# Breaking News About N=2* SYM On Four-Manifolds, Without Spin

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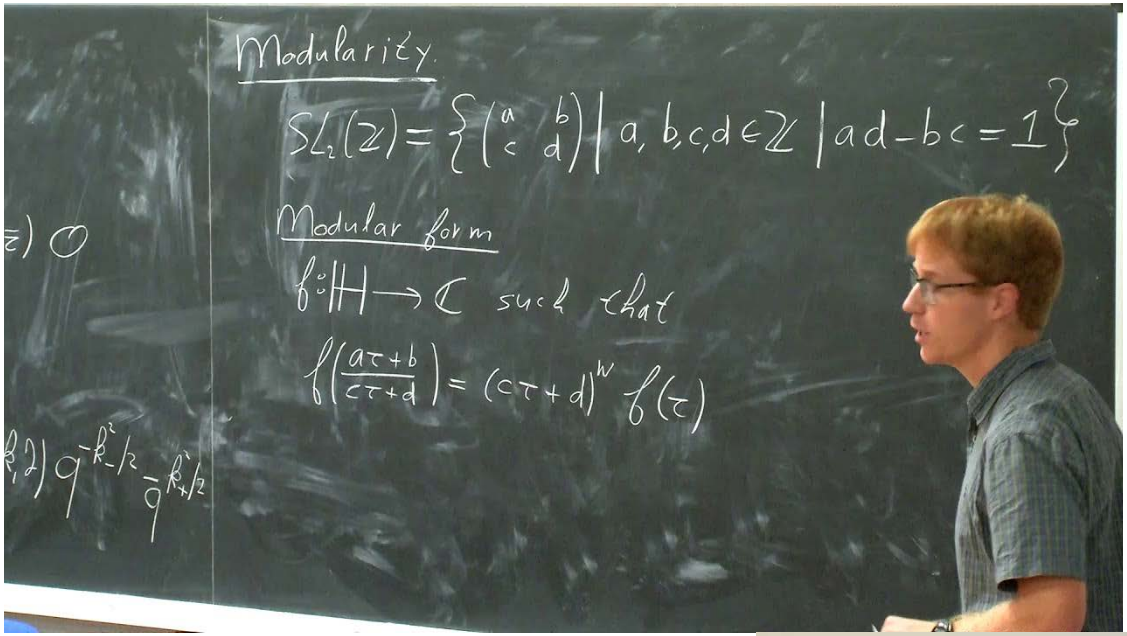
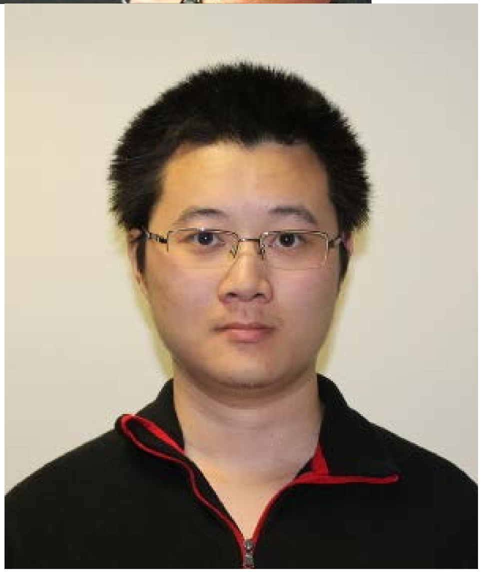
# Work with JAN MANSCHOT




+ related work with JM
and XINYU ZHANG



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# Nontechnical Summary

We study a TQFT in 4d whose partition function generalizes both the Donaldson invariants and the Vafa-Witten invariants, and interpolates between them.

The theory depends on a choice of background spin-c structure $\mathfrak{s}$. This dependence has not previously been discussed. Including it turns out to be nontrivial. We believe we have solved the problem completely.

I gave a preliminary report at the end of my talk at StringMath 2018, where the last slide said…

# Surprise!!

It doesn't work!

Correct version appears to be non-holomorphic.

With Jan Manschot we have an alternative which is currently being checked.

Does the u-plane integral make sense for ***ANY*** family of Seiberg-Witten curves ?

どうもありがとうございます

 

And then – oh Muse! – after many years wandering o'er stormy seas, 'twixt whirlpools o' singularities and monstrous mock modular forms, unnumbered toils we did endure through dark and dismal nights, 'till with the rosy-fingered dawn, in the safe haven of explicit formulae, with full many consistency checks, we did arrive.

# Intro & Main Claims – 1/6

d=4 N=2* SYM. $G = SU(2), SO(3)$

$X$: Smooth, compact, oriented, $\partial X = \emptyset$ , $b_2^+ > 0$,

For simplicity: Connected, $\pi_1(X) = 0$, ignore torsion

**<u>Data needed to formulate the invariants:</u>**

$\tau_0 \in \mathcal{H}$ ; $q_0 := e^{2\pi i \tau_0}$ $m \in \mathbb{C}$

(UV) Spin-c structure: $\mathfrak{s}$, $c_{uv} := c_1(\mathfrak{s}) \in H^2(X, \mathbb{Z})$

$\nu \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ Orientation of $H^2(X; \mathbb{R})$

# Intro & Main Claims – 2/6

Path integral defines a ``function''

$$Z_\nu(\tau_0, m, c_{uv}) : H_*(X;\mathbb{Z}) \to \mathbb{C}$$

$$Z_\nu(\tau_0, m, c_{uv})(x) = \sum_{k\geq 0} q_0^k \int_{\mathcal{M}_k} e^{\mu(x)}\ Eul(\mathcal{E}_{\mathfrak{s}}; m)$$

$\mathcal{M}_k$: Moduli of ASD connections on a principal $SO(3)$ bundle $P \to X$ with $\nu = w_2(P)$ and instanton no. $= k$

$$\mu : H_*(X, \mathbb{Z}) \to H^{4-*}(\mathcal{M}_k; \mathbb{Q})$$

$\mathcal{E}_{\mathfrak{s}}$ : U(1)-equivariant virtual bundle over moduli space of instantons

# Intro & Main Claims – 3/6

Special cases were studied in
[Moore & Witten 1997; Labastida & Lozano 1998 ]

Those studies were limited to spin manifolds
with trivial spin-c structure.

Related work: Dijkgraaf, Park, Schroers 1998
N=1 deformation of N=4 SYM, twisted using
Kahler structure for Kahler 4-folds with $b_2^+ \geq 3$.

Recently an important contribution to related issues appeared in [18]. It would be fruitful to apply the physical methods of [18] to the problems addressed here. In that way one could compute the entire generating functional of the $N = 2$ theory with a massive adjoint hypermultiplet, and work on more general four-manifolds.

# Intro & Main Claims – 4/6

An acs $\mathcal{J}$ defines a canonical spin-c structure $\mathfrak{s}(\mathcal{J})$ .

(Use canonical homomorphism $U(2) \to Spin^c(4).$)

1A: For such a spin-c structure and $m \to 0$

$$Z_\nu(\tau_0, m, c_{uv}) \to Z_\nu^{VW}(\tau_0)$$

1B: $m \to \infty \,\&\, q_0 \to 0 \;\; with \;\; \Lambda^4 := 4m^4 q_0$ fixed:

$$Z_\nu^{renorm}(\tau_0, m, c_{uv}) \to Z_\nu^{DW}$$

# Intro & Main Claims – 5/6

$$Z_\nu = Z_\nu^{CB} + Z_\nu^{SW}$$

$Z_\nu^{CB}$ : Coulomb branch integral

2a: Writing a single-valued measure requires nonholomorphic interactions with $\mathfrak{s}$

(⇒ implications for class S generalization )

2b: ***Integrand*** is a total derivative of a Maass-Jacobi form. (``Mock Jacobi form'' )

2c: ***Value*** of the integral is a nonholomorphic completion of a mock modular form.

2d: VW expression for $\mathbb{CP}^2$ is a special case

# Intro & Main Claims – 6/6

For $b_2^+ > 1$ $Z_\nu$ is a linear combination of SW invariants with coefficients in a ring of modular forms for $\tau_0$

Corollary: VW invariants vanish if $X = Y_1 \# Y_2$ with $b_2^+(Y_i) > 0$

(VW invariants only rigorously defined for algebraic surfaces: Tanaka-Thomas 2017; Sheshmani-Yau 2019)

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# $\mathcal{N} = 2^*$ Theory

## Bosonic Fields:

Vectormultiplet $A \in \mathcal{A}(P)$ $\phi \in \Gamma(adP \otimes \mathbb{C})$

$$S = \int \overline{\tau_0}\, tr(F_+^2) + \tau_0 tr(F_-^2) + \cdots$$

Adjoint HM: $q, \tilde{q} \in \Gamma(adP \otimes \mathbb{C})$

$$W = Tr(\, q\, (Ad(\phi) + m)\tilde{q}\, )$$

$\Rightarrow U(1)_b$ symmetry: Charge$(q, \tilde{q}) = (1, -1)$

# Topological Twisting

Couple to background $SO(3)_R$ bundle with connection.

Choose an isomorphism with SO(3) bundle with connection associated to $(\Lambda^{2,+}TX, \nabla^{LC})$

*Magically, all metric dependence is Q-exact (Witten 1988):*

$$S = \int \tau_0 \, tr(F^2) + Q(*)$$

N.B. Also holomorphic in $\tau_0$

With adjoint hypers topological twisting only makes sense if they couple to a background spin-c structure $\mathfrak{s}$ and spin-c connection [Labastida-Marino 95]

# Topological Twisting

HM bosons $(q, \tilde{q}^*) \Rightarrow \quad M \in \Gamma(W^+ \otimes adP \otimes \mathbb{C})$

$W^+ \to \quad X :$ Positive chirality rank two bundle associated to uv spin-c structure $\mathfrak{s}$

$Q$ –fixed point equations

$$F^+ + [M, \bar{M}] = 0 \qquad\qquad \boldsymbol{D}M = 0$$

``Nonabelian monopole/SW equations”

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

$U(1)_b$ acts on the moduli space $\mathcal{M}_Q$ of these eqs.

Fixed point set: $M = 0$ is $\mathcal{M}_{asd}(P) = \mathcal{M}_k$

# Observables

$$\mathcal{O}: H_*(X,\mathbb{Z}) \to \quad Q - coho$$

$$\mathcal{O}(p) = [\, Tr\, \phi^2(p)]$$

$$\mathcal{O}(S) = [\, \int_S Tr(\phi F + \psi^2)]$$

Q- coho $\cong$ $H^*_{U(1)_b}(\mathcal{M}_Q)$

$m$ : $U(1)_b$ equivariant parameter

= deg 2 generator of $S^*(\mathfrak{u}_b(1))$

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

# Localization

$Q$: Path integral $\rightarrow \int_{\mathcal{M}_Q} \cdots$

$U(1)_b$: $\int_{\mathcal{M}_Q} \cdots \rightarrow \int_{\mathcal{M}_{asd}} \cdots$

$$\langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*} = \sum_{k\geq 0} q_0^k \int_{\mathcal{M}_k} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}}; m)$$

$\mathcal{E}_{\mathfrak{s}}$ : Obstruction bundle for elliptic complex, pulled back to $\mathcal{M}_k$.

# Index Computations

$$v \dim \mathcal{M}_Q = \dim G \, \frac{c_{uv}^2 - (2\chi + 3\,\sigma)}{4}$$

N.B. Independent of instanton number $k$ !

$$\dim \mathcal{M}_k = 8k - \frac{3}{2}(\chi + \sigma)$$

$\Rightarrow$ Partition function is an infinite $q_0$ - series even without insertion of observables.

$$Index\ \boldsymbol{D} = -8k + \frac{3}{8}(c_{uv}^2 - \sigma)$$

Conjecture: $\mathcal{E}_{\mathfrak{s}}$ =ker($\boldsymbol{D}^*$) for large $k$

# Relation To Vafa-Witten Equations-1/2

$$A \in \mathcal{A}(P) \qquad C \in \Gamma(ad\ P) \qquad B^+ \in \Omega^{2,+}(ad\ P)$$

$$F^+ + [B^+, B^+] + [C, B^+] = 0$$

$$D_\mu C + D^\nu B^+_{\nu\mu} = 0$$

ACS $\mathcal{J} \Rightarrow \quad \Lambda^{2,+} T^* X \cong \underline{\underline{\mathbb{R}}} \oplus K_{\mathbb{R}}$

$\mathcal{J}$ also determines a
canonical spin-c structure $\mathfrak{s}(\mathcal{J})$

$$W^+ \cong \underline{\underline{\mathbb{R}}} \otimes \mathbb{C} \oplus K$$

# Relation To Vafa-Witten Equations -2/2

DW twist of N=4 SYM is inequivalent to the SW twist.

Nevertheless, for $\mathfrak{s}(\mathcal{J})$ Q-fixed point eqs coincide

Seiberg-Witten $\cong$ Vafa-Witten

# Mass Limits

$$Lim_{m\to 0}[N = 2^* \, SYM] = [N = 4 \, SYM]$$

SW94:

$$m \to \infty \,\&\, q_0 \to 0$$

$$\Lambda_0^4 = 4\, m^4 q_0$$

$$\Rightarrow pure\ SYM$$

# Mass Limits – 2

$$\langle e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*} = \sum_{k\geq 0} q_0^k \int_{\mathcal{M}_k} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}}; m)$$

$$Eul(\mathcal{E}_{\mathfrak{s}}; m) = \prod_i (x_i + m) = m^{-Index(D)} \sum_{\ell} \frac{c_\ell(\mathcal{E}_{\mathfrak{s}})}{m^\ell}$$

Leading term for $m \to 0$ : $c_{top}(\mathcal{E}_{\mathfrak{s}})$

For $\mathfrak{s}(\mathcal{J})$: $\mathcal{E}_{\mathfrak{s}} \cong T^*\mathcal{M}_k \Rightarrow$ ``Euler character of $\mathcal{M}_k$ ''

Leading term for $m \to \infty$ : $c_0(\mathcal{E}_{\mathfrak{s}})$= 1

$\Rightarrow$ Donaldson invariants

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# Coulomb Branch Integral

This is a useful and nontrivial test case for a more general very interesting open problem: Generalize DW theory to class S.

CB = Base of a Hitchin system $\mathcal{B}$

Here: $u \in \mathbb{C} \cong \mathcal{B}$

Physics described by special geometry of a family of Abelian varieties over $\mathcal{B}$

SW94: Jacobians of a holo family of curves

Equipped with meromorphic differential

# Seiberg-Witten Geometry

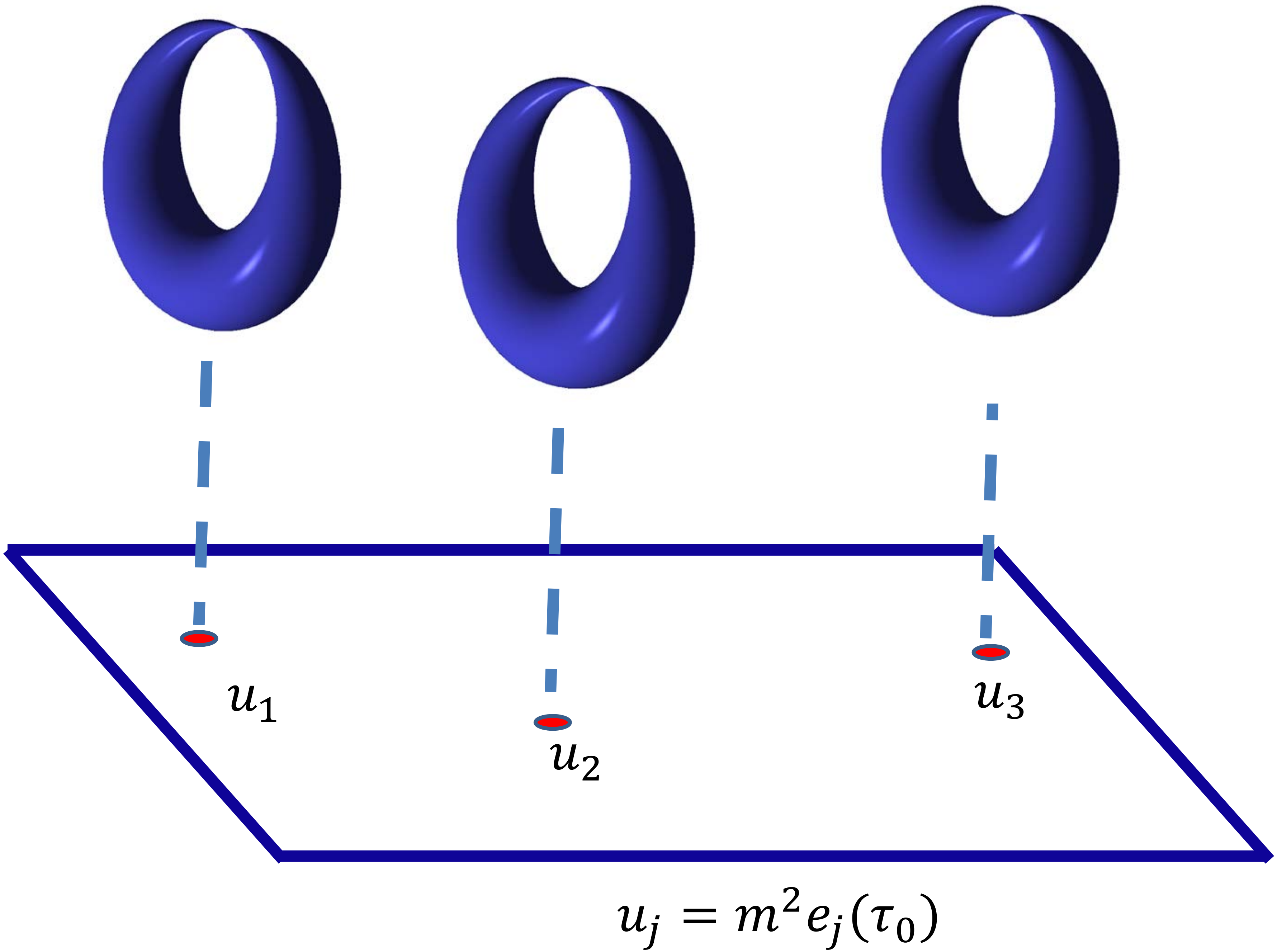
$E_u \quad y^2 = \prod_{i=1}^{3}(x - \alpha_i) \qquad \alpha_i = u\, e_i(\tau_0) + \frac{m^2}{4} e_i(\tau_0)^2$

$e_i(\tau_0)$ half-periods of $E_{\tau_0} = \mathbb{C}/(\mathbb{Z} + \tau_0\mathbb{Z})$

$$e_i(\tau_0) \in \{\frac{1}{3}(\vartheta_3^4(\tau_0) + \vartheta_4^4(\tau_0), \dots\}$$

N.B. After choice of duality frame $E_u$ has a $\tau(u, m, \tau_0)$ which should not be confused with $\tau_0$

$\lim_{m \to 0} \tau(u, m, \tau_0) = \tau_0 \qquad \lim_{u \to \infty} \tau(u, m, \tau_0) = \tau_0$


$u_1$
$u_2$
$u_3$
$u_j = m^2 e_j(\tau_0)$

# Path Integral Of U(1) LEET

LEET: U(1) Maxwell + N=2 superpartners with topological couplings

$$Z_{\nu}^{CB} = \int_{\mathcal{B}} d^2u \, B(u)^{\sigma} A(u)^{\chi} \, Z_{Maxwell}$$

$$B = \prod_i (u - u_i)^{\frac{1}{8}}$$

⇒ Potential problems with single-valued measure.

# CB Measure: 1997-1998

$$Z_{\nu}^{CB}(p,S) = \int_{\mathcal{B}} d^2u \, B^{\sigma} \, e^{pu} e^{S^2 T} A^{\chi} \, \Psi_{\nu}$$

$$A = \left(\frac{da}{du}\right)^{-\frac{1}{2}} \qquad \Psi_{\nu} \sim \sum_{fluxes} e^{-\int \bar{\tau} F_{dyn,+}^2 + \tau F_{dyn,-}^2}$$

Depend on duality frame –

- but the local system has nontrivial monodromy.

# CB Measure Only SV For $X$ Spin

$A^{\chi} e^{S^2 T} \Psi_{\nu}$ *is independent of duality frame, up to* $8^{th}$ *roots of unity.*

On a spin manifold, $\sigma = 0\ mod\ 8$ :
The measure is single-valued.

If $X$ is not spin the above measure is not single-valued ….

# CB Measure: New Interactions

We need to include the background spin-c structure $\mathfrak{s}$

There are couplings to the UV spin-c connection:

$$\Delta S_{LEET} = \int c(u) F_b^2 + d(u)\, F_b F_{dyn}$$

[Shapere-Tachikawa, 2008]

Surprise! No choice of holomorphic coupling makes the measure single-valued!

# Resolution

``Weakly gauge" the $U(1)_b$ symmetry:

Gauge group: $U(1)_b \times G$ $G \in \{SU(2), SO(3)\}$

$\Rightarrow$ Rank TWO gauge group.

Take UV coupling of $U(1)_b$ to zero:

Freezes $U(1)_b$ vm fields to classical values

$$m = \langle a_b \rangle$$

Nati Seiberg & Ann Nelson – 1993

# Non-Holomorphic Coupling

Rank 2 Maxwell action: $F_I = (F_b, F_{dyn})$

$$\sim \int \bar{\tau}_{IJ} F_+^I F_+^J + \tau_{IJ} F_-^I F_-^J + \cdots$$

$$\tau_{IJ} = \begin{pmatrix} \frac{d^2\mathcal{F}}{da^2} & \frac{d^2\mathcal{F}}{dadm} \\ \frac{d^2\mathcal{F}}{dadm} & \frac{d^2\mathcal{F}}{dm^2} \end{pmatrix} \qquad v = \frac{d^2\mathcal{F}}{dadm}$$

$$\int \bar{v}\, F_b^+ F_{dyn}^+ + v\, F_b^- F_{dyn}^-$$

# Remark: SW limit $m \to \infty$

$$e^{\int \bar{v}\, F_b^+ F_{dyn}^+ + v\, F_b^- F_{dyn}^-}$$

Metric dependent & nonholomorphic, varying continuously on $\mathcal{B}$

$$\to e^{i\,\pi \int w_2(X) \frac{F_{dyn}}{2\pi}}$$

Important implications for the generalization of CB integral to class S theories: We do not want a $\mathbb{Z}_2$ –valued QRIF.

# Coulomb Branch Measure: 2019 -2020

$$Z_\nu^{CB} = \int_{\mathcal{B}} \Omega$$

$$\Omega = du \wedge d\bar{u}\; B^{\sigma} e^{pu} e^{S^2 T} A^{\chi} C^{c_{uv}^2} \Psi_\nu$$

Nontrivial question: Is this single-valued ?

Step 1: Use modular parametrization.

Identify $\mathcal{B}$ with the modular curve $\mathcal{H}/\Gamma(2)$

# Modular Parametrization

## Weak coupling duality frame:

Nekrasov: Instanton partition function

$$\mathcal{F}(a,m) = \frac{1}{2}\tau_0 a^2 + $$

$$+ m^2 f_1(\tau_0) \left( \log\left(\frac{2a}{m}\right) - \frac{3}{4} + \frac{3}{2}\log\left(\frac{m}{\Lambda}\right) \right)$$

$$+ a^2 \sum_{n=2}^{\infty} f_n(\tau_0) \left(\frac{m}{a}\right)^{2n}$$

$f_n(\tau_0)$: polynomials:

$E_2, E_4, E_6$ wt $= 2n - 2$

[Minhahan, Nemeschansky, Warner; Dhoker, Phong]

$\Lambda, m$ dependence (also A,B couplings):

[Manschot, Moore, Xinyu Zhang 2019]

# Modular Parametrization

$$\tau = \frac{d^2\mathcal{F}}{da^2} \qquad \frac{da}{du} = \oint_A \frac{dx}{y}$$

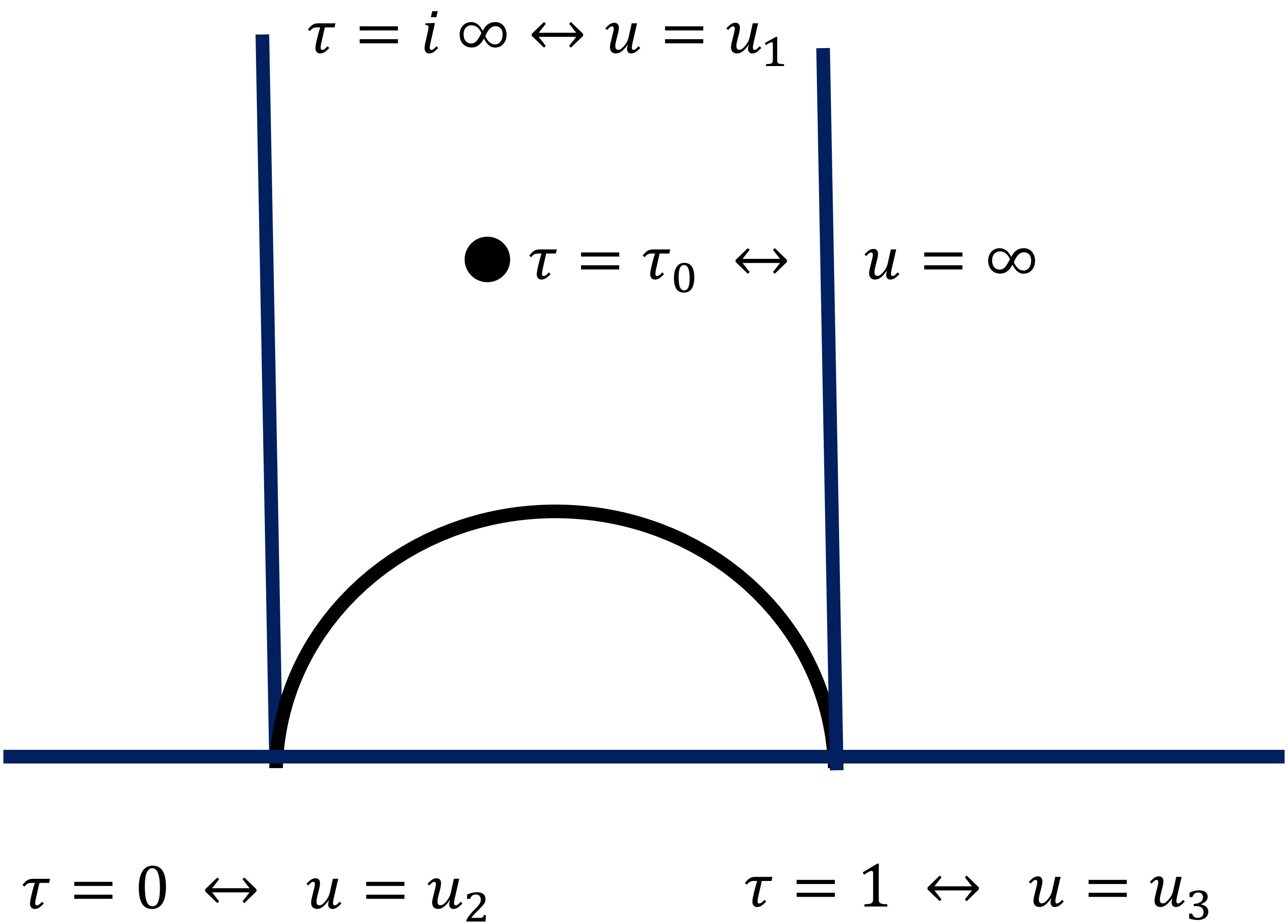
$$m^2 \left(\frac{da}{du}\right)^2 = \frac{\vartheta_4^4(\tau)\vartheta_3^4(\tau_0) - \vartheta_3^4(\tau)\vartheta_4^4(\tau_0)}{\eta^6(\tau_0)}$$

$$m^{-2}\, u(\tau,\tau_0) = \frac{e_1^2(\tau_0)\, e_{23}(\tau) + cycl.}{e_1(\tau_0)e_{23}(\tau) + cycl}$$

[Huang, Kashani-Poor,Klemm]

$$\mathcal{B} \;\cong\; \mathcal{H}/\Gamma(2)$$

# Modular Parametrization



# Two New Nontrivial Identities

$$C := \exp\left(-2\,\pi\, i\,\frac{d^2\mathcal{F}}{dm^2}\right) = \left(\frac{\Lambda}{m}\right)^{\frac{3}{2}} \frac{\vartheta_1(2\tau, 2v)}{\vartheta_2^2(\tau_0)\vartheta_4(2\tau)}$$

$$v := \frac{d^2\mathcal{F}}{dadm}$$

$$\frac{\vartheta_2(2\tau, v)}{\vartheta_3(2\tau, v)} = \frac{\vartheta_2(2\tau_0, 0)}{\vartheta_3(2\tau_0, 0)}$$

Determines $v(\tau, \tau_0)$

# The ``Period Point'' $J$

$$b_2^+ > 1 \Rightarrow Z_\nu^{CB} = 0$$

$Z_{Maxwell}$:

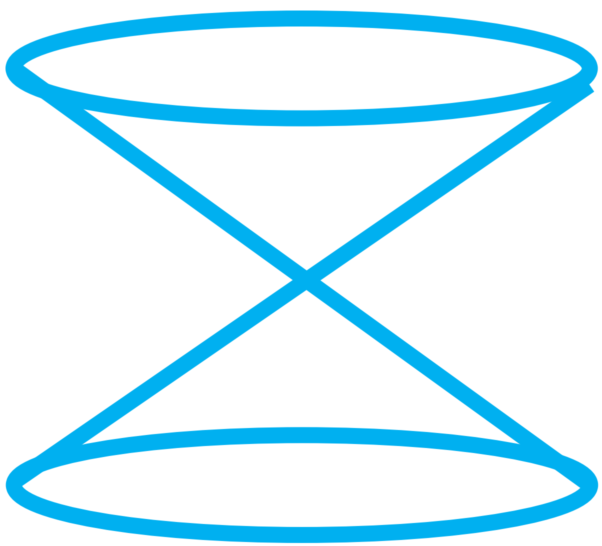
***Frame dependent.***

***Not holomorphic.***

***Metric dependent.***

$$b_2^+ = 1 \quad Z_\nu^{CB} \neq 0$$

$H^2(X;\mathbb{R})$



$$* J = J$$

$$J^2 = 1$$

$$J^0 > 0$$

# Maxwell Partition Function

Sum over the first Chern class $\lambda \in 2L + \nu$ ,

$L = H^2(X;\mathbb{Z})$ (Simplicity: Put $S = 0.$)

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \; e^{\pi i \, \lambda \cdot c_{uv}} \; v(\tau, \tau_0)$$

$$E_\lambda^J = Erf(x_\lambda) \qquad Erf(x) := \int_0^x e^{-\pi t^2} dt$$

$$x_\lambda = \sqrt{Im\tau}(\lambda + \frac{Im v}{Im\tau} c_{uv}) \cdot J$$

With all these ingredients we can now check that the CB measure is indeed monodromy invariant and hence well-defined.

The definition of the integral is still rather subtle. One must define naively divergent expressions like

$$Z_{\mu}^{CB} = \int_{\mathcal{H}/\Gamma(2)} d^2\tau \, (Im\, \tau)^{-s} q^{n} \bar{q}^{\tilde{n}}$$

with $n < 0$ and $\tilde{n} < 0$

It can be done in a satisfactory way: Korpas, Manschot, Moore, Nidaiev 2019

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# Evaluation Of CB Integral ?

$$Z_\nu^{CB} = \int_{\mathcal{H}/\Gamma(2)} \Omega$$

(For simplicity: $p = S = 0$ .)

$$\Omega = d\tau \wedge d\bar{\tau} \;\; B^\sigma A^\chi C^{c_{uv}^2} \Psi_\nu^J$$

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \;\; q^{-\frac{1}{4}\lambda^2} \, e^{\pi i \, \lambda \cdot c_{uv}} \;\; v(\tau, \tau_0)$$

$$\Omega = d\,\Lambda \qquad\qquad \Lambda = d\tau \, B^\sigma A^\chi \, C^{c_{uv}^2} \;\; \hat{G}$$

$$\Psi_\nu^J = \partial_{\bar{\tau}} \, \hat{G}$$

# Evaluation Of CB Integral ?

$$\Psi^J_\nu = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E^J_\lambda \; q^{-\frac{1}{4}\lambda^2} \; e^{\pi i\, \lambda \cdot c_{uv}} \; \upsilon(\tau,\tau_0)$$

$$\Psi^J_\nu = \partial_{\bar{\tau}} \hat{G}$$

$$\hat{G} = \sum_{\lambda \in 2L+\nu} E^J_\lambda \; q^{-\frac{1}{4}\lambda^2} \; e^{\pi i\, \lambda \cdot c_{uv}} \; \upsilon(\tau,\tau_0)$$

??? NO!!! $\lim_{\lambda^2 \to +\infty} E^J_\lambda = \pm 1$

# Evaluating Difference Of CB Integrals

$$\Psi^{J_1} - \Psi^{J_2} = \partial_{\bar{\tau}} \hat{G}^{J_1,J_2}$$

$$\widehat{G^{J_1,J_2}} = \sum_{\lambda \in 2L+\nu} E_\lambda^{J_1,J_2} q^{-\frac{1}{4}\lambda^2} \cdots$$

$$E_\lambda^{J_1,J_2} = Erf\left(x_\lambda^{J_1}\right) - Erf\left(x_\lambda^{J_2}\right)$$

Converges nicely!

$\Rightarrow$ Can use this to evaluate the difference $Z_\nu^{CB,J_1} - Z_\nu^{CB,J_2}$ by a sum of residues.

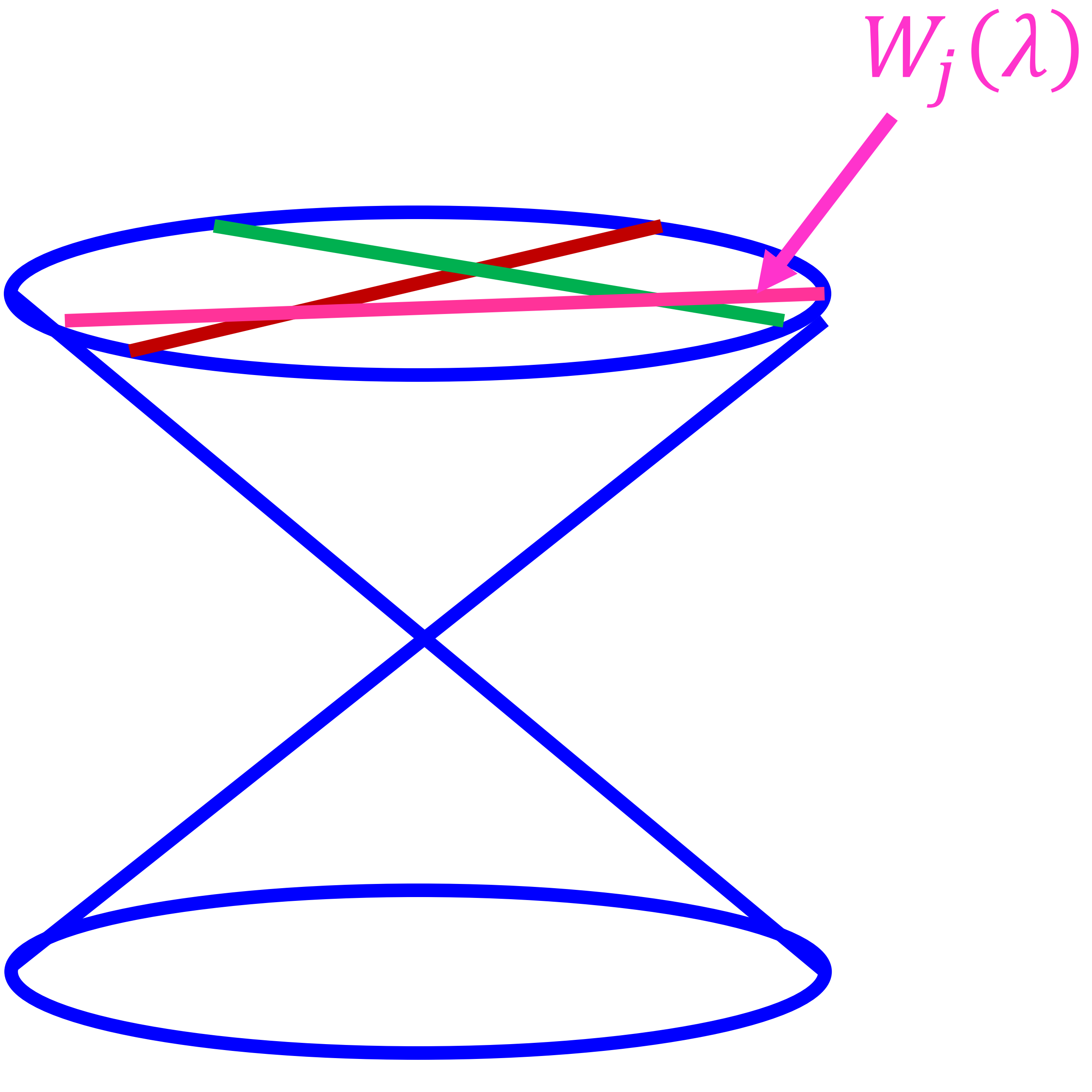
# Wall-Crossing

As the contour approaches the cusp $u_j$: $E_\lambda^{J_1,J_2}$ limits to

$$sign\left[\left(\lambda \pm \frac{1}{2}\, c_{uv}\right)\cdot J_1\right] - sign\left[\left(\lambda \pm \frac{1}{2}\, c_{uv}\right)\cdot J_2\right]$$

$\Rightarrow$ $Z_\nu^{CB,J}$ is piecewise constant as function of $J$ but has nontrivial chamber dependence.

Chambers defined by various walls $W_j(\lambda)$


$W_j(\lambda)$

# Continuous Metric Dependence

For the boundary at $u \to \infty$ the modular parameter $\tau \to \tau_0$. This leads to ***continuous*** metric dependence.

For $\mathfrak{s}(\mathcal{J})$ one finds:

$$\eta(\tau_0)^{-2\chi} \sum_{\lambda} [E(\sqrt{y_0}\lambda \cdot J_1) - E(\sqrt{y_0}\lambda \cdot J_2)](\lambda \cdot c_{uv})\, q_0^{-\lambda^2}$$

+ Another Term

Closely related: Nonholomorphic: $y_0 = Im(\tau_0)$

# The Special Period Point

For any manifold with $b_2^+ = 1$ $\exists$ special $J_0$ such that

$$\Omega = d\,\Lambda \qquad \Lambda = d\tau\, B^{\sigma} A^{\chi}\, C^{c^2_{uv}}\ \hat{G}$$

Where we can write $\hat{G}$ explicitly so that $\Lambda$ is:

1. Well-defined
2. Nonsingular away from $\tau \in \{0, 1, i\infty, \tau_0\}$
3. Modular: Good $q_i$ expansion near cusps

# Mock Jacobi-Maass Forms

These conditions determine $\hat{G}$ uniquely.

It is a Jacobi-Maass form evaluated at $z = c_{uv}\, v(\tau, \tau_0)$

After doing the integration by parts we obtain mock modular forms as functions of $\tau_0$

For $X = \mathbb{CP}^2$ and $\mathfrak{s}(\mathcal{J})$ we reproduce exactly the mock modular forms used in Vafa-Witten.

+ many generalizations

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# LEET Near Cusps $u_j$

In the region of each cusp $u_j$ , $j = 1,2,3$

the LEET changes:

We have a U(1) VM coupled to a charge 1 HM.

(In the appropriate duality frame) [Seiberg-Witten 94]

There is a separate contribution to the path integral coming from the path integral of these three LEET.

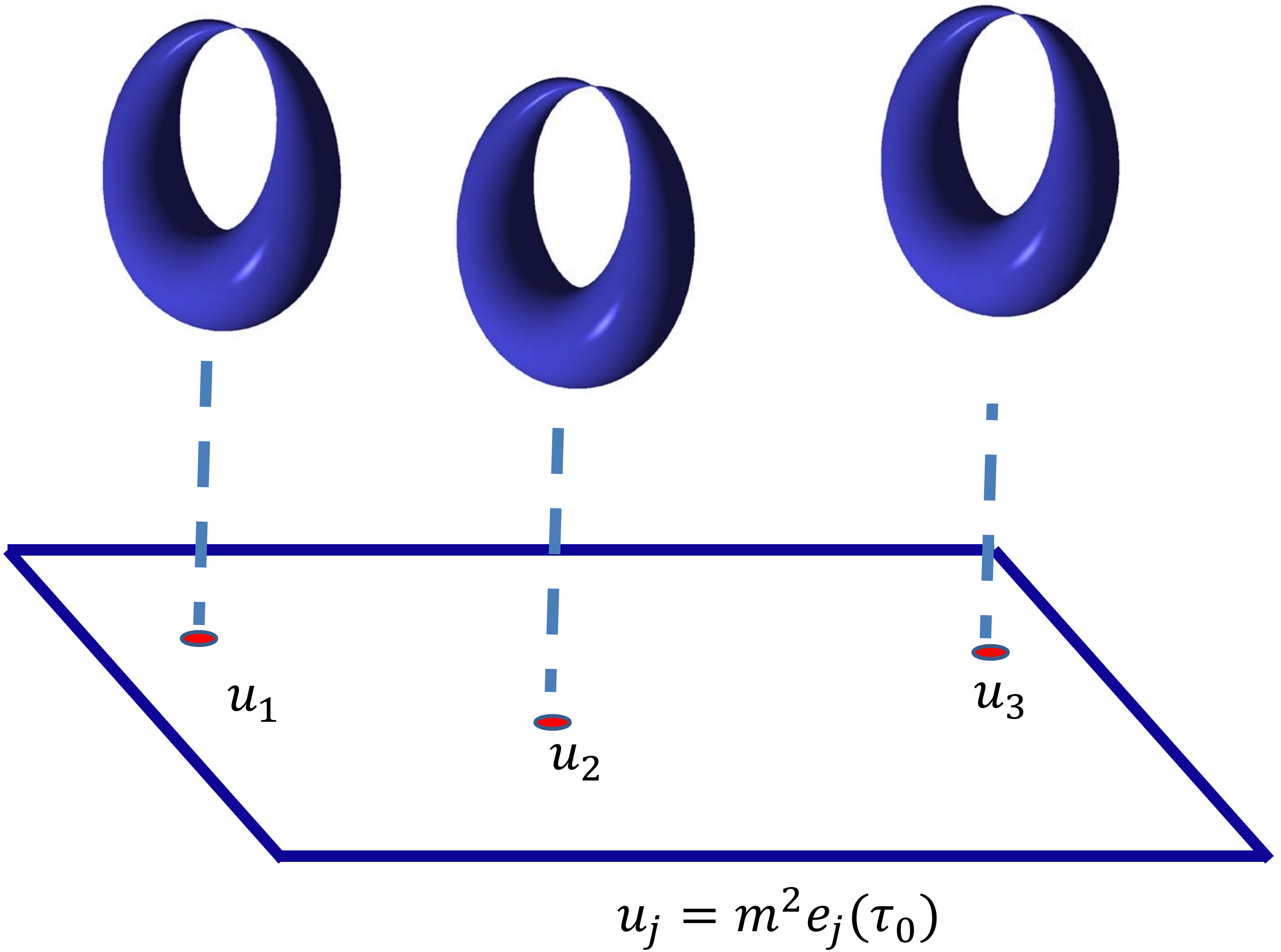
We add the contributions, because we sum over vacua:

$$Z_\nu = Z_\nu^{CB} + \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

When $b_2^+ > 1$ $\mathrm{Z}_\nu^{\mathrm{CB}}$ vanishes –
- we get true topological invariants:

$$Z_\nu = \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

So it is quite interesting to determine
The three effective actions


$u_1$
$u_2$
$u_3$
$u_j = m^2 e_j(\tau_0)$

# General Form Of Effective Action Near $u_j$

$a$: Local special coordinate vanishing at $u_j$

$$S_{LEET,j}^{SW} =$$

$$\int \alpha_j(a) Eul(X) + \beta_j(a) Sig(X) + \gamma_j(a) F_{dyn}^2$$

$$+ \int \delta_j(a) F_{dyn} \wedge F_b + \varepsilon_j(a) F_b \wedge F_b$$

$$+ Q(*)$$

# Determination Of Effective Action

MW97: The terms in the effective action at $u_j$ can be determined from the contribution to the wall-crossing behavior $Z_{\nu}^{CB}$ from $u_j$

$$Z_{\nu,j}^{SW} = \sum_{c_{ir}=w_2(X) mod 2} SW(c_{ir}) \left[ \mathcal{A}_j^{\chi} \mathcal{B}_j^{\sigma} \mathcal{C}_j^{c_{uv}^2} \mathcal{D}_j^{c_{uv}\cdot c_{ir}} \mathcal{E}_j^{c_{ir}^2} \right]_{q_j^0}$$

There is a prescription for including the homology observables $e^{\mu(x)}$

$$Z_\nu = \kappa_\nu \left(\frac{\Lambda}{m}\right)^{3/8(c_{uv}^2-2\chi-3\sigma)} \eta(\tau_0)^{3(2\chi+3\sigma)} \Bigg\{ \eta(\tau_0/2)^{-5\chi-6\sigma-c_{uv}^2} e^{S^2\mathcal{K}_1+\frac{p}{4}\frac{m^2}{\Lambda^2}e_2(\tau_0)}$$

$$\sum_{c_{ir}} SW(c_{ir}) e^{\frac{\mathrm{i}\pi}{2}(c_{ir}-c_{uv})\cdot\nu\chi_h} \left(\frac{\vartheta_3(\tau_0/2)}{\vartheta_4(\tau_0/2)}\right)^{\frac{1}{2}c_{uv}\cdot c_{ir}} e^{\frac{m}{\Lambda}(\vartheta_2\vartheta_3(\tau_0))^2 c_{ir}\cdot S}$$

$$+\cdots \Bigg\}$$

$X = K3$ @ $p = 0$ & $S = 0$

$$Z_\mu = 2^{12}\left(\frac{\Lambda}{m}\right)^{\frac{3}{8}c_{uv}^2} \left[\frac{\delta_{\mu=\frac{1}{2}c_{uv}\mathrm{mod}2}}{(\vartheta_2\eta)^p} + \frac{e^{\mathrm{i}\pi\mu\cdot c_{uv}/2}}{(\vartheta_4\eta)^p} - \frac{e^{\mathrm{i}\pi\mu\cdot c_{uv}/2-\mathrm{i}\pi\mu^2/4}}{(\vartheta_3\eta)^p}\right]$$

$p = 12 + \frac{1}{2}c_{uv}^2$ $\quad\quad$ $\vartheta_i\eta$ evaluated at $\tau_0$

# Relation To Previous Results

For $c_{uv}^2 = 2\chi + 3\,\sigma$ and $m \to 0$ we recover and generalize formulae of [VW;DPS] for VW invariants.

For $c_{uv} = 0$ we recover formulae of Labastida-Lozano

For $m \to \infty\,, \mathrm{q}_0 \to 0$ *after suitable renormalization* we recover the ``Witten conjecture'' for the Donaldson invariants in terms of the Seiberg-Witten invariants.

A generalization and unification of the 1990's formulae:
Vafa-Witten; Witten; Moore-Witten;
Dijkgraaf-Park-Schroers; Labastida-Lozano

# Concluding Remarks

Twisted $N = 2^*$ on four-manifolds with a spin-c structure unifies and generalizes previous expressions for invariants of 4-manifolds derived from SYM.

Some technical points are still being sorted out.

Non-simply connected generalization and implications for three-manifold invariants?

Hamiltonian formulation (Floer theory)?

Derivation from 6d (2,0) theory?

# S-Duality

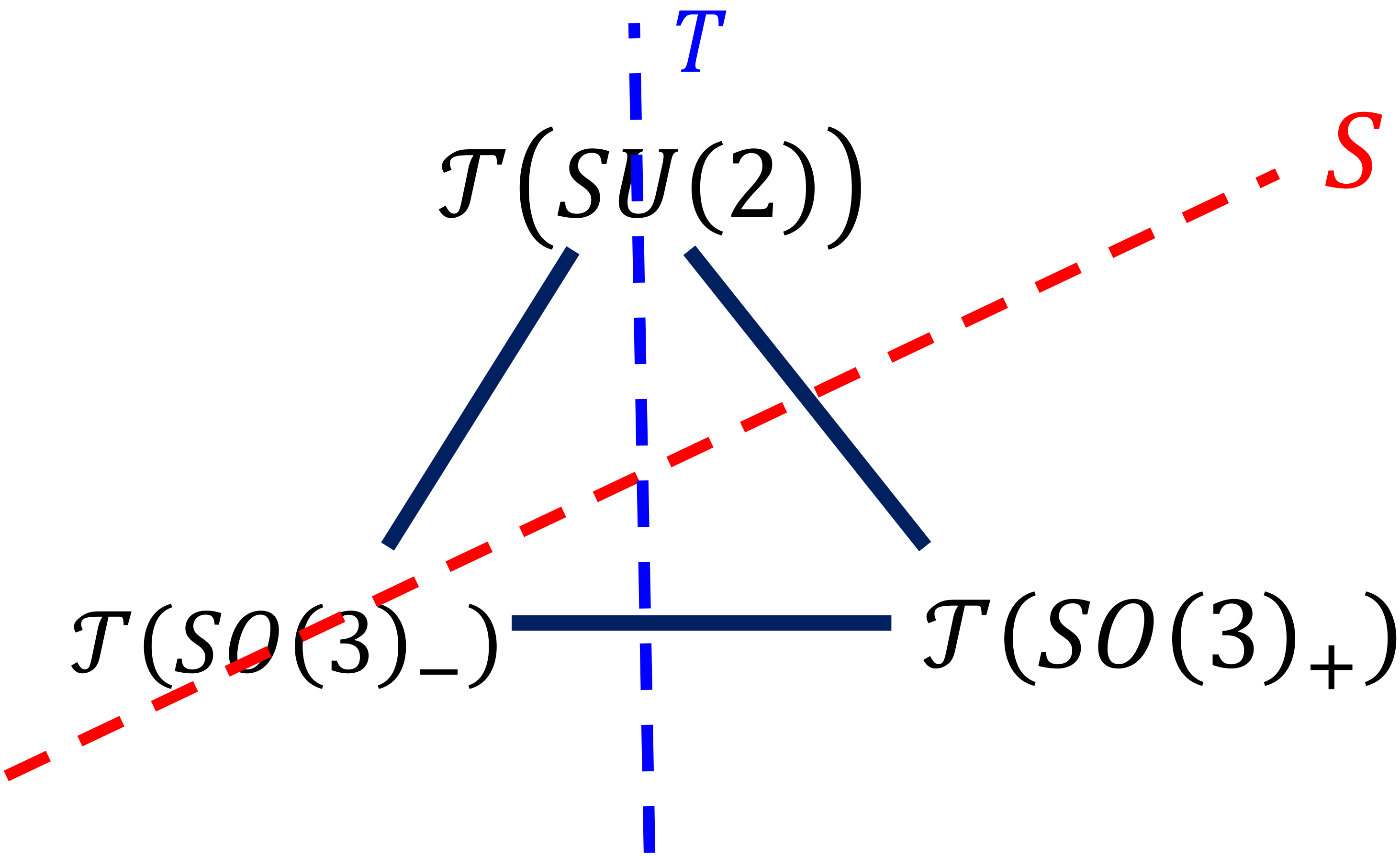
In the $SU(2)$ theory $Z_\nu$ is the partition function in the presence of 't Hooft flux

``Partition function in a background field for a magnetic $\mathbb{Z}_2$ 1-form symmetry."

The $Z_\nu$ span a vector space $\mathcal{V}$

But arbitrary linear combinations aren't physically meaningful

# Three Distinct Theories




Gaiotto, Moore, Neitzke 2009;
Aharony, Seiberg, Tachikawa 2013

# Partition Functions For The $SO(3)_\pm$ Theories

$$Z_\nu^{SO(3)_+} = \sum_\rho e^{i\,\pi\,\nu\cdot\rho} Z_\rho$$

$$Z_\nu^{SO(3)_-} = \sum_\rho e^{\frac{i\pi}{2}\rho^2 + i\pi\nu\cdot\rho} Z_\rho$$

$$\Delta S = \frac{i\pi}{2}\int P_2\big(w_2(P)\big)$$

Aharony, Seiberg, Tachikawa 2013

# S-Duality Transformations

$$T: Z_\nu \to \xi_\nu \, Z_\nu$$

$$S: Z_\nu \to (-i\,\tau_0)^w \sum_\rho e^{i\,\pi\nu\cdot\rho} Z_\rho$$

$$w = \frac{1}{2}(\chi + 3\,\sigma - c_{uv}^2)$$

$$\xi_\nu = \omega^{\,-\frac{1}{2}(2\chi - 3\sigma + c_{uv}^2 + 12\nu^2)} \qquad \omega = e^{\frac{2\pi i}{24}}$$

Derivation from 6d ?

# Orbit Of Partition Functions -1/2

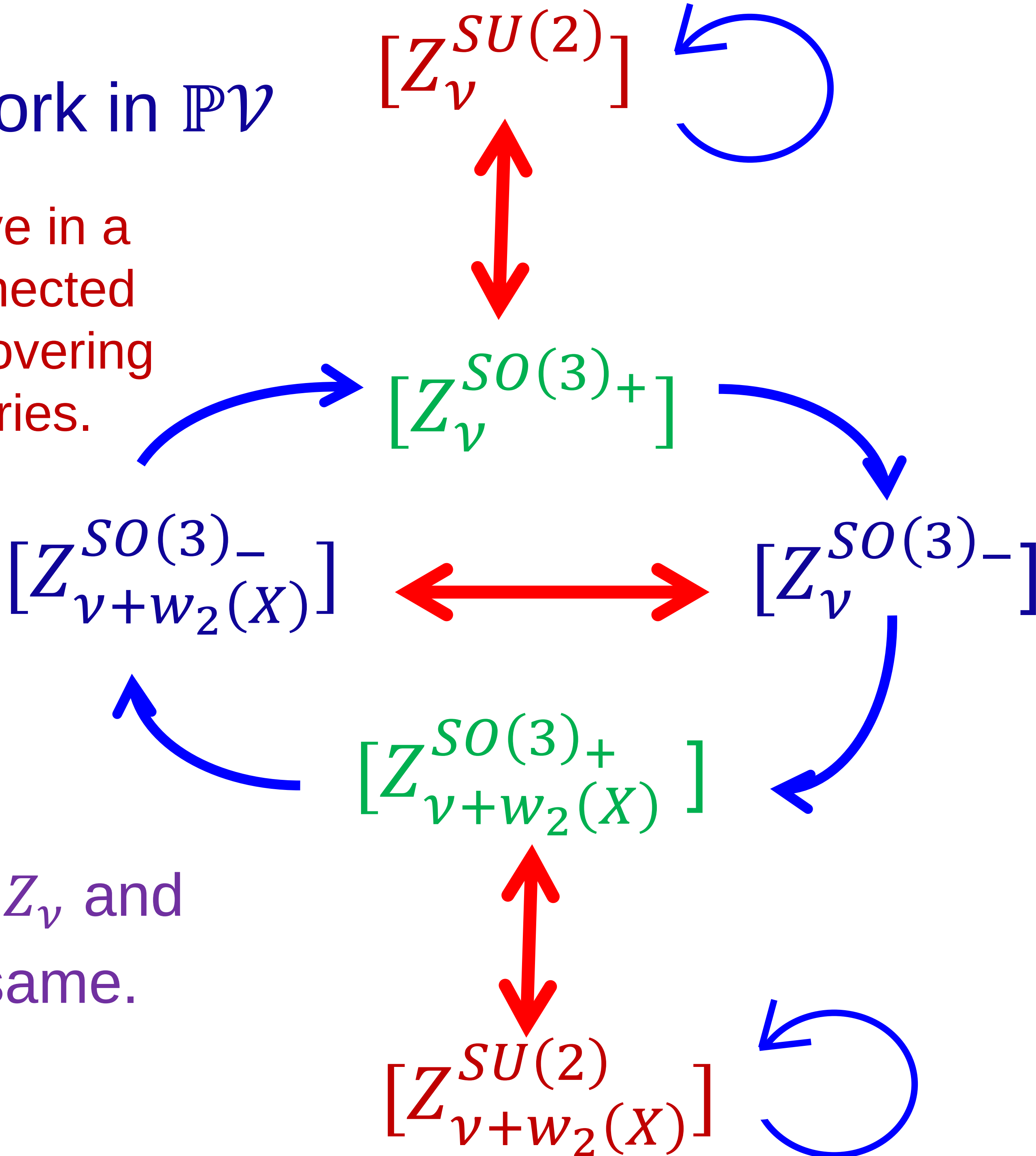
The $Z_\nu$ span a vector space $\mathcal{V}$

The physical partition functions of the theories form an *orbit* in that vector space.

It is a finite covering of the triangle of theories.

For simplicity, work in $\mathbb{P}\mathcal{V}$

Partition functions live in a disjoint union of connected orbits, each double-covering the triangle of theories.




Orbits of $Z_\nu^{SU(2)} = Z_\nu$ and $Z_{\nu+w_2(X)}^{SU(2)}$ are the same.

# REMARKS ON CLASS S: SLIDES FROM MY STRING MATH 2018 TALK IN SENDAI, JAPAN

# u-plane for class S: General Remarks

UV interpretation is not clear in general.
These theories might give new 4-manifold invariants.

The u-plane is an integral over the base $\mathcal{B}$ of a Hitchin fibration with a theta function associated to the Hitchin torus. It will have the form

$$Z_u = \int_{\mathcal{B}} du\, d\bar{u}\, \mathcal{H}\, \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

**$\Psi$: *NOT holomorphic*** and ***metric- DEPENDENT***
***``theta function''***

# Class S: General Remarks

$$\mathcal{H} = \alpha^{\chi}\,\beta^{\sigma}\,\det\left(\frac{da^i}{du_j}\right)^{1-\frac{\chi}{2}}\;\Delta_{phys}^{\frac{\sigma}{8}}$$

$\Delta_{phys}$ a holomorphic function on $\mathcal{B}$ with first-order zeros at the loci of massless BPS hypers

$\alpha, \beta$ will be automorphic forms on Teichmuller space of the UV curve $C$

$\alpha, \beta$ are related to correlation functions for fields in the (0,2) QFT gotten from reducing 6d (0,2)

# Class S: General Remarks

$$\Psi \sim \sum_{\lambda} e^{i\pi\lambda\cdot\xi} e^{-i\pi\bar{\tau}(\lambda_+,\lambda_+)-i\pi\tau(\lambda_-\cdot\lambda_-)+\cdots}$$

$$\Gamma \subset H^1(\Sigma;\mathbb{Z})$$

Lagrangian sublattice

$$\lambda \in \lambda_0 + \Gamma \otimes H^2(X;\mathbb{Z})$$

$$\xi \in \Gamma \otimes H^2(X;\mathbb{R})$$

***If*** $\xi = \rho \otimes w_2(X)\ mod\ 2$ ***then*** WC from interior of $\mathcal{B}$ will be cancelled by SW invariants

$\Rightarrow$ No new four-manifold invariants…

$\Psi$ comes from a ``partition function" of a level 1 SD 3-form on $M_6 = \Sigma \times X$

Quantization: Choose a QRIF $\Omega$ on $H^3(M_6; \mathbb{Z})$

Natural choice: [Witten 96,99; Belov-Moore 2004]

$$\Omega(x) = \exp\left( i\pi \; WCS(\theta \cup x; \; S^1 \times M_6 ) \right)$$

Choice of weak-coupling duality frame + natural choice of $spin^c$ structure gives

$$\xi = \rho \otimes w_2(X)$$

# Case Of SU(2) $\mathcal{N} = 2^*$

Using the tail-wagging-dog argument, analogous formulae were worked out for $\mathcal{N} = 2^*$, by Moore-Witten and Labastida-Lozano in 1998, but *only* in the case when $X$ is spin.

L&L checked S-duality for the case $b_2^+ > 1$

The generalization to $X$ which is NOT spin is nontrivial: The standard expression from Moore-Witten and Labastida-Lozano is NOT single-valued on the u-plane.

This is not surprising: The presence of external $U(1)_{baryon}$ gauge field $F_{baryon} \sim c_1(\mathfrak{s})$ means there should be new interactions:

$$e^{\,\kappa_1(u) c_1(\mathfrak{s})^2 + \kappa_2(u)\lambda\cdot c_1(\mathfrak{s})}$$

Shapere & Tachikawa

Holomorphy, 1-loop singularities, single-valuedness forces:

$$(u - u_1)^{-\frac{c_1(\mathfrak{s})^2}{8}} e^{-i\frac{\partial a_D}{\partial m}\lambda\cdot c_1(\mathfrak{s})}$$

# Surprise!!

It doesn’t work!

Correct version appears to be non-holomorphic.

With Jan Manschot we have an alternative which is currently being checked.

Does the u-plane integral make sense for ***<u>ANY</u>*** family of Seiberg-Witten curves ?

# MORE DETAILS ABOUT MOCK MODULAR FORMS :
# SLIDES FROM MY
# JMM TALK
# JANUARY, 2020, DENVER

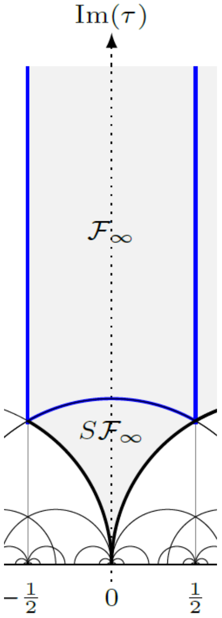
# Relation To Mock Modular Forms -1.1

$Z_u$ : A sum of integrals of the form :

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, (Im\,\tau)^{-s}\, f(\tau,\bar{\tau})$$

Support of $c$ is bounded below

$$f(\tau,\bar{\tau}) = \sum_{m-n\in\mathbb{Z}} c(m,n) q^m\, \bar{q}^n$$




Strategy: Find $\hat{h}(\tau,\bar{\tau})$ such that

$$\partial_{\bar{\tau}}\hat{h} = (Im\,\tau)^{-s}\, f(\tau,\bar{\tau})$$

$\hat{h}\,(\tau,\bar{\tau})$ is modular of weight (2,0)

# Relation To Mock Modular Forms – 1.2

$$\hat{h}(\tau,\bar{\tau}) = h(\tau) + R$$

We choose an explicit solution

$$\partial_{\bar{\tau}} R \; = (Im\tau)^{-s} f(\tau,\bar{\tau})$$

vanishing exponentially fast at $Im\tau \to \infty$

$h(\tau)$ : mock modular form

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m \qquad q = e^{2\pi i\tau}$$

$$h\left(-\frac{1}{\tau}\right) = \tau^2 h(\tau) \; + \tau^2 \int_{-i\infty}^{0} \frac{f(\tau,\bar{v})}{(\bar{v}-\tau)^s} d\bar{v}$$

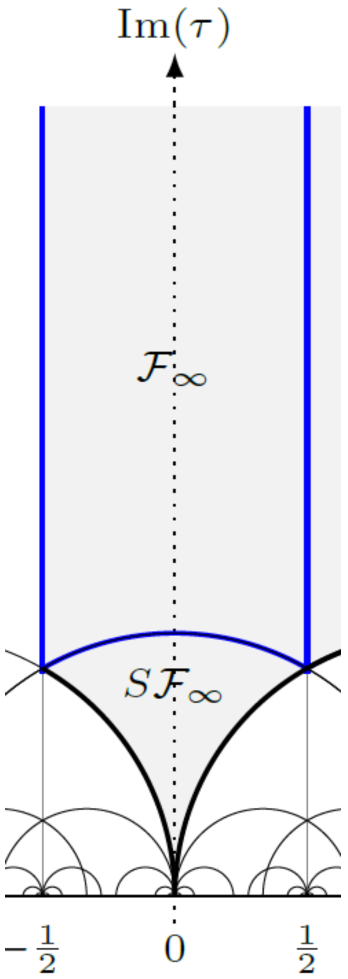
# Doing The Integral

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, y^{-s}\, f(\tau,\bar{\tau})$$

$$\partial_{\bar{\tau}} \hat{h} = y^{-s}\, f(\tau,\bar{\tau})$$

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m$$

$$I_f = d(0)$$




Note: $d(0)$ undetermined by diffeq but fixed by the modular properties: Subtle!

∃ Long history of the definition & evaluation of such integrals with singular modular forms – refs at the

# Examples 1.1

$$X = \mathbb{CP}^2: \quad b_2 = b_2^+ = 1$$

$$Z_u = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau} \ \mathcal{H} \ \Psi$$

$$\mathcal{H} = \frac{\vartheta_4^{12}}{\eta^9} \exp[\, S^2 T(\tau) \,]$$

$$\Psi = e^{-2\pi y \, b^2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \partial_{\bar{\tau}} \Big( \sqrt{y} \, (k + b) \Big) (-1)^k \, \bar{q}^{k^2} \, e^{-2\pi i \, \bar{z} \, k}$$

$$y = Im(\tau) \qquad z = \frac{S}{\omega} \qquad b = \frac{Im(z)}{y}$$

# Examples 1.2

$$h(\tau,z) = \frac{r}{\vartheta_4(\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n^2}{2}-\frac{1}{8}}}{1-r^2 q^{n-\frac{1}{2}}}$$

$$r = e^{i\,\pi\,z} \qquad z = \frac{S}{\omega} \qquad \omega = \vartheta_2(\tau)\vartheta_3(\tau)$$

$$Z_u = Z_{DW}(S) = [\,\mathcal{H}\,h(\tau,z)]_{q^0}$$

$$Z_{DW}(S) = -\frac{3}{2}S + S^5 + 3\,S^9 + 54\,S^{13} + 2540\,S^{17} + \cdots$$