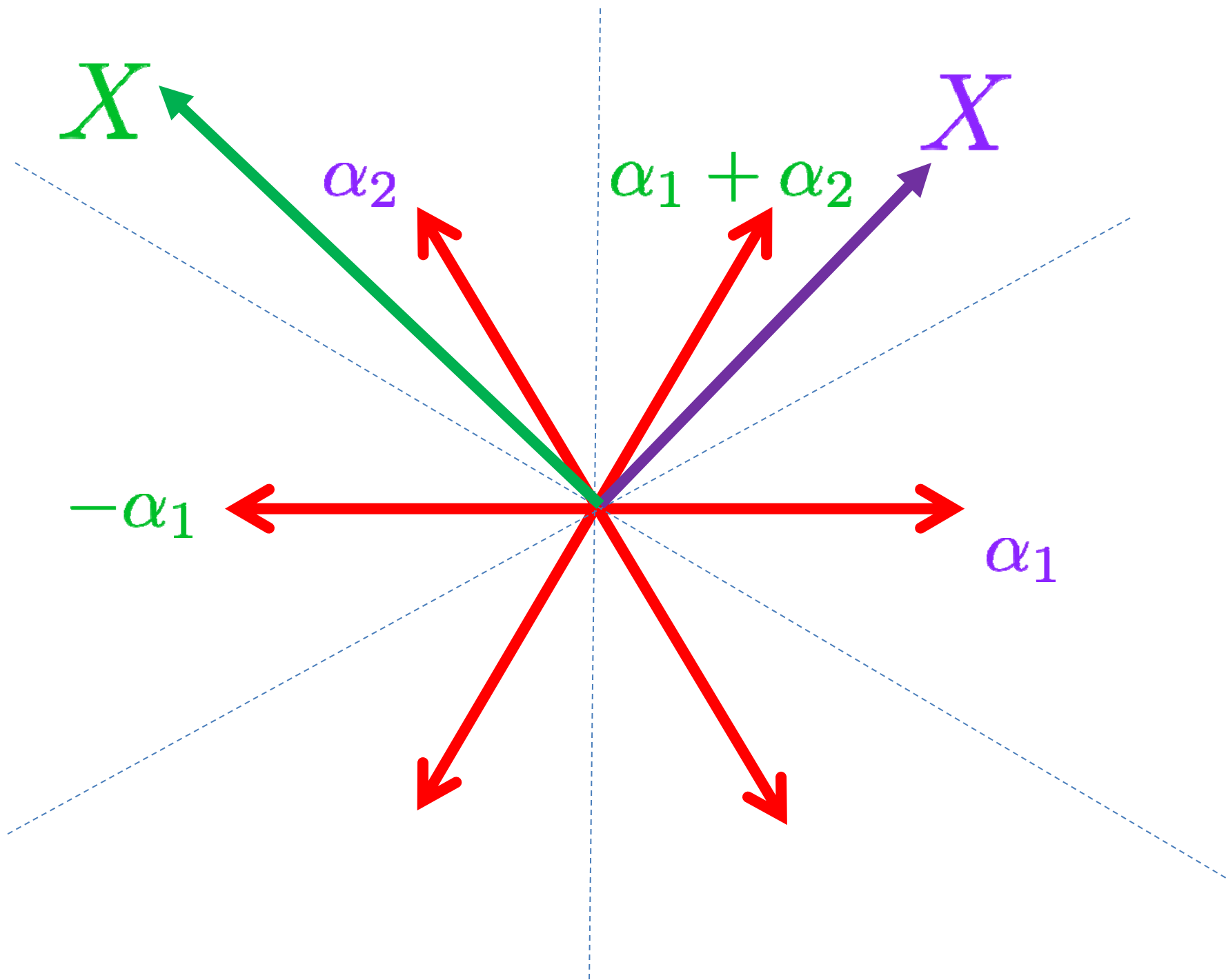
Lie Algebra Review: 2/4

Moreover, a regular element X
determines a set of simple roots $\alpha_I \in \mathfrak{t}^\vee$

and simple coroots $H_{\alpha_I} := H_I \in \mathfrak{t}$

$$\Lambda_{rt} = \bigoplus_I \mathbb{Z} \alpha_I \subset \mathfrak{t}^\vee$$

$$\Lambda_{cr} = \bigoplus_I \mathbb{Z} H_I \subset \mathfrak{t}$$



Nonabelian Monopoles

Yang-Mills-Higgs system for compact simple G

$$(A, X) \quad \int_{\mathbb{R}^4} \text{Tr}(F * F + DX * DX)$$

$$F = *DX \quad \text{on } \mathbb{R}^3$$

$$F = \gamma_m \text{vol}(S^2) + \cdots \quad X \rightarrow X_\infty - \frac{\gamma_m}{2r} + \cdots$$

$$X_\infty \in \mathfrak{g} \quad \text{regular} \quad \longrightarrow \quad \mathfrak{t} \quad \alpha_I \quad H_I$$

$$\gamma_m \in \Lambda_{cr} \subset \mathfrak{t} \subset \mathfrak{g}$$

$$\gamma_m = \sum_{I=1}^r n_m^I H_I \quad n_m^I \in \mathbb{Z}$$

Monopole Moduli Space

$$\mathcal{M}(\gamma_m; X_\infty) \quad \text{SOLUTIONS/GAUGE TRANSFORMATIONS}$$

If $\mathcal{M}$ is nonempty then [Callias; E. Weinberg]:

$$\dim \mathcal{M}(\gamma_m; X_\infty) = 4 \sum_I n_m^I$$

Known: $\mathcal{M}$ is nonempty iff all magnetic charges nonnegative and at least one is positive (so $4 \leq \dim \mathcal{M}$)

$\mathcal{M}$ has a hyperkahler metric. Group of isometries with Lie algebra:

$$\mathbb{R}^3 \oplus \mathfrak{so}(3) \oplus \mathfrak{t}$$

Translations

Rotations

Global gauge transformations

Action Of Global Gauge Transformations

$$H \in \mathfrak{t} \longrightarrow G(H) \quad \text{Killing vector field on } \mathcal{M}$$

$$\hat{A} = A_i dx^i + X dx^4 \quad \hat{F} = * \hat{F}$$

Directional derivative
along $G(H)$ at

$$[\hat{A}] \in \mathcal{M} \quad \frac{d\hat{A}}{ds} = -\hat{D}\epsilon$$

$$\epsilon : \mathbb{R}^3 \rightarrow \mathfrak{g}$$

$$\lim_{x \rightarrow \infty} \epsilon(x) = H \quad \hat{D}^2 \epsilon = 0$$

Strongly Centered Moduli Space

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

Orbits of translations

Orbits of $G(X_\infty)$

$$\mathcal{M}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0}{\mathbb{Z}}$$

Higher rank is different!

$$\mathcal{M}(\gamma_m; X_\infty) \neq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_0}{\mathbb{Z}_r}$$

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Singular Monopoles

$$F = \gamma_m \text{vol}(S^2) + \dots \quad X \rightarrow X_\infty - \frac{\gamma_m}{2r} + \dots$$
$$\vec{x} \rightarrow \infty$$

AND

$$F = P \text{vol}(S^2) + \dots \quad X \rightarrow -\frac{P}{2r} + \mathcal{O}(r^{-1/2})$$
$$\vec{x} \rightarrow 0$$

Use: construction of 't Hooft line defects ('line operators')

Kapustin; Kapustin-Witten

Example: A Singular Nonabelian SU(2) Monopole

$$X = \frac{1}{2} h(r) H_\alpha \quad A = \frac{1}{2} (\pm 1 - \cos \theta) d\phi H_\alpha \\ + \frac{1}{2} f(r) [e^{\pm i\phi} (-d\theta - i \sin \theta d\phi) E_+ + c.c.]$$

Bogomolnyi eqs:

$$f'(r) + f(r) h(r) = 0 \\ r^2 h'(r) + f(r)^2 - 1 = 0$$

$$h(r) = m_W \coth(m_W r + c) - \frac{1}{r} \quad f(r) = \frac{m_W r}{\sinh(m_W r + c)}$$

('t Hooft; Polyakov; Prasad & Sommerfield took $c = 0$)

$c > 0$ is the singular monopole: Physical interpretation?

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Singular Monopole Moduli Space

$$\overline{\mathcal{M}}(P; \gamma_m; X_\infty) \quad \begin{array}{l} \text{SOLUTIONS/GAUGE} \\ \text{TRANSFORMATIONS} \end{array}$$

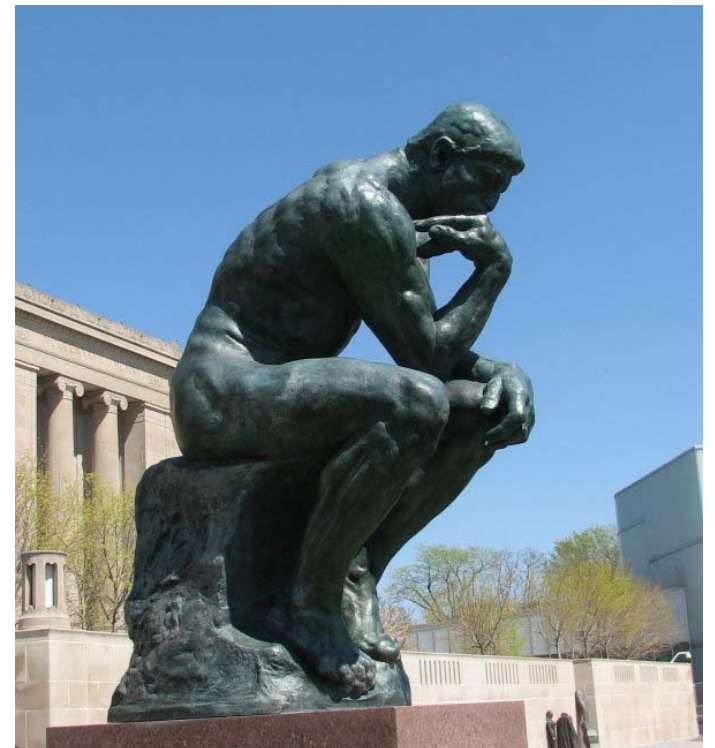
When is it nonempty?

What is the dimension?

If $P = \gamma_m$ is P screened or not ?

Is the dimension zero?

or not?



Dimension Formula

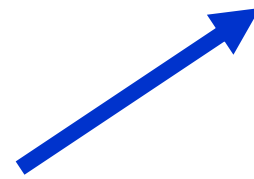
Assuming the moduli space is nonempty repeat computation of Callias; E. Weinberg to find:

$$\dim \overline{\mathcal{M}} = 2\text{ind}(L) = \lim_{\epsilon \rightarrow 0^+} \text{Tr} \left(\frac{\epsilon}{L^\dagger L + \epsilon} - \frac{\epsilon}{L L^\dagger + \epsilon} \right)$$

For a general 3-manifold we find:

$$\dim \overline{\mathcal{M}} = \int_{M_3 - \mathcal{S}} dJ^{(\epsilon)} = 4 \sum_I \tilde{n}_m^I$$

Relative magnetic charges.



Dimension Formula

$$\dim \overline{\mathcal{M}} = 4 \sum_I \tilde{n}_m^I$$

$$\sum_I \tilde{n}_m^I H_I = \gamma_m - P^-$$

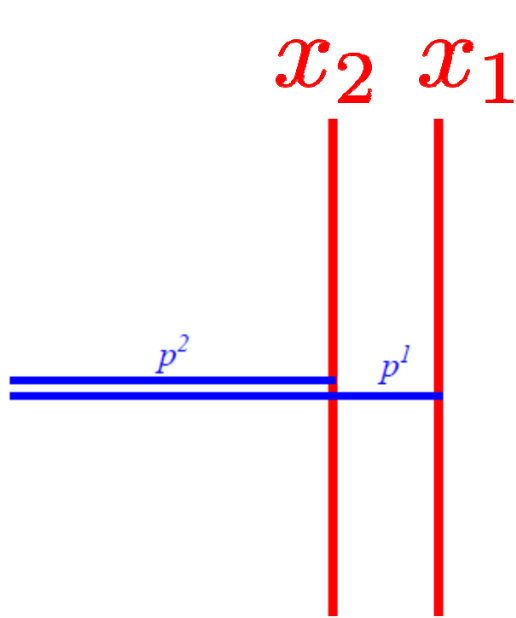
γ_m from $r \rightarrow \infty$ and $-P^-$ from $r \rightarrow 0$

P^- : Weyl group image such that $\langle \alpha_I, P^- \rangle \leq 0$

(Positive chamber determined by X_∞)

D-Brane Interpretation

Intuition for relative charges comes from D-branes. Example: Singular SU(2) monopoles from D1-D3 system

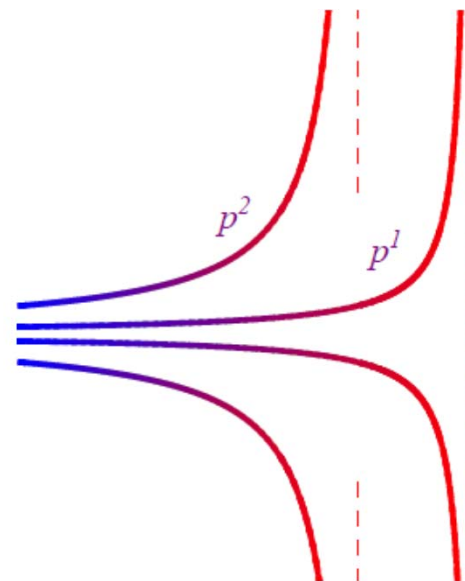


$$X = \begin{pmatrix} x_1 - \frac{p_1}{2r} & 0 \\ 0 & x_2 - \frac{p_2}{2r} \end{pmatrix}$$

$$\gamma_m = P = (p^1 - p^2)H_\alpha$$

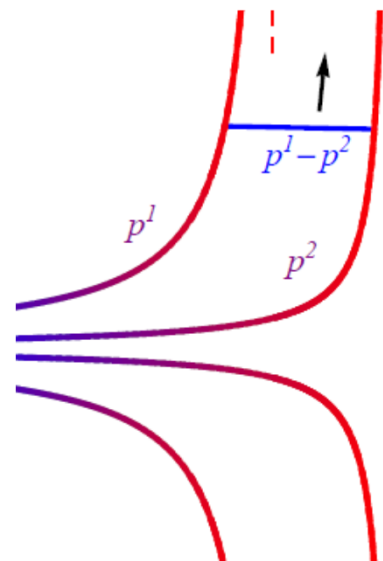
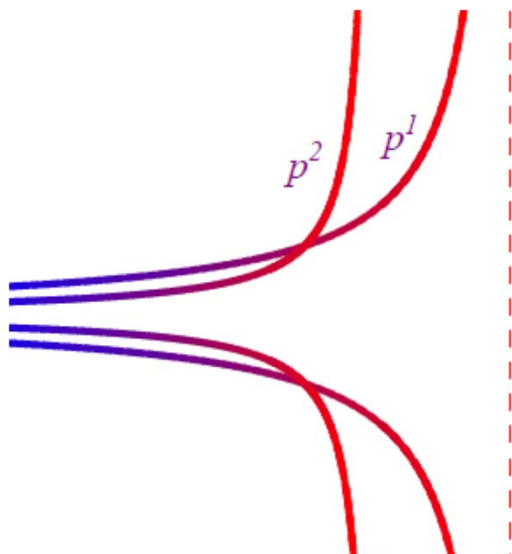
$$p^1 < p^2 \longrightarrow \gamma_m = P^-$$

$$\longrightarrow \dim \overline{\mathcal{M}} = 0$$



$$p^1 > p^2 \longrightarrow \gamma_m = -P^-$$

$$\longrightarrow \dim \overline{\mathcal{M}} = 4(p^1 - p^2)$$



Application: Meaning Of The Singular 't Hooft-Polyakov Ansatz

$$X = (m_W \coth(m_W r + c) - \frac{1}{r}) \frac{1}{2} H$$

$$\gamma_m = P = H \Rightarrow \tilde{n}_m = 2$$

$$\Rightarrow \dim \overline{\mathcal{M}} = 8$$

Two smooth monopoles in the presence of minimal SU(2) singular monopole.

They sit on top of the singular monopole but have a relative phase: $e^{-c} = \sin(\psi/2)$

Two D6-branes on an O6⁻ plane;
Moduli space of d=3 N=4 SYM with two massless HM

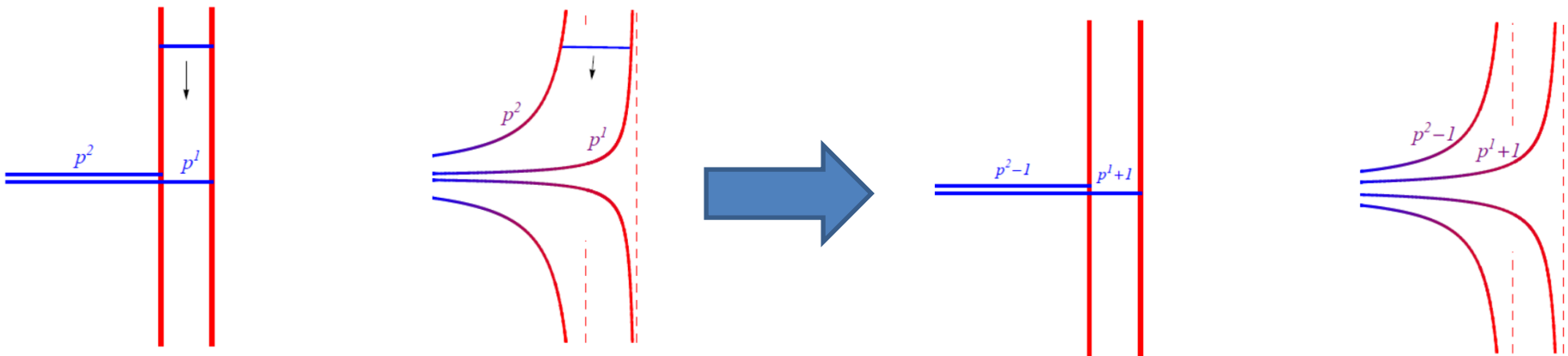
Properties of $\overline{\mathcal{M}}$

Conjecture:

$$\overline{\mathcal{M}}(P; \gamma_m; X_\infty) \neq \emptyset \iff \forall I, \tilde{n}_m^I \geq 0$$

$\overline{\mathcal{M}}$ Hyperkähler (with singular loci - monopole bubbling)

[Kapustin-Witten]



Isometries as before, but without translation.

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

$\mathcal{N}=2$ Super-Yang-Mills

Second real adjoint scalar Y

Vacuum requires $[X_\infty, Y_\infty]=0$.

$$\zeta^{-1}\varphi = Y + iX$$

Meaning of ζ : BPS equations on $\mathbb{R}^3$ for preserving

$$Q + \zeta^{-1}\bar{Q}$$

$$F = B = *DX \qquad E = DY$$

ζ And BPS States

BPS bound: $E \geq -\text{Re}(\zeta^{-1} Z_\gamma) \quad \gamma \in \Gamma$

Definition: **BPS states** saturate the bound.

Framed BPS states: Phase ζ is part of the data describing 't Hooft-Wilson line defect L

$$\overline{\mathcal{H}}^{\text{BPS}}(L, \gamma; u) \quad u \in \mathcal{M}_{\text{Coulomb}}$$

Smooth/unframed/vanilla BPS states:

$$\zeta = -Z_\gamma(u)/|Z_\gamma(u)| \quad \mathcal{H}^{\text{BPS}}(\gamma; u)$$

Semiclassical Regime

Definition: Series expansions for $a_D(a; \Lambda)$ converges: $|\langle \alpha, a \rangle| > c|\Lambda|$

Local system of charges has natural duality frame:

$$\Gamma \subset \Lambda_{wt} \oplus \Lambda_{mw} \quad (\text{Trivialized after choices of cuts in logs for } a_D.)$$
$$\gamma = \gamma^e \oplus \gamma_m$$

$$\Lambda(t) = e^{-\pi t/h^\vee} \Lambda_0 \quad \lim_{t \rightarrow +\infty} \mathcal{H}^{\text{BPS}}(\gamma; u_t)$$

In this regime there is a well-known semiclassical approach to describing BPS states.

Collective Coordinate Quantization

At weak coupling BPS monopoles are heavy:
Use moduli space approximation

The semiclassical states at (u, ζ) with electromagnetic charge $\gamma^e \oplus \gamma_m$ should be described in terms of supersymmetric quantum mechanics on

$$\overline{\mathcal{M}}(P, \gamma_m; X_\infty) \quad \text{OR} \quad \mathcal{M}(\gamma_m; X_\infty)$$

What sort of SQM? How is (u, ζ) related to X_∞ ?

How does γ^e have anything to do with it?

What Sort Of SQM?

(Sethi, Stern, Zaslow; Gauntlett & Harvey ; Tong; Gauntlett, Kim, Park, Yi;
Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi)

N=4 SQM on $\mathcal{M}(\gamma_m, X_\infty)$ with a potential:

$$S = \int \left(\| \dot{z} \|^2 - \| G(\mathcal{Y}_\infty^{\text{cl}}) \|^2 + \dots \right)$$

$$\mathcal{Y}_\infty^{\text{cl}} := \frac{4\pi}{g_0^2} Y_\infty + \frac{\theta_0}{2\pi} X_\infty$$

$\{Q, z^\mu\} \sim \chi^\mu \quad \longrightarrow \quad \text{States are spinors on } \mathcal{M}$

$$Q_4 = \chi^\mu (D + G(\mathcal{Y}_\infty^{\text{cl}}))_\mu := \mathbf{D}$$

How is (u, ζ) related to X_∞ ?

Need to write $X_\infty, \mathcal{Y}_\infty$ as functions on the Coulomb branch

$$X_\infty := \text{Im}(\zeta^{-1} a(u)) := X$$

$$\mathcal{Y}_\infty := \text{Im}(\zeta^{-1} a_D(u; \Lambda)) := \mathcal{Y}$$

Nicely encodes quantum corrections, e.g.

$$G(\mathcal{Y}_\infty^{\text{cl}}) \rightarrow G(\mathcal{Y}_\infty)$$

goes beyond the weak-potential approximation.

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Semiclassical BPS States: Overview

$$Q_4 = \chi^\mu (D + G(\mathcal{V}))_\mu := \mathbf{D}$$

Semiclassical framed or vanilla BPS states
with magnetic charge γ_m will be:

a Dirac spinor Ψ on $\mathcal{M}(\gamma_m)$ or $\overline{\mathcal{M}}(\gamma_m)$ $\mathbf{D}\Psi = 0$

Must be suitably normalizable: $\ker_{L^2} \mathbf{D}$

Must be suitably equivariant...

Many devils in the details....



States Of Definite Electric Charge

$\overline{\mathcal{M}}$ has a $\mathfrak{t}$ -action: $G(H)$ commutes with $\mathbf{D}$

$$\exp[2\pi G(H)] \cdot \Psi = \exp[2\pi i \langle \gamma^e, H \rangle] \Psi$$

$$\gamma^e \in \mathfrak{t}^\vee$$

Cartan torus T of adjoint group acts on $\overline{\mathcal{M}}$

$$T = \mathfrak{t} / \Lambda_{mw} \longrightarrow \gamma^e \in \Lambda_{mw}^\vee \cong \Lambda_{rt}$$

Organize L^2 -harmonic spinors by T -representation:

$$\ker_{L^2} \mathbf{D} = \bigoplus_{\gamma^e} \ker_{L^2}^{\gamma^e} \mathbf{D}$$

Framed BPS States: The Answer

L is an 't Hooft line defect of charge P and phase ζ

$$\begin{aligned}\underline{\mathcal{H}}^{\text{BPS}}(L, \gamma; u) &= ??? \\ &= \ker_{L^2}^{\gamma^e} \mathbf{D}\end{aligned}$$

$$X = \text{Im}(\zeta^{-1} a(u)) \Rightarrow \underline{\mathcal{M}}(P; \gamma_m; X)$$

$$\mathcal{Y} = \text{Im}(\zeta^{-1} a_D(u; \Lambda)) \Rightarrow \mathbf{D}$$

Vanilla BPS States & Smooth Monopoles

Begin analysis on universal cover: $\mathcal{M}^\sim$

Physical states Ψ must descend to $\mathcal{M}$

-- Electric Charge --

T acts on $\mathcal{M}$, and $T = t/\Lambda_{mw}$

States Ψ of definite electric charge transform with a definite character of t :

$$\gamma^e \in \Lambda_{mw}^\vee \cong \Lambda_{rt}$$

Separating The Center Of Mass

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

$$\mathbf{D} = \mathbf{D}_{\text{com}} + \mathbf{D}_0$$

$$\Psi = \Psi_{\text{com}} \otimes \Psi_0$$

$$\mathbf{D}_{\text{com}} \Psi_{\text{com}} = 0 \quad \mathbf{D}_0 \Psi_0 = 0$$

Separating the COM involves the nontrivial identity:

$$(G(X_\infty), G(H))_{\text{metric}} = (\gamma_m, H)_{\text{Killing}}$$

L^2 - Condition

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

No L^2 harmonic spinors on $\mathbb{R}^4$
Only “plane-wave-normalizable” in $\mathbb{R}^4$

$$\Psi_0 \in \ker_{L^2} \mathbf{D}_0$$

Note: The L^2 condition is crucial!
We do not want “extra” internal d.o.f.

Contrast this with the hypothetical
“instanton particle” of 5D SYM.

Semiclassical Smooth BPS States

????

$$\mathcal{H}^{\text{BPS}}(\gamma; u) = (\ker(\mathbf{D}_{\text{com}}) \otimes \ker_{L^2} \mathbf{D}_0)^{\gamma^e}$$

$$X = \text{Im}(\zeta^{-1} a(u))$$

$$\mathcal{Y} = \text{Im}(\zeta^{-1} a_D(u; \Lambda))$$

$$\zeta = -Z_\gamma(u)/|Z_\gamma(u)|$$

Tricky Subtlety: 1/3

Spinors must descend to $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{D}$

$\mathbb{D} \cong \mathbb{Z}$ Generated by isometry ϕ

Subtlety: Imposing electric charge quantization only imposes invariance under a proper subgroup of the Deck group:

$$\exp[2\pi G(\lambda)]\Psi = \Psi \quad \lambda \in \Lambda_{mw}$$

$$\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$$

Put differently:

T acts on $\mathcal{M}$, so choosing a point $m_0 \in \mathcal{M}$

$$f : T \rightarrow T \cdot m_0 \subset \mathcal{M}$$

$$f_* : \pi_1(T, 1) \rightarrow \pi_1(\mathcal{M}, m_0)$$

$$f_* : \Lambda_{mw} \rightarrow \mathbb{Z}$$

$$f_* := \mu$$

Using the relation of $\mathcal{M}$ to rational maps from $\mathbb{P}^1$ to the flag variety we prove:

$$\mu(\lambda) = (\lambda, \gamma_m)$$

Tricky Subtlety: 3/3

 $\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$

only generate a subgroup $r\mathbb{Z}$, where r is, roughly speaking, the gcd(magnetic charges)

 Extra restriction to $\mathbb{Z}/r\mathbb{Z}$ invariant subspace:

$$\left((\ker(\mathbf{D}_{\text{com}}) \otimes \ker_{L^2} \mathbf{D}_0)^{\gamma_e} \right)^{\mathbb{Z}/r\mathbb{Z}}$$

Combine above picture with
results on $N=2, d=4$:

No Exotics Theorem

Wall-Crossing

Exact Results On Line Defect VEV's

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 **Application 1: No Exotics & Generalized Sen Conjecture**
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Exotic (Framed) BPS States

$$\overline{\mathcal{H}}_{\gamma}^{\text{BPS}} \quad \mathcal{H}_{\gamma}^{\text{BPS}} \quad \mathfrak{so}(3)_{\text{rot}} \oplus \mathfrak{su}(2)_R \text{ -reps}$$

Vanilla BPS: $\mathcal{H}_{\gamma}^{\text{BPS}} = \rho_{hh} \otimes \mathfrak{h}(\gamma)$

Half-Hyper from COM: $\rho_{hh} = (\frac{1}{2}; 0) \oplus (0; \frac{1}{2})$

Framed BPS: No HH factor:

$$\overline{\mathcal{H}}_{\gamma}^{\text{BPS}} = \mathfrak{h}(\gamma)$$

Definition: *Exotic BPS states*: States in $\mathfrak{h}(\gamma)$ transforming nontrivially under $\mathfrak{su}(2)_R$

No Exotics Conjecture/Theorem

Conjecture _[GMN]: $\mathfrak{su}(2)_R$ acts trivially on $\mathfrak{h}(\gamma)$: exotics don't exist.

Theorem: It's true!

Diaconescu et. al. : Pure $SU(N)$ vanilla and framed
(for pure 't Hooft line defects)

Sen & del Zotto: Simply laced G (vanilla)

Cordova & Dumitrescu: Any theory with ``Sohnius''
energy-momentum supermultiplet (vanilla, so far...)

Geometry Of The R-Symmetry

$$\dim_{\mathbb{R}} \mathcal{M} = 4N$$

Riemannian holonomy: $SO(4N)$

Hyperkähler holonomy: $USp(2N)$

$SU(2)_R$ is the commutant of $USp(2N)$

Collective coordinate expression
for generators of $\mathfrak{su}(2)_R$

$$I^r \sim \omega_{\mu\nu}^r \chi^\mu \chi^\nu$$

This defines a lift to the spin bundle.

Generators do not commute with
Dirac, but do preserve kernel.

$\overline{\mathcal{M}}$ $\mathcal{M}$ have $\mathfrak{so}(3)$ action of rotations. Suitably defined, it commutes with $\mathfrak{su}(2)_R$.

Again, the generators do not commute with $\mathbf{D}_0$, $\mathbf{D}$, but do preserve the kernel.

$$\overline{\mathcal{H}}^{\text{BPS}}(L, \gamma, u) \cong \ker_{L^2}^{\gamma_e} \mathbf{D}$$

$$\mathfrak{h}^{\text{BPS}}(\gamma; u) \cong \ker_{L^2}^{\gamma_0^e} \mathbf{D}_0$$

Equality of $\mathfrak{so}(3)_{\text{rot}} \oplus \mathfrak{su}(2)_R$ - reps.

Geometrical Interpretation Of The No-Exotics Theorem -1

$$\rho : SU(2)_R \times USp(2N) \rightarrow \text{Spin}(4N)$$

$$\rho : (-1, 1) \rightarrow \text{vol} := \Gamma^1 \dots \Gamma^{4N}$$



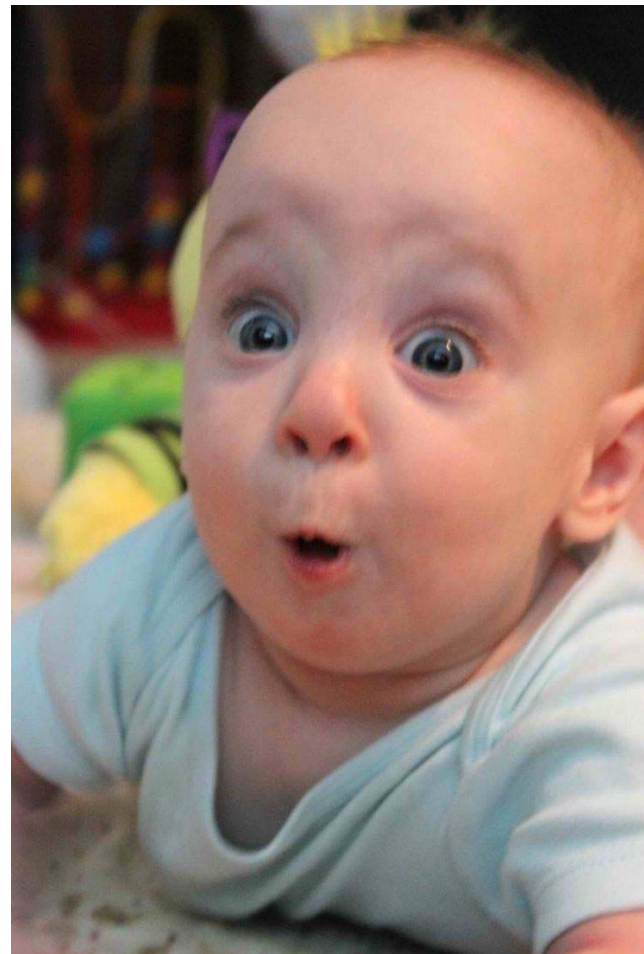
All spinors in the kernel
have chirality +1



$$\text{Ind} \mathbf{D}_0^+ = \dim \ker \mathbf{D}_0$$

So, the *absolute* number of BPS states is the same as the BPS *index*!

This kind of question arises frequently in BPS theory...



Geometrical Interpretation Of The No-Exotics Theorem - 2

Choose any complex structure on $\mathcal{M}$.

$$\mathcal{S} \cong \Lambda^{0,*}(T\mathcal{M}) \otimes K^{-1/2}$$

$$Q_3 + \mathrm{i}Q_4 \sim \bar{\partial} + G^{0,1}(\mathcal{Y}) \wedge$$

$\mathfrak{su}(2)_R$ becomes “Lefschetz $\mathfrak{sl}(2)$ ”

$$I^3|_{\Lambda^{0,q}} = \frac{1}{2}(q - N)\mathbf{1}$$

$$I^+ \sim \omega^{0,2} \wedge \qquad I^- \sim \iota(\omega^{2,0})$$

Geometrical Interpretation Of The No-Exotics Theorem - 3

$H_{L^2}^{0,q}(\bar{\partial} + G^{0,1}(\mathcal{V}))$
vanishes except in the middle degree $q = N$,
and is primitive wrt “Lefschetz $\mathfrak{sl}(2)$ ”.

$$\forall \mathcal{V} \in \mathfrak{t}$$

Adding Matter-1/2

(Manton & Schroers; Sethi, Stern & Zaslow; Gauntlett & Harvey ;
Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi)

Add matter hypermultiplets in a quaternionic representation R of G .

Bundle of hypermultiplet fermion zero modes
defines a vector bundle $\mathcal{E}$ over $\mathcal{M}$:

$\mathcal{E}$ = associated bundle to the universal bundle.

Universal connection is hyperholomorphic

Adding Matter-2/2

(work with Daniel Brennan)

Real rank of $\mathcal{E}$

$$d = \sum_{\mu} \left[\text{sign}(\langle \mu, X \rangle + m_I) \langle \mu, \gamma_m \rangle + |\langle \mu, P \rangle| \right]$$

Sum over weights μ of R . $m_I := \text{Im}(\zeta^{-1}m)$

$$\mathcal{E} \otimes \mathbb{C} \cong \mathcal{W} \oplus \overline{\mathcal{W}}$$

States are now L^2 -sections of

$$S \otimes \Lambda^* \mathcal{W} \rightarrow \mathcal{M}_0, \overline{\mathcal{M}}$$

Geometrical Interpretation Of The No-Exotics Theorem - 4

$$H_{L^2}^{0,q}(\bar{\partial} + G^{0,1}(\mathcal{V}); \Lambda^* \mathcal{W})$$

vanishes except in the middle degree $q = N$,
and is primitive wrt “Lefschetz $\mathfrak{sl}(2)$ ”.

$SU(2)$ $N=2^*$ $m \longrightarrow 0$ recovers the famous
Sen conjecture

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Semiclassical Wall-Crossing: Overview

Easy fact: There are no L^2 harmonic spinors for ordinary Dirac operator on a noncompact hyperkähler manifold.

➡ $\exists$ Semiclassical chamber ($\mathcal{Y}_\infty=0$) where all populated magnetic charges are just simple roots ($\mathcal{M}_0 = \text{pt}$)

Other semiclassical chambers have nonsimple magnetic charges filled.

➡ Nontrivial semi-classical wall-crossing
(Higher rank is different.)

➡ Interesting math predictions

Jumping Index

The L^2 -kernel of D jumps.

No exotics theorem 

Harmonic spinors have definite chirality

 L^2 index jumps! How?!

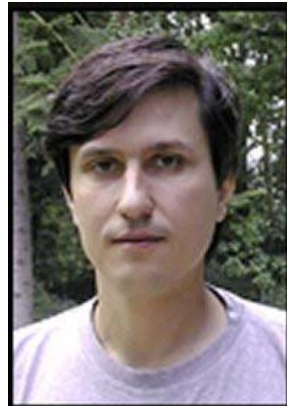
Along hyperplanes in $\mathcal{Y}$ -space zeromodes mix with continuum and D^+ fails to be Fredholm.

(Similar picture proposed by M. Stern & P. Yi)

We give explicit formulae for these hyperplanes.

How Does The BPS Space Jump?

Unframed/
smooth/
vanilla:



&



Framed:



&



Framed Wall-Crossing: 1/2

$$\underline{\bar{\Omega}}(L, \gamma; X, \mathcal{Y}) = \text{Tr}_{\underline{\mathcal{H}}} y^{2J_3}$$

“Protected spin characters”

$$F(L) = \sum_{\gamma \in \Gamma} \underline{\bar{\Omega}}(L, \gamma; X, \mathcal{Y}) V_\gamma$$

Where does it jump?

$$\mathcal{W}(\gamma_h) := \{(X, \mathcal{Y}) : (\gamma_{h,m}, \mathcal{Y}) + \langle \gamma_{h,e}, X \rangle = 0\}$$

$$\mathcal{H}^{\text{BPS}}(\gamma_h, u) \neq 0$$

Framed Wall-Crossing: 2/2

$$F(L) = \sum_{\gamma \in \Gamma} \overline{\Omega}(L, \gamma; X, \mathcal{Y}) V_{\gamma}$$

How does it jump across $\mathcal{W}(\gamma_h)$?

$$V_{\gamma_1} V_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} V_{\gamma_1 + \gamma_2}$$

$$F(L) \rightarrow S F(L) S^{-1}$$

S is an operator-valued function of V_{γ_h}

Example: Semiclassical Vanilla Wall Crossing

Does not exist for $\mathfrak{g} = \mathfrak{su}(2)$ (Seiberg & Witten 1994)

$$\mathfrak{g} = \mathfrak{su}(3) \quad [\text{Gauntlett, Kim, Lee, Yi (2000)}]$$

$$\gamma_m = H_1 + H_2 = \gamma_{1,m} + \gamma_{2,m}$$

$$\gamma^e = n_1 \alpha_1 + n_2 \alpha_2 = \gamma_1^e + \gamma_2^e$$

$$\mathcal{H}(\gamma_i; X, \mathcal{Y}) \neq 0$$

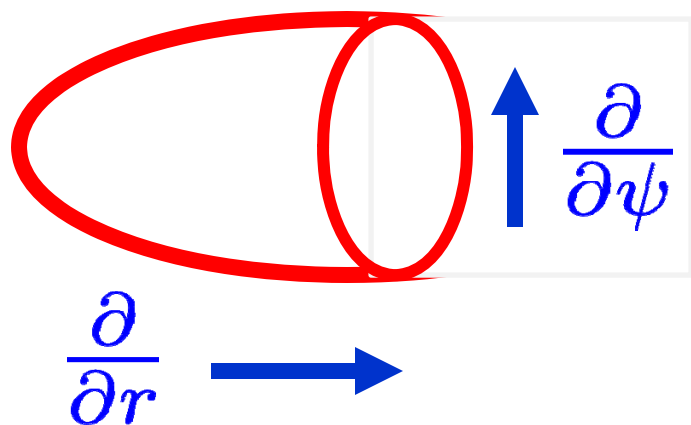
“Constituent BPS states exist”

Why choose $\gamma_m = H_1 + H_2$??

$\mathcal{M}_0(X; \gamma_m) = \text{Taub-NUT:}$

Zeromodes of $\mathbf{D}_0$ can be explicitly computed
[C. Pope, 1978]

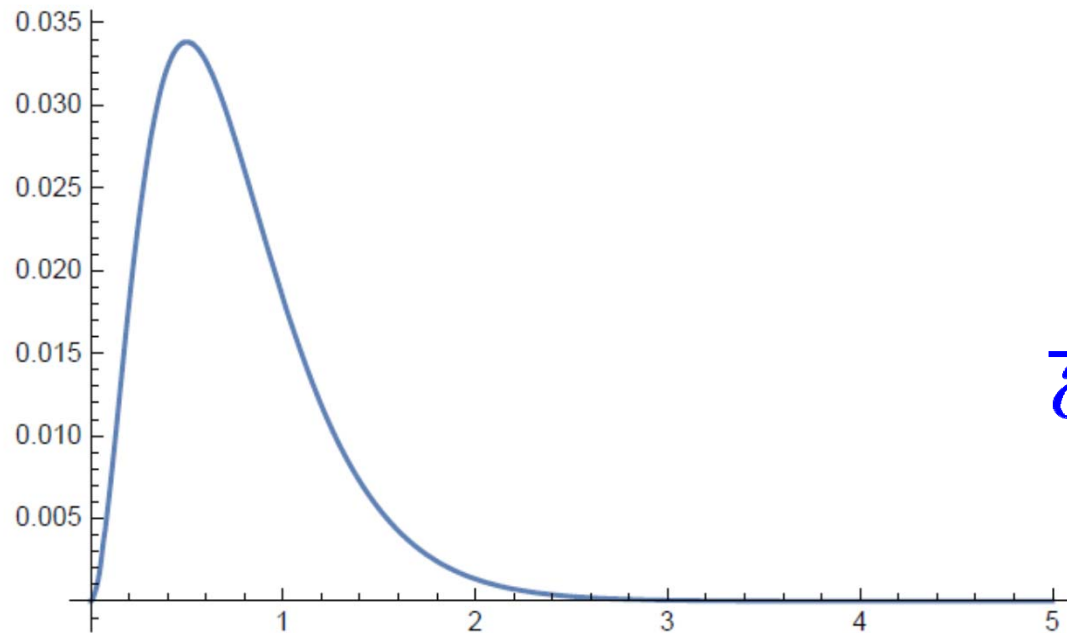
What do the zeromodes look like ??



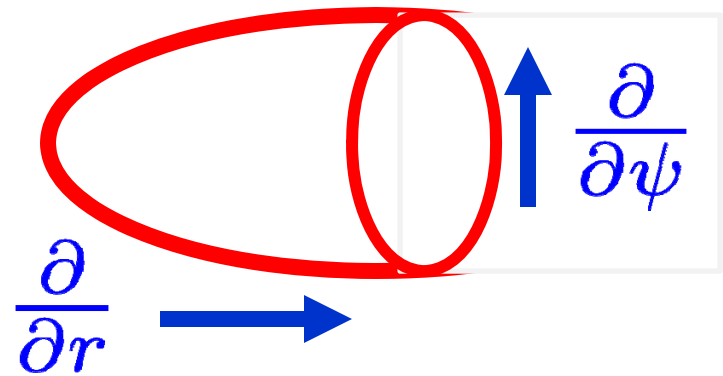
$$G(\mathcal{Y}) = C(\mathcal{Y}) \frac{\partial}{\partial \psi}$$

$$\begin{aligned} L_{\frac{\partial}{\partial \psi}} \Psi_0 &= i(n_1 - n_2) \Psi_0 \\ &= i\mu \Psi_0 \end{aligned}$$

$$\Psi_0 \sim r^{(\mu-1)/2} e^{-|C-\mu|r/2}$$



$$r_{\max} = \frac{\mu}{|C-\mu|} = r_{\text{Denef}}$$

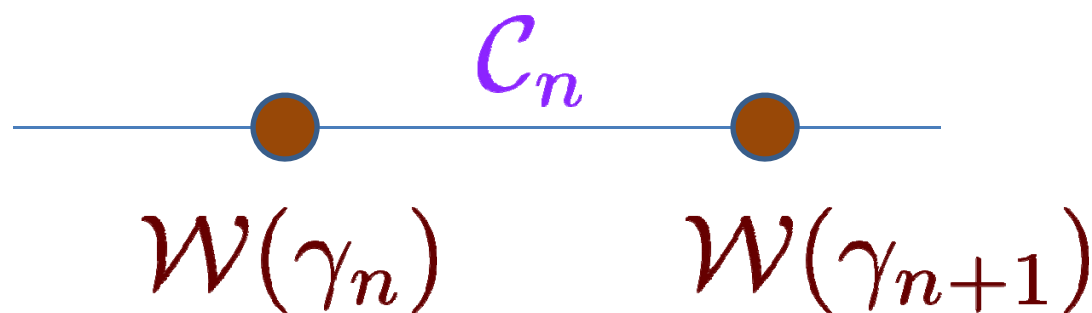


Example: Semiclassical Framed Wall-Crossing

$$\mathfrak{g} = \mathfrak{su}(2) \quad \mathfrak{t} \cong \mathbb{R}$$

Well-known spectrum of
smooth BPS states
[Seiberg & Witten]:

$$\gamma_n = n\alpha \oplus H_\alpha$$



$$\mathcal{W}(\gamma_h) := \{\mathcal{Y} | (\gamma_{h,m}, \mathcal{Y}) + \langle \gamma_{h,e}, X \rangle = 0\}$$

Line defect L: $P = \frac{p}{2} H_\alpha$

$$F(L) = \sum_{\gamma \in \Gamma} \overline{\Omega}(L, \gamma; X, \mathcal{Y}) V_\gamma$$

Explicit Generator Of PSC's

$$V_1 V_2 = y V_2 V_1$$

$$V_\gamma = V_{n^e \alpha + n_m H} = y^{-\frac{1}{2} n^e n_m} V_2^{n^e} V_1^{n_m}$$

$$F(C_\ell) = [y^{2\ell} V_1^{-1} V_2^{-\ell} (\mathcal{U}_\ell(f_\ell) - y^2 V_2^{-1} \mathcal{U}_{\ell-1}(f_\ell))]^p$$

$$\mathcal{U}_\ell(\cos \theta) := \frac{\sin((\ell+1)\theta)}{\sin \theta}$$

$$f_\ell = \frac{1}{2} [y^{-2} V_2 + y^2 V_2^{-1} (1 + y^{-1} V_1^2 V_2^{2\ell+2})]$$



Predictions for $\ker \mathbf{D}$ for infinitely many moduli spaces of arbitrarily high magnetic charge.

- 1 Introduction
- 2 Monopoles & Monopole Moduli Space
- 3 Singular Monopoles
- 4 Singular Monopole Moduli: Dimension & Existence
- 5 Semiclassical $N=2$ $d=4$ SYM: Collective Coordinates
- 6 Semiclassical (Framed) BPS States
- 7 Application 1: No Exotics & Generalized Sen Conjecture
- 8 Application 2: Wall-crossing & Fredholm Property
- 9 Future Directions

Future Directions -1/4

Add matter and arbitrary Wilson-'t Hooft lines. (In progress with Daniel Brennan)

Understand better how Fredholm property fails by using asymptotic form of the monopole metric.

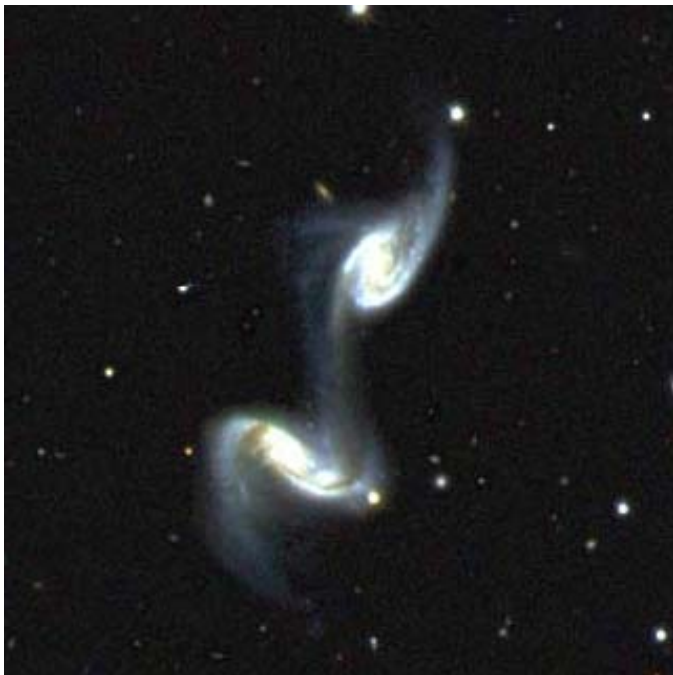
Combine the localization result of Ito, Okuda, Taki with ``Darboux expansion'' of GMN to get an interesting L^2 -index theorem on (noncompact!) monopole moduli spaces ?

Future Directions -2/4

(In progress with Daniel Brennan & Andy Royston)

Understand better how Fredholm property of $\mathbf{D}$ fails:

Use asymptotic metric for moduli space for widely separated monopoles with charges $\in \{\text{simple roots}\}$



$$\mathcal{M}_0^N \downarrow \mathcal{M}_0^{N_1} \times \mathcal{M}_0^{N_2} \times TN$$

Future Directions -3/4

(In progress with Anindya Dey)

There are methods to compute vev's of susy line defects on $\mathbb{R}^3 \times S^1$ exactly.

$$\langle L \rangle = \sum_{\gamma} \bar{\Omega}(L, \gamma) \mathcal{Y}_{\gamma} \quad \text{GMN-2010}$$

$$L_{\zeta} = \text{Tr}_2 \text{Pexp} \int_{\mathbb{R}_t \times \vec{0}} (\zeta^{-1} \varphi + A + \zeta \bar{\varphi})$$

Weak coupling expression
+ known nonperturbative
corrections.

Surprising
nonperturbative
correction

$$\langle \text{Tr}_2 L_{\zeta} \rangle = \underbrace{\sqrt{\mathcal{Y}_{\gamma_e}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_e}}}}_{\text{Weak coupling expression + known nonperturbative corrections}} + \sqrt{\mathcal{Y}_{\gamma_m + \gamma_e}} \quad \text{(Surprising nonperturbative correction)}$$

Future Directions -4/4

Localization computations of the same quantities by Ito, Okuda, Taki (2011) give expressions like:

$$\langle L \rangle = \sum_v e^{2\pi i(v, \mathfrak{b})} Z(P, v, \mathfrak{a})$$

$\mathfrak{a}, \mathfrak{b}$: complexified Fenchel-Nielsen coordinates

$Z(P, v, \mathfrak{a})$ Involves integrals over moduli spaces of singular monopoles of characteristic classes.



So, What Did He Say?

Recent new old

Recent results on $N=2$ $d=4$
imply new results about the
differential geometry of old
monopole moduli spaces.